Examples of Ch. 1

EX.1 Solve the 3x3 system
$$\begin{cases} x_1 + 2x_2 + x_3 = 3 \\ 3x_1 - x_2 - 3x_3 = -1 \\ 2x_1 + 3x_2 + x_3 = 4 \end{cases}$$

STEP 1 form the augmented matrix
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right)$$

STEP 2 elimination: perform elementary row operations on the rows of the augmented matrix to obtain a reduced system:

1) eliminate the entries in column 1 below the 1st entry:

pivot $= 1$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right) \xrightarrow[R_2 - \alpha \cdot R_1 \to R_2]{\alpha = \frac{3}{1}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 2 & 3 & 1 & 4 \end{array}\right) \xrightarrow[R_3 - \alpha \cdot R_1 \to R_3]{\alpha = \frac{2}{1}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & -1 & -1 & -2 \end{array}\right)$$

2) eliminate the entries in column 2 below the 2nd entry:

pivot $= -7$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & -1 & -1 & -2 \end{array}\right) \xrightarrow[R_3 - \alpha \cdot R_2 \to R_3]{\alpha = \frac{-1}{-7}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & 0 & \frac{-1}{7} & \frac{-4}{7} \end{array}\right)$$

STEP 3 solve the obtained strictly triangular system by back substitution:

$$\begin{cases} x_1 + 2x_2 + x_3 = 3 & (1) \\ -7x_2 - 6x_3 = -10 & (2) \\ -\frac{1}{7}x_3 = \frac{-4}{7} & (3) \end{cases}$$

(3) $\Rightarrow \boxed{x_3 = 4}$

(2) $\Rightarrow -7x_2 - 6 \times 4 = -10 \Rightarrow \boxed{x_2 = -2}$

(1) $\Rightarrow x_1 + 2 \times (-2) + 4 = 3 \Rightarrow \boxed{x_1 = 3}$

$\boxed{\text{Ex. 2}}$ Solve the 3x3 system
$$\begin{cases} \quad\;\; -x_2 - x_3 = 1 \\ x_1 + x_2 + x_3 = 4 \\ 3x_1 + 4x_2 + x_3 = -2 \end{cases}$$

STEP 1 augmented matrix
$$\left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 1 & 1 & 1 & 4 \\ 3 & 4 & 1 & -2 \end{array}\right)$$

STEP 2 elimination

1) column 1

Pivot $= 0 \Rightarrow$ interchange the first two rows $\Rightarrow$ pivot $= 1$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -1 & 1 \\ 3 & 4 & 1 & -2 \end{array}\right) \xrightarrow[R_3 - \alpha \cdot R_1 \to R_3]{\alpha = \frac{3}{1}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & -2 & -14 \end{array}\right)$$

2) column 2

pivot $= -1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & -2 & -14 \end{array}\right) \xrightarrow[R_3 - \alpha \cdot R_2 \to R_3]{\alpha = \frac{1}{-1}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -3 & -13 \end{array}\right)$$

STEP 3 back substitution

$$\begin{cases} x_1 + x_2 + x_3 = 4 & (1) \\ \quad\;\; - x_2 - x_3 = 1 & (2) \\ \qquad\quad -2x_3 = -14 & (3) \end{cases}$$

$(1) \Rightarrow \boxed{x_3 = 7}$

$(2) \Rightarrow -x_2 - 7 = 1 \Rightarrow \boxed{x_2 = -8}$

$(3) \Rightarrow x_1 - 8 + 7 = 4 \Rightarrow \boxed{x_1 = 5}$

$\boxed{\text{EX.3}}$ Consider the system represented by the augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right]$$

Solve the system by reducing it to row echelon (RE) form.

<u>STEP 1</u> elimination

1) <u>column 1</u> : pivot $=1$

$$\xrightarrow{\text{after some work}} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right]$$

2) <u>column 2</u> : Since all entries below the 2nd entry are zero, we move over to column 3 and eliminate the entries below the 2nd entry.

