2 Probability

CHAPTER OUTLINE

LEARNING OBJECTIVES

After careful study of this chapter you should be able to do the following:

1. Understand and describe sample spaces and events for random experiments with graphs, tables, lists, or tree diagrams
2. Interpret probabilities and use probabilities of outcomes to calculate probabilities of events in discrete sample spaces
3. Use permutation and combinations to count the number of outcomes in both an event and the sample space.
4. Calculate the probabilities of joint events such as unions and intersections from the probabilities of individual events
5. Interpret and calculate conditional probabilities of events
6. Determine the independence of events and use independence to calculate probabilities
7. Use Bayes' theorem to calculate conditional probabilities
8. Understand random variables

2-1 Sample Spaces and Events

2-1.1 Random Experiments

Definition

An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a **random experiment.**

2-1 Sample Spaces and Events

2-1.1 Random Experiments

Example: Toss a coin.

Coin may land on heads (H) or tails (T).

2-1 Sample Spaces and Events

2-1.2 Sample Spaces

Definition

The set of all possible outcomes of a random experiment is called the **sample space** of the experiment. The sample space is denoted as S.

2-1 Sample Spaces and Events

2-1.2 Sample Spaces

- **Discrete S={1, 2, 3}**
- **Continuous**

Example 2-1

Consider an experiment in which you select a molded plastic part, such as a connector, and measure its thickness. The possible values for thickness depend on the resolution of the measuring instrument, and they also depend on upper and lower bounds for thickness. However, it might be convenient to define the sample space as simply the positive real line

$$S = R^{+} = \{x \mid x > 0\}$$

because a negative value for thickness cannot occur.

2-1 Sample Spaces and Events

Example: Toss a coin continued.

In this example, the sample space is S={H,T}

Tree Diagrams

- Sample spaces can also be described graphically with **tree diagrams.**
  - When a sample space can be constructed in several steps or stages, we can represent each of the n_1 ways of completing the first step as a branch of a tree.
  - Each of the ways of completing the second step can be represented as n_2 branches starting from the ends of the original branches, and so forth.

