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MATH 505, Fall 2022 – HW # **4**, Posted: Nov. 28, 2022

THE UNIVERSITY OF NEW MEXICO

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Consider the smooth traget function

$$f(x) = \frac{1}{1 + 25\,x^2}, \qquad x \in [-1, 1].$$

Historically, this is the function that Carl Runge (German mathematician, 1856-1927) studied to illustrate the instability of polynomial interpolation at equi-spaced points.

Problem 1. (50 points) In this problem, we will study the effect of different measures on the least squares solutions and get insight about the relation between the degree of the approximating polynomial (m) and the number of data points (n).

- **Data generation and measures**: Generate two sets of $n = 200$ i.i.d. samples $\{x_1, \ldots, x_n\}$, one set with respect to the uniform measure $\rho_1 := dx/2$, and one set with respect to the Chebyshev measure $\rho_2 := dx/(\pi\sqrt{1 - x^2})$. For both cases, compute observations $y_i = f(x_i)$, with $i = 1, \ldots, n$.

- **Least squares approximation**: Use the above data sets to compute the least squares approximating polynomial of degree $m < n$, for a range of values $m = 0, 1, 2, \ldots, 100$. Note that for each fixed m you should get two approximating polynomials, one polynomial for each data set.

- **Error plots**: Compute the approximation error in $L^2([-1, 1], \rho)$ norm, with ρ the corresponding measure in which the samples have been drawn. You can either use a quadrature rule to approximate the integral of the norm, or you can use built-in commands in MATLAB and PYTHON to compute the integrals. Plot the error versus m for both cases in the same figure. Make sure your figure has a clear legend and the figure axes are labeled and readable. Use log-scale for the error and a linear scale for m. For what value of m you obtain the best approximation to f for each choice of samples?

- **Discussion**: Discuss your results and observations.

Problem 2. (50 points) We next study the effect of n on the best choice of m. Recall from our theoretical results that we expect $m = \mathcal{O}(\sqrt{n})$ for uniform measures and $m = \mathcal{O}(n)$ for Chebyshev measures.

- Vary the sample size from $n = 1$ to $n = 100$, with steps of 10. That is, take $n = 1, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100$. For each fixed n repeat the process in Problem 1, with a range of values for the polynomial degree $m < n$ for which the least squares problem is stable, and find the best choice of m that corresponds to the polynomial degree for which you obtain the best approximation to f. Note that the optimal m for uniform measure will be different from the optimal m for the Chebyshev measure. Also note that as you change n, the optimal m will also change.

- Plot the optimal m values versus n for both measures in the same figure. Make sure your figure has a clear legend and the figure axes are labeled and readable. Use linear scales for both axes. Does your plots agree with theoretical results? Discuss your observations.