3) <u>column 3</u> : pivot $=1$

$$\xrightarrow{\text{after some work}} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

4) <u>column 4</u> : since all entries below the 3rd entry are zero, we move over to column 5 and eliminate the entries below the 3rd entry.

5) <u>column 5</u> : pivot $= 1$

$$\xrightarrow{\text{after some work}} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right]$$

Note that since the first nonzero entries in each nonzero row are already one, we do not need to perform any row operation of type II to make them one.

<u>STEP 2</u> solution:

Since the RE form of the augmented matrix contains rows of the form $[0\ 0\ 0\ 0\ 0 \mid a \neq 0]$, the system is <u>inconsistent</u>.

$\boxed{\text{EX. 4}}$ Consider the system represented by the augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{array}\right]$$

Solve the system by reducing it to row echelon (RE) form.

STEP 1 elimination

after some work (an exercise for you) we obtain the augmented matrix in RE form:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

STEP 2 Solution

the last two rows of the augmented matrix are of the form $[0\,0\,0\,0\,0|0]$. therefore the last two equations of the system will always be satisfied for any five-tuple $(x_1, x_2, x_3, x_4, x_5)$.

we consider the first three equations:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 1 \\ x_3 + x_4 + 2x_5 = 0 \\ x_5 = 3 \end{cases}$$

lead variables: x_1, x_3, x_5

free variables: x_2, x_4

we transfer the free variables to the right-hand-side. Let $\begin{cases} x_2 = a \\ x_4 = b \end{cases}$

$$\begin{cases} x_1 + x_3 + x_5 = 1 - a - b & (1) \\ x_3 + 2x_5 = -b & (2) \\ x_5 = 3 & (3) \end{cases}$$

back substitution: $\boxed{x_5 = 3}$, $\boxed{x_3 = -b-6}$, $\boxed{x_1 = 4-a}$

Therefore there are many solutions $(4-a, a, -b-6, b, 3)$.

EX. 5 Which of the following matrices are in RE form:

$$A=\begin{bmatrix}1&3&0\\0&1&3\\0&0&1\end{bmatrix}\quad B=\begin{bmatrix}1&3&0\\0&1&3\\0&0&0\end{bmatrix}\quad C=\begin{bmatrix}0&0&0\\1&3&0\\0&1&3\end{bmatrix}\quad D=\begin{bmatrix}1&3&0\\0&1&3\\0&0&2\end{bmatrix}\quad E=\begin{bmatrix}1&3&0&1\\0&0&1&2\\0&0&1&0\end{bmatrix}$$

1) Only A and B are in RE form. Both of them satisfy the three conditions of RE form.

2) C is not in RE form, because the first row is a zero row, which is not below the nonzero rows.

   Note that by performing row operation of type I (exchange row 1 & 3), we can transform C into RE form.

3) D is not in RE form, because the first nonzero entry in row 3 is not one.

   Note that by performing row operation of type II (multiply row 3 by $\frac{1}{2}$), we can transform D into RE form.

4) E is not in RE form, because the number of leading zero entries in row 3 is not greater than the number of leading zero entries in row 2.

$\boxed{\text{Ex. 6}}$ Consider a reduced system with the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

a) Is the augmented matrix in RE form?

b) Is the system consistent?

c) Is there a unique solution?

Motivate your answers.

a) Yes, the augmented matrix is in RE form, because:

1) the first nonzero entry in each nonzero row (rows 1,2,3,4) is one.

2) the number of leading zero entries in row 4 is greater than that of row 3, and the number of leading zero entries in row 3 is greater than that of row 2, and the number of leading zero entries in row 2 is greater than that of row 1.

3) the only zero row (row 5) is below nonzero rows (rows 1,2,3,4).

b) Yes, the system is consistent, because the augmented matrix is in RE form, and it does not contain a row of form $[0\ 0\ 0\ 0 \mid a \neq 0]$.

c) Yes, the solution is unique, because the last equation is always satisfied, and the first four equations in four unknowns correspond to the first four rows of the augmented matrix, which is in strictly triangular form.