2-1 Sample Spaces and Events

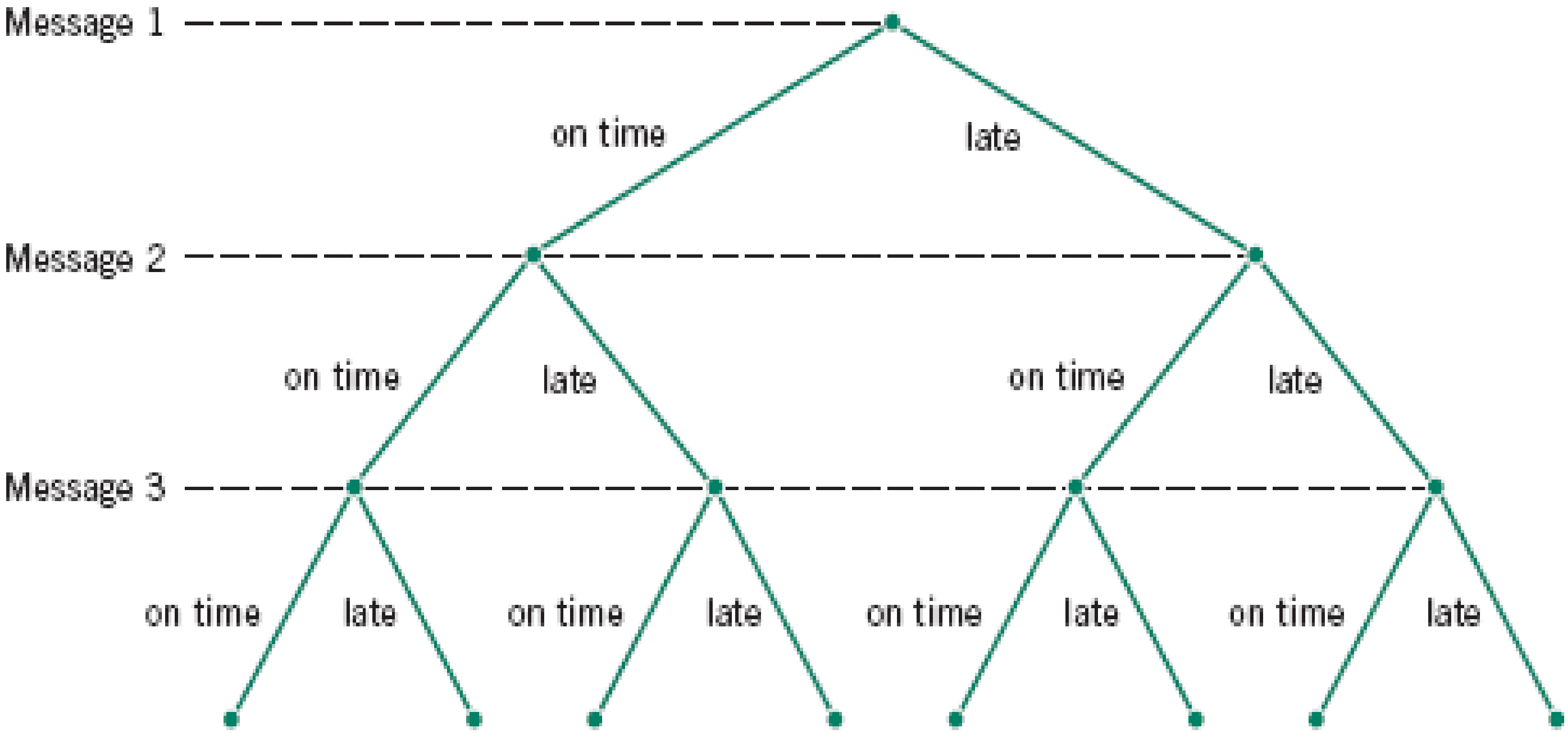


Figure 2-5 Tree diagram for three messages.

2-1 Sample Spaces and Events

Example 2-3, S={OT,OT,OT; OT, OT, L; OT,L,OT; OT,L,L; L,OT,OT; L,OT,L; L,L,OT; L,L,L}

Each message in a digital communication system is classified as to whether it is received within the time specified by the system design. If three messages are classified, use a tree diagram to represent the sample space of possible outcomes.

Each message can either be received on time or late. The possible results for three messages can be displayed by eight branches in the tree diagram shown in Fig. 2-5.

2-1 Sample Spaces and Events

2-1.3 Events

Definition

An **event** is a subset of the sample space of a random experiment.

2-1 Sample Spaces and Events

2-1.3 Events

Basic Set Operations

- The **union** of two events is the event that consists of all outcomes that are contained in either of the two events. We denote the union as $E_1 \cup E_2$.
- The **intersection** of two events is the event that consists of all outcomes that are contained in both of the two events. We denote the intersection as $E_1 \cap E_2$.
- The **complement** of an event in a sample space is the set of outcomes in the sample space that are not in the event. We denote the component of the event E as E'.

2-1 Sample Spaces and Events

2-1.3 Events

Example: Toss a coin continued.

S={H,T}

E1={Head appears}={H}

E2={Tail appears}={T}

E1 and E2 are subsets of the sample space S.

