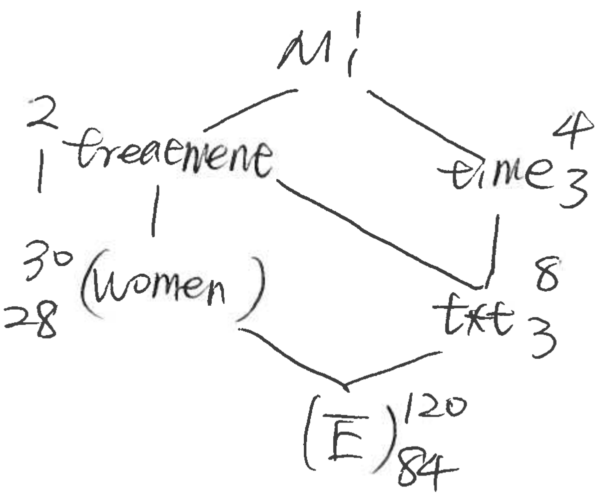
Week 10. 03/20/2018. Tuesday. ①

split plot design.

Example: 30 women are randomly assigned to 2 treatment groups. (treatment group and placebo). Their heart rates were measured at 4 time points after injection of a drug (or placebo).

	women	time1	t_2	t_3	t_4
treatment	1	y_{11}	y_{12}	y_{13}	y_{14}
	2	y_{21}	y_{22}	y_{23}	y_{24}
	⋮	⋮			
	15				
Control	16				
	17				
	⋮				
	30	$y_{(30)1}$	$y_{(30)2}$	$y_{(30)3}$	$y_{(30)4}$



Model

$h = 1, 2 \cdots m$
$l = 1, 2 \cdots g$ (# of groups)
$j = 1, 2 \cdots n$ (# of time points)

$$y_{hlj} = \mu_{lj} + \varepsilon_{hlj}$$

y_{hlj}:
- h: unit h
- l: lth group
- j: time j

μ_{lj}: mean response for l^{th} group at time j

ε_{hlj}: deviation of response above mean.

$$\mu_{lj} = \mu + \tau_l + \delta_j + (\tau\delta)_{lj}$$

- μ: overall mean
- τ_l: group effect
- δ_j: time effect
- $(\tau\delta)_{lj}$: interaction.

$$\varepsilon_{hlj} = b_{hl} + e_{hlj}$$

- b_{hl}: subject specific random effect
- e_{hlj}: within unit deviation from time j

Assume:
$$\left.\begin{aligned} b_{hl} &\overset{iid}{\sim} N(0, \sigma_B^2) \\ e_{hlj} &\overset{iid}{\sim} N(0, \sigma_e^2) \end{aligned}\right\} \text{mutually independent.}$$

b_{hl}: main plot term

e_{hlj}: split plot term

Hierachical specification

$$y_{hlj} \mid b_{hl} \sim N(\mu_{ij} + b_{hl}, \sigma_e^2)$$

$$b_{hl} \sim N(0, \sigma_B^2)$$

marginally.

$$y_{hlj} \sim N(\mu_{ij}, \sigma_e^2 + \sigma_B^2)$$

Two ~~ob~~ observations on same individual units are correlated.

$$Cov(y_{hlj}, y_{hlk}) = \sigma_B^2 \quad j \neq k.$$

$$Corr(y_{hlj}, y_{hlk}) = \frac{\sigma_B^2}{\sigma_B^2 + \sigma_e^2}.$$

observations on distinct individuals are uncorrelated.

thinking from a multivariate perspective.

$$\underset{\sim}{y_{hl}} = \begin{pmatrix} y_{hl1} \\ y_{hl2} \\ \vdots \\ y_{hln} \end{pmatrix}$$

$$E(\underset{\sim}{y_{hl}}) = \underset{\sim}{\mu_l} = \begin{pmatrix} \mu_{l1} \\ \mu_{l2} \\ \vdots \\ \mu_{ln} \end{pmatrix}$$ mean vector for individuals in group l.

$$\mu_{lj} = \mu + \tau_l + \delta_j + (\tau\delta)_{lj}.$$

$$\underset{\sim}{\varepsilon_{hl}} = \begin{pmatrix} \varepsilon_{hl1} \\ \varepsilon_{hl2} \\ \vdots \\ \varepsilon_{hln} \end{pmatrix}$$

$$\underset{\sim}{\varepsilon_{hl}} = \underset{\sim}{b_{hl}} + \underset{\sim}{e_{hlj}}$$

write model as.

$$\underset{\sim}{y_{hl}} = \underset{\sim}{\mu_l} + \underset{\sim}{\varepsilon_{hl}}$$

$$Var(\underset{\sim}{y_{hl}}) = Var(\underset{\sim}{\varepsilon_{hl}}) = \begin{bmatrix} \sigma_B^2+\sigma_e^2 & & \sigma_B^2 \\ & \sigma_B^2+\sigma_e^2 & \\ & & \ddots & \\ \sigma_B^2 & & & \sigma_B^2+\sigma_e^2 \end{bmatrix} = \Sigma$$

$$\Sigma = \sigma_e^2 I_n + \sigma_B^2 J_n.$$

$\searrow$ $n \times n$ matrix of 1s

$$= \begin{bmatrix} \sigma^2 & & \sigma^2\rho \\ & \sigma^2 \ddots & \\ \sigma^2\rho & & \sigma^2 \end{bmatrix}$$

$$\sigma^2 = \sigma_e^2 + \sigma_B^2$$

$$\rho = \frac{\sigma_B^2}{\sigma_B^2 + \sigma_e^2} = \frac{\sigma_B^2}{\sigma^2}$$

$$= \sigma^2(1-\rho) I_n + \rho\sigma^2 J_n$$

$$\underset{\sim}{y}_{hi} \sim N(\underset{\sim}{\mu}_i, \Sigma)$$

$$f(\underset{\sim}{y}) = (2\pi)^{-n/2} |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\underset{\sim}{y} - \underset{\sim}{\mu}_i)' \Sigma^{-1} (\underset{\sim}{y} - \underset{\sim}{\mu}_i)\right\}$$

profile analysis uses the multivariate form to make inferences on the mean vectors, allowing Σ to be arbitrary.