Therefore a unique solution can be obtained by back substitution.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

strictly triangular

$\boxed{\text{EX. 7}}$ Using elimination, investigate if the following overdetermined system is consistent or not:

$$\begin{cases} x_1 + x_2 = 1 \\ x_1 - x_2 = 0 \\ -x_1 + 2x_2 = 5 \end{cases}$$

STEP 1 augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{array}\right)$$

STEP 2 elimination

1) column 1 - pivot = 1

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{array}\right) \xrightarrow[R_2 - \alpha \cdot R_1 \to R_2]{\alpha = \frac{1}{1}} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & -1 \\ -1 & 2 & 5 \end{array}\right) \xrightarrow[R_3 - \alpha R_1 \to R_3]{\alpha = \frac{-1}{1}} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 3 & 6 \end{array}\right)$$

2) column 2 - pivot = -2

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 3 & 6 \end{array}\right) \xrightarrow[R_3 - \alpha \cdot R_2 \to R_3]{\alpha = \frac{3}{-2}} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & \frac{9}{2} \end{array}\right)$$

STEP 3 If needed, we can transform the reduced matrix into RE form

For this, we just need to apply row operation of type II on the 2nd row, that is, we multiply the 2nd row by $\frac{-1}{2}$ and obtain

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{9}{2} \end{array}\right)$$

STEP 4 solution

Since the RE form of the augmented matrix contains a row of the form $[0 \quad 0 \mid a \neq 0]$, the system is inconsistent.

$\boxed{\text{EX.8}}$ Using elimination, investigate whether the following underdetermined system is consistent or not:

$$\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 2x_1 + 4x_2 + 2x_3 = 3 \end{cases}$$

<u>STEP 1</u> augmented matrix: $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right)$

<u>STEP 2</u> elimination

1) <u>Column 1</u> – pivot $=1$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right) \xrightarrow[R_2 - \alpha \cdot R_1 \rightarrow R_2]{\alpha = \frac{2}{1}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

the reduced matrix is already in RE form.

<u>STEP 3</u> solution

Since the RE form of the augmented matrix contains a row of the form $[0 \quad 0 \quad 0 \mid a \neq 0]$, the system is inconsistent.

$\boxed{\text{EX. 9}}$ Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \end{pmatrix}$.

Represent A both in terms of its column vectors and in terms of its row vectors.

1) Colum vector representation:

Let $a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $a_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $a_3 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Then $A = (a_1, a_2, a_3)$.

2) row vector representation:

Let $\bar{a}_1 = (1 \;\; 1 \;\; 3)$, $\bar{a}_2 = (0 \;\; 1 \;\; 2)$. Then $A = \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \end{pmatrix}$.

$\boxed{\text{EX. 10}}$ Find AB, wher $A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 1 \\ -3 & 2 \end{pmatrix}$.

We first note that $A \in \mathbb{R}^{3\times 2}$ and $B \in \mathbb{R}^{2\times 2}$. Hence, it is possible to multiply A and B and obtain $AB \in \mathbb{R}^{3\times 2}$.

$$AB = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1\times(-2) - 1\times(-3) & 1\times 1 - 1\times 2 \\ 2\times(-2) + 2\times(-3) & 2\times 1 + 2\times 2 \\ 2\times(-2) \times 3\times(-3) & 2\times 1 + 3\times 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -10 & 6 \\ -13 & 8 \end{pmatrix}$$

$\boxed{\text{Ex. 11}}$ Find BA, with A and B given in previous example.

Since $B \in \mathbb{R}^{2\times 2}$ and $A \in \mathbb{R}^{3\times 2}$, the number of columns of B is not equal to the number of rows of A, and hence BA is not defined.

$\boxed{\text{Ex. 12}}$ Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Verify that $AB \neq BA$.

We first note that since $A, B \in \mathbb{R}^{2\times 2}$, both AB and BA are defined.