2-1 Sample Spaces and Events

Definition

Two events, denoted as E_1 and E_2, such that

$$E_1 \cap E_2 = \varnothing$$

are said to be **mutually exclusive.**

2-1 Sample Spaces and Events

Venn Diagrams

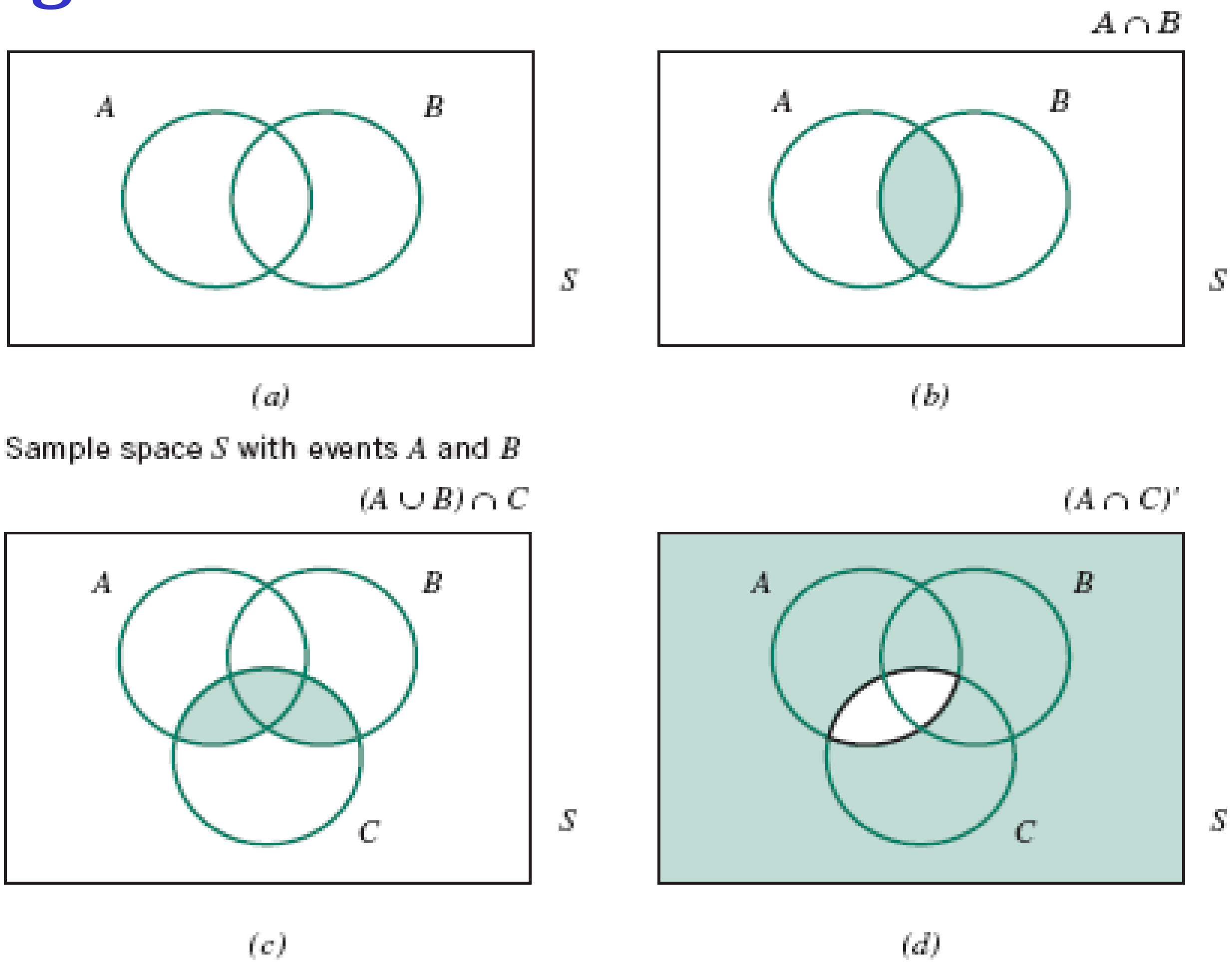


Figure 2-8 Venn diagrams.

2-1 Sample Spaces and Events

2-1.4 Counting Techniques

Multiplication Rule

If an operation can be described as a sequence of k steps, and

if the number of ways of completing step 1 is n_1, and

if the number of ways of completing step 2 is n_2 for each way of completing step 1, and

if the number of ways of completing step 3 is n_3 for each way of completing step 2, and so forth,

the total number of ways of completing the operation is

$$n_1 \times n_2 \times \cdots \times n_k$$

2-1 Sample Spaces and Events

2-1.4 Counting Techniques

Permutations

The number of **permutations** of n different elements is $n!$ where

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \qquad (2\text{-}1)$$

2-1 Sample Spaces and Events

2-1.4 Counting Techniques

Permutations : Example 2-10

A printed circuit board has eight different locations in which a component can be placed. If four different components are to be placed on the board, how many different designs are possible?

