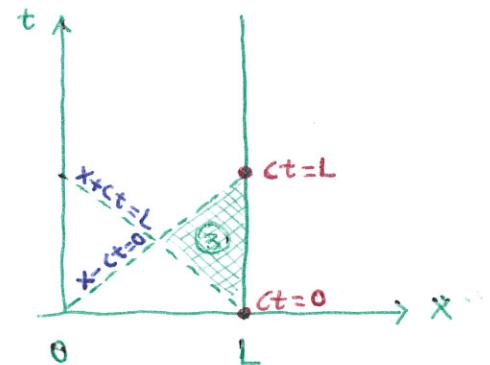


$$\left| \begin{array}{l} u_{tt} = c^2 u_{xx} \quad x \in [0, L] , \quad t \geq 0 \\ u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \\ BC1 \quad u(0, t) = 0 \\ BC2 \quad u(L, t) = 0 \end{array} \right.$$



We want to find the solution in region ③ by the method of characteristic^{es}

In region ③: $\begin{cases} 0 \leq x - ct \leq L & \text{and} \\ L \leq x + ct \leq 2L \end{cases}$

STEP 1. General solution: $u(x, t) = F(x - ct) + G(x + ct)$

where: $\begin{cases} F(x) = \frac{f(x)}{2} - \frac{1}{2c} \int_0^x g(\alpha) d\alpha & \text{for } 0 \leq x \leq L \\ G(x) = \frac{f(x)}{2} + \frac{1}{2c} \int_0^x g(\alpha) d\alpha & \text{for } 0 \leq x \leq L \end{cases}$ (1) (2)

STEP 2. We find $F(x - ct)$ and $G(x + ct)$:

Since $0 \leq x - ct \leq L \Rightarrow$ we can use (1) and obtain:

$$F(x - ct) = \frac{f(x - ct)}{2} - \frac{1}{2c} \int_0^{x-ct} g(\alpha) d\alpha$$

Since $L \leq x + ct \leq 2L \Rightarrow$ we cannot use (2) to obtain $G(x + ct)$.

We need to use BC2: $u(L, t) = 0$

$$\Rightarrow F(L - ct) + G(L + ct) = 0 \Rightarrow G(L + ct) = -F(L - ct)$$

let $z = ct \xrightarrow{\{0 \leq ct \leq L\}} G(L+z) = -F(L-z)$

Now let $L+z = x+ct \Rightarrow z = x+ct-L \Rightarrow L-z = 2L-x-ct$. Then

$$G(x+ct) = -F(\underbrace{2L-x-ct}_{0 \leq \cdot \leq L}) \Rightarrow \text{Now we can use (1)}$$

and obtain $G(x+ct) = -\frac{f(2L-x-ct)}{2} + \frac{1}{2c} \int_0^{2L-x-ct} g(\alpha) d\alpha.$

STEP 3

We finally get:

$$u(x,t) = \frac{f(x-ct)}{2} - \frac{1}{2c} \int_0^{x-ct} g(\alpha) d\alpha - \frac{f(2L-x-ct)}{2} + \frac{1}{2c} \int_0^{2L-x-ct} g(\alpha) d\alpha$$