1) $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1\times1+2\times0 & 1\times0+2\times2 \\ 3\times1+4\times0 & 3\times0+4\times2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 8 \end{pmatrix}$

2) $BA = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1\times1+0\times3 & 1\times2+0\times4 \\ 0\times1+2\times3 & 0\times2+2\times4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix}$

Since $\begin{pmatrix} 1 & 4 \\ 3 & 8 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix}$, therefore $AB \neq BA$.

$\boxed{\text{Ex. 13}}$ Write the following system of equations in matrix form:

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 4 \\ x_1 - x_2 + 2x_3 = 1 \end{cases}$$

In matrix form, the system reads: $\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

If we set $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, then the system reads $A\mathbf{x} = \mathbf{b}$.

$\boxed{\text{Ex. 14}}$ Consider the system $\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 1 \end{cases}$.

a) Write the system in matrix form $A\mathbf{x} = \mathbf{b}$

b) By showing that $\mathbf{b}$ cannot be written as a linear combination of column vectors of A, verify that the system is not consistent.

a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) Let us form a linear combination of column vectors of A:

$$x_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} + \begin{pmatrix} 2x_2 \\ 4x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 2(x_1 + 2x_2) \end{pmatrix}$$

Obviously, the 2nd entry of the resulting vector is double the 1st entry. Hence we can write $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in this form $\Rightarrow$ By Theorem $\boxed{\text{T1}}$, the system is inconsistent.

$\boxed{\text{Ex. 15}}$ Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Find A^2 and A^3.

$$A^2 = AA = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

$\boxed{\text{Ex. 16}}$ Is $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ the inverse of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$?

We need to check whether $AB = I$?

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2\times1+1\times(-1) & 2\times(-1)+1\times2 \\ 1\times1+1\times(-1) & 1\times(-1)+1\times2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Therefore B is the inverse of A.

<u>Note</u>: We can also check whether $BA = I$. However, only one check is enough.
Here: $BA = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.

$\boxed{\text{Ex. 17}}$ Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

We need to show that $(ABC)(C^{-1}B^{-1}A^{-1}) = I$.

We write: $(ABC)(C^{-1}B^{-1}A^{-1}) = (AB)(CC^{-1})(B^{-1}A^{-1})$

here we use associative law in Theorem $\boxed{T2}$

$$= (AB)I(B^{-1}A^{-1})$$
$$= A(BB^{-1})A^{-1}$$
$$= A\,I\,A^{-1}$$
$$= AA^{-1}$$
$$= I.$$

EX. 18 Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Determine the type of the following elementary matrices, and show and state what happens if we premultiply A by them:

$$E_1 = \begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

1) E_1 is of type (III): E_1 is obtained from I by adding the second row of I to 4 times its first row.

$$E_1 A = \begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 12 \\ 3 & 4 \end{pmatrix}$$

By premultiplying A by E_1, the 2nd row of A is added to 4 times its first row.

2) E_2 is of type (II): E_2 is obtained from I by multiplying the second row of I by 2.

$$E_2 A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix}$$

By premultiplying A by E_2, the 2nd row of A is multiplied by 2.

3) E_3 is of type (I): E_3 is obtained by interchanging the rows of I.

$$E_3 A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

It interchanges the rows of A.

4) E_4 is of type (III): E_4 is obtained from I by adding 3 times the 1st row of I to its 2nd row.

$$E_4 A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 6 & 10 \end{pmatrix}$$

It adds 3 times the 1st row of A to the 2nd row of A.

$\boxed{\text{EX. 19}}$ Let $E = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ be an elementary matrix.

Verify that $E^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and that E^{-1} is an elementary matrix of the same type as E.

1) We need to verify that $EE^{-1} = I$:

$$EE^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1\times1+0\times0 & 1\times0+0\times\frac{1}{2} \\ 0\times1+2\times0 & 1\times0+2\times\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

2) Both E and E^{-1} are of type (II). E is obtained from I by multiplying the 2nd row of I by 2, and E^{-1} is obtained from I by multiplying the 2nd row of I by $\frac{1}{2}$.