Each design consists of selecting a location from the eight locations for the first component, a location from the remaining seven for the second component, a location from the remaining six for the third component, and a location from the remaining five for the fourth component. Therefore,

$$P_4^8 = 8 \times 7 \times 6 \times 5 = \frac{8!}{4!} = 1680 \text{ different designs are possible.}$$

2-1 Sample Spaces and Events

2-1.4 Counting Techniques

Permutations of Subsets

The number of permutations of subsets of r elements selected from a set of n different elements is

$$P_r^n = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!} \qquad (2\text{-}2)$$

2-1 Sample Spaces and Events

2-1.4 Counting Techniques

Permutations of Similar Objects

The number of permutations of $n = n_1 + n_2 + \cdots + n_r$ objects of which n_1 are of one type, n_2 are of a second type, ... , and n_r are of an rth type is

$$\frac{n!}{n_1!\, n_2!\, n_3! \ldots n_r!} \tag{2-3}$$

2-1 Sample Spaces and Events

2-1.4 Counting Techniques

Permutations of Similar Objects: Example 2-12

A part is labeled by printing with four thick lines, three medium lines, and two thin lines. If each ordering of the nine lines represents a different label, how many different labels can be generated by using this scheme?

From Equation 2-3, the number of possible part labels is

$$\frac{9!}{4!\,3!\,2!} = 2520$$

2-1 Sample Spaces and Events

2-1.4 Counting Techniques

Combinations

The number of combinations, subsets of size r that can be selected from a set of n elements, is denoted as $\binom{n}{r}$ or C_r^n and

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad (2\text{-}4)$$

2-1 Sample Spaces and Events

2-1.4 Counting Techniques

Combinations: Example 2-13

A printed circuit board has eight different locations in which a component can be placed. If five identical components are to be placed on the board, how many different designs are possible?

Each design is a subset of the eight locations that are to contain the components. From Equation 2-4, the number of possible designs is

$$\frac{8!}{5!\,3!} = 56$$

2-2 Interpretations of Probability

2-2.1 Introduction

Probability

- Used to quantify likelihood or chance
- Used to represent risk or uncertainty in engineering applications
- Can be interpreted as our **degree of belief** or **relative frequency**

2-2 Interpretations of Probability

Equally Likely Outcomes

Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$.

2-2 Interpretations of Probability

Definition

For a discrete sample space, the probability of an event E, denoted by P(E) = number of outcomes in E/ number of possible outcomes in S

Example: Toss a coin continued

$P(E1) = P(H) = \frac{1}{2}$

$P(E2) = P(T) = \frac{1}{2}$

2-2 Interpretations of Probability

2-2.2 Axioms of Probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

If S is the sample space and E is any event in a random experiment,

(1) $P(S) = 1$

(2) $0 \leq P(E) \leq 1$

(3) For two events E_1 and E_2 with $E_1 \cap E_2 = \emptyset$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Review

Random Experiment

↓

Sample Space (S)

↓

Events (E)

↓

Probability P(E)

Example. A random experiment can results in one of the following outcomes {a, b, c, d}.

Let A={a,b}, B={b,c}, C={d},

Find P(A),P(B),P(C), P(A∩B) and P(AUB)

Solution: S={a,b,c,d}

P(A)=2/4=1/2

P(B)=2/4=1/2

P(C)=1/4

A∩B={b}, P(A∩B)=1/4

AUB={a,b,c}, P(AUB) = ¾

P(AUB) = P(A)+P(B)-P(A∩B)

2-3 Addition Rules

Probability of a Union

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad (2\text{-}5)$$

Mutually Exclusive Events

If A and B are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B) \qquad (2\text{-}6)$$

Example 2-16

A random experiment can result in one of the outcomes $\{a, b, c, d\}$ with probabilities 0.1, 0.3, 0.5, and 0.1, respectively. Let A denote the event $\{a, b\}$, B the event $\{b, c, d\}$, and C the event $\{d\}$. Then,

$$P(A) = 0.1 + 0.3 = 0.4$$

$$P(B) = 0.3 + 0.5 + 0.1 = 0.9$$

$$P(C) = 0.1$$

Also, $P(A') = 0.6$, $P(B') = 0.1$, and $P(C') = 0.9$. Furthermore, because $A \cap B = \{b\}$, $P(A \cap B) = 0.3$. Because $A \cup B = \{a, b, c, d\}$, $P(A \cup B) = 0.1 + 0.3 + 0.5 + 0.1 = 1$. Because $A \cap C$ is the null set, $P(A \cap C) = 0$.

Practice: The following table lists the history of 940 wafers in a semiconductor manufacturing process, suppose one wafer is selected at random.

	center	Edge	
low contamination	514	68	582
High contamination	112	246	358
	626	314	940

H = {wafer contains high levels of contamination}
C= {wafer is from the center of the spattering tool}
What is the probability that a wafer is from the center of the spattering tool or contains high levels of contamination (or both)?

