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MATH 505, Fall 2022 – HW # **1**, Posted: Sep. 7, 2022

THE UNIVERSITY OF NEW MEXICO

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1. Polynomial Interpolation

   (a) Consider a generic polynomial of degree n in the form

   $$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \ldots + c_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

   The problem of finding the polynomial $p(x)$ that interpolates a function f at the points $x_0,\ x_1,\ \ldots, x_n$ is equivalent to solving a linear system of the form

   $$A\mathbf{c} = \mathbf{f} \tag{1}$$

   with $\mathbf{c} = (c_0, c_1, \ldots, c_n)^\top$ and $\mathbf{f} = (f(x_0),\ f(x_1),\ \ldots, f(x_n))^\top$. Write the corresponding matrix A.

   (b) Implement a function `interpolate` that computes the coefficients $\{c_i\}_{i=0}^n$ of the interpolating polynomial $p(x)$ for a given function f by solving the system (1). The function takes as inputs the vector $(x_0,\ x_1,\ \ldots\ x_n)$ and the function f, and returns the vector $(c_0, c_1, \ldots, c_n)$.

   (c) Implement a function `evaluate` that evaluates the polynomial $p(x)$ at a set of $m+1$ points $\{x_i^{\text{ev}}\}_{i=0}^m$. Input: a vector of evaluation points $(x_0^{\text{ev}},\ x_1^{\text{ev}},\ \ldots\ x_m^{\text{ev}})$, the vector $(x_0,\ x_1,\ \ldots\ x_n)$, and the vector of coefficients $(c_0, c_1, \ldots, c_n)$. To improve efficiency you may use the recursive formula:

   $$b_n = c_n, \qquad b_{n-i} = (x - x_{n-i})b_{n-i+1} + c_{n-i},\ i = 1, 2, \ldots, n, \qquad p(x) = b_0.$$

   . How many additions and multiplications are needed to evaluate the polynomial?

   (d) Verify your `interpolate` and `evaluate` functions by taking f to be a polynomial of low degree and by plotting f and p. Also try $f(x) = sin(2\pi x)$ over $[0, 2\pi]$. Make sure your plots have readible and labeled axes with a legend.

2. Let $f(x) = \frac{1}{1+x^2}$ on $[-5, 5]$.

   (a) Find and plot the n-degree polynomial $p_n(x)$ that interpolates f on the equally spaced points $x_i = -5 + \frac{10i}{n}$, $i = 0, 1, \ldots n$. Consider $n = 3, 6, 10$.

   (b) Find and plot the n-degree polynomial $p_n(x)$ that interpolates f on the points $x_i = 5\cos\left(\frac{\pi(2i+1)}{2(n+1)}\right)$, $i = 0, 1, \ldots n$. Consider $n = 3, 6, 10$.

   (c) Plot the function $w_n(x) = \prod_{j=0}^{n}(x - x_j)$ for $n = 2, 3, \ldots, 10$, $x \in [-5, 5]$, where the points x_j are as in part (a), and when the points x_j are as in part (b).

   (d) Using interpolation theorem, discuss your observations in parts (a) and (b) on the basis of w_n graphs.

   Note: if you were not able to write the functions `evaluate` and `interpolate` in the first exercise, you can use Matalb/pyhton functions like `polyval` and `polyfit` in this exercise.

3. Consider the functions $f(x) = x - x^3$ and $q(x) = \frac{1}{4}x$, with $x \in [-1,\, 1]$.

   (a) Using the *Oscillation Theorem*, show that q is the minimax polynomial of degree 1 for the function f on the interval $[-1,\, 1]$. State the theorem hypotheses and sho that they are satisfied.

   (b) Show that q is *also* the minimax polynomial of degree (at most) 2 for the function f, over the interval $[-1,\, 1]$.

   (c) Write the interpolation error $f - p$, where p is the polynomial of degree (at most) 2 interpolating f at the points $x_0,\, x_1,\, x_2$ over the interval $[-1,\, 1]$. Show that p is the minimax polynomial if $x_0,\, x_1,\, x_2$ are taken to be the Chebyshev points. Why is this statement false for a generic (smooth) function f?

   (d) Calculate the polynomial p of degree (at most) 2 that interpolates f at the Chebyshev points and verify that, indeed, p is the minimax polynomial for u, that is, $p(x) \equiv q(x)$ on $[-1, 1]$.