$\boxed{\text{EX. 20}}$ Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$. Find A^{-1}.

We form the augmented matrix $(A|I)$ and redue it to obtain $(I|A^{-1})$.

$$(A|I) = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right) \xrightarrow[R_2+R_1\to R_2]{} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \end{array}\right) \xrightarrow[R_1-\frac{2}{3}R_2\to R_1]{} \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & \frac{-2}{3} \\ 0 & 3 & 1 & 1 \end{array}\right) \xrightarrow[\frac{1}{3}R_2\to R_2]{} \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & \frac{-2}{3} \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \end{array}\right) = (I|A^{-1})$$

$$\Rightarrow \quad A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

<u>check</u>: $AA^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3}+\frac{2}{3} & \frac{-2}{3}+\frac{2}{3} \\ \frac{-1}{3}+\frac{1}{3} & \frac{2}{3}+\frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ O.K.

$\boxed{\text{EX. 21}}$ Find $EA = LU$ factorization of $A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$.

We will perform elimination with partial pivoting.

1) Column 1

partial pivoting $\Rightarrow$ pivot $= \max(2, 4, 1) = 4$

$\Rightarrow$ interchange rows 1 and 2 $\Rightarrow E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$EA = \begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[R_2 - \alpha . R_1 \to R_2]{\alpha = \frac{2}{4} = \frac{1}{2}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[R_3 - \alpha . R_1 \to R_3]{\alpha = \frac{1}{4}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ \frac{1}{4} & 2 & 2 \end{bmatrix}$$

> Here, instead of putting a zero in the place of eliminated entry, we store the multiplier α inside a big ◯. This way ensures us that the multipliers will stay with their row in case future row exchanges are made.

Column 2

partial pivoting $\Rightarrow$ pivot $= \max(|-1|, 2) = 2$

$\Rightarrow$ interchange rows 2 and 3 of the most recent E

$\Rightarrow E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$$EA = \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -1 & 7 \end{bmatrix} \xrightarrow{\alpha = \frac{-1}{2}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 8 \end{bmatrix}$$

Hence: $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$ $U = \begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}$

check: $EA = LU$

$\boxed{\text{EX. 22}}$ Consider the system $A\mathbf{x} = \mathbf{b}$, where A is given in the previous example, and $\mathbf{b} = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$. Using $EA = LU$ factorization find the solution $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$A\mathbf{x} = \mathbf{b} \xrightarrow{\text{multiply by } E} EA\mathbf{x} = E\mathbf{b} \xrightarrow{\text{use } EA = LU} LU\mathbf{x} = E\mathbf{b}$$

Let $U\mathbf{x} = \mathbf{c} \Rightarrow L\mathbf{c} = E\mathbf{b}$

1) Solve $L\mathbf{c} = E\mathbf{b}$ for $\mathbf{c}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & \frac{-1}{2} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & \frac{-1}{2} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 5 \end{bmatrix} \Rightarrow \begin{cases} c_1 = 0 \\ \frac{1}{4}c_1 + c_2 = 6 \Rightarrow c_2 = 6 \\ \frac{1}{2}c_1 - \frac{1}{2}c_2 + c_3 = 5 \Rightarrow c_3 = 8 \end{cases}$$

2) Solve $U\mathbf{x} = \mathbf{c}$ for $\mathbf{x}$:

$$\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 8 \end{bmatrix} \Rightarrow \begin{cases} 4x_1 + 4x_2 - 4x_3 = 0 & (1) \\ 2x_2 + 2x_3 = 6 & (2) \\ 8x_3 = 8 & (3) \end{cases}$$

$(3) \Rightarrow \boxed{x_3 = 1}$

$(2) \Rightarrow 2x_2 + 2 \times 1 = 6 \Rightarrow \boxed{x_2 = 2}$

$(3) \Rightarrow 4x_1 + 4 \times 2 - 4 \times 1 = 0 \Rightarrow \boxed{x_1 = -1}$

$\Rightarrow$ solution $\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$.