Three Events

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \quad (2\text{-}7)$$

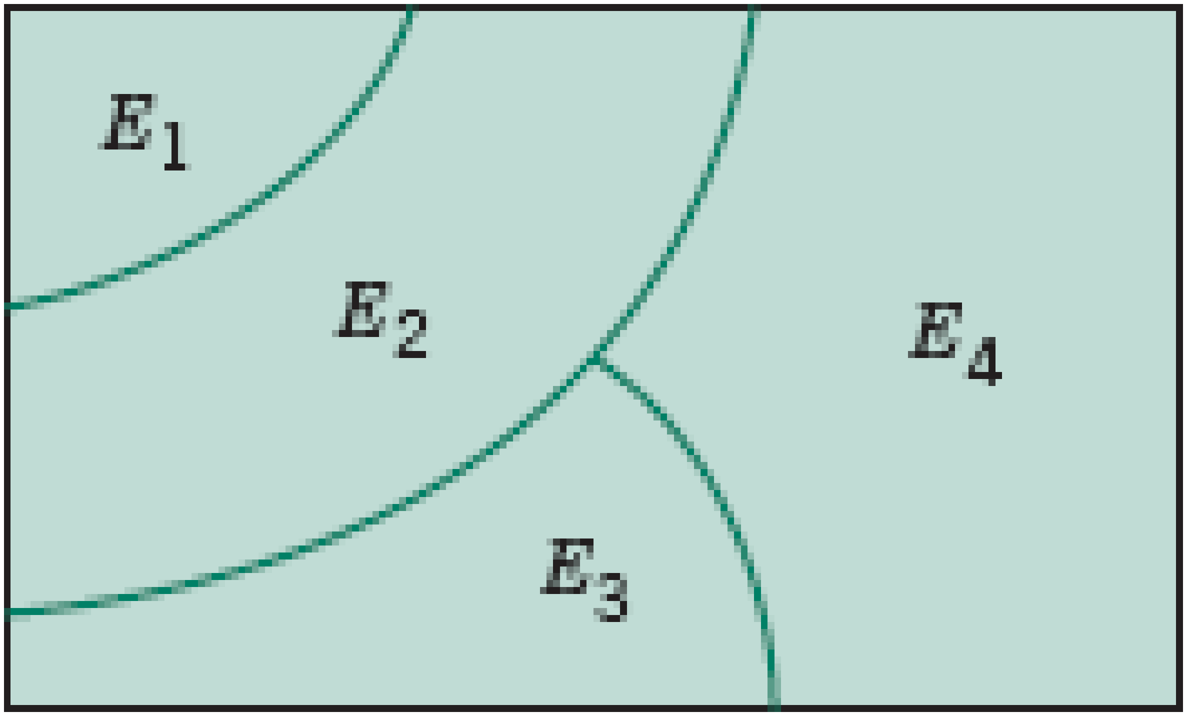


Figure 2-12 Venn diagram of four mutually exclusive events

A collection of events, $E_1, E_2, \ldots, E_k$, is said to be **mutually exclusive** if for all pairs,

$$E_i \cap E_j = \varnothing$$

For a collection of mutually exclusive events,

$$P(E_1 \cup E_2 \cup \ \ldots \ \cup E_k) = P(E_1) + P(E_2) + \ldots P(E_k) \tag{2-8}$$

2-4 Conditional Probability

- To introduce conditional probability, consider an example involving manufactured parts.
- Let *D* denote the event that a part is defective and let *F* denote the event that a part has a surface flaw.
- Then, we denote the probability of *D* given, or assuming, that a part has a surface flaw as $P(D|F)$. This notation is read as the **conditional probability** of *D* given *F*, and it is interpreted as the probability that a part is defective, given that the part has a surface flaw.

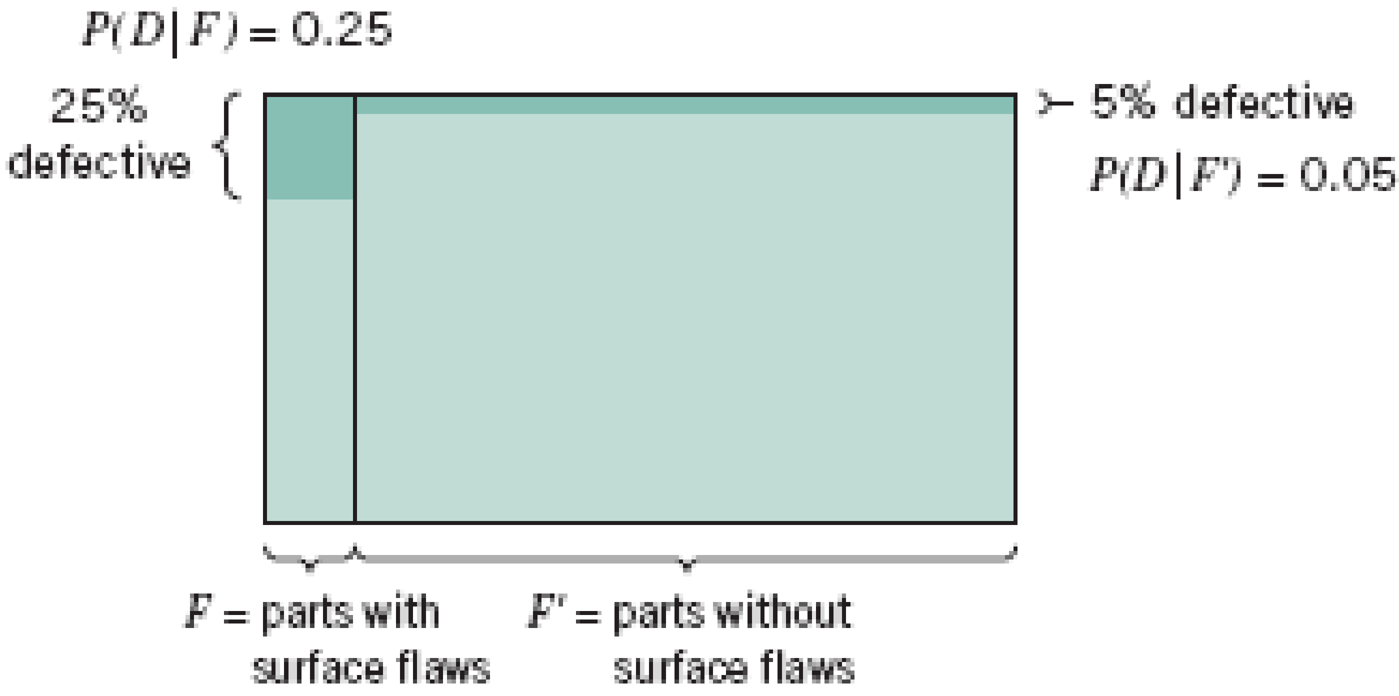


Figure 2-13 Conditional probabilities for parts with surface flaws

Definition

The **conditional probability** of an event B given an event A, denoted as $P(B|A)$, is

$$P(B|A) = P(A \cap B)/P(A) \quad (2\text{-}9)$$

for $P(A) > 0$.

Example: 400 parts are classified by surface flaws and defective.

Find the probability that the part is defective given it has surface flaws.

And the probability that the part is defective given it doesn’t have surface flaws.

	Surface flaws (F)	No surface Flaws (F′)	
Defective (D)	10	18	28
Not defective (D′)	30	342	372
	40	360	400(Total)

Want to find $P(D|F)$ and $P(D|F')$

$P(D \cap F) = 10/400$, $P(F) = 40/400$

$P(D|F) = P(D \cap F)/P(F) = (10/400)/(40/400) = 1/4 = .25$

$P(D \cap F') = 18/400$, $P(F') = 360/400$

$P(D|F') = P(D \cap F')/P(F') = (18/400)/(360/400) = .05$

Practice: A batch contain 10 parts from tool one and 40 parts from tool two. If two parts are selected randomly without replacement. Given that the 1^{st} selected is from tool one, what is the probability that the second part selected is from tool two?

Review:

$P(AUB)=P(A)+P(B)-P(A\cap B)$

$P(A|B) = P(A\cap B)/P(B)$

Can derive

$P(A|B)+P(A'|B) =1$

Proof: $P(A|B)+ P(A'|B)$

$= P(A\cap B)/P(B) + P(A'\cap B)/P(B)$

$=P(B)/P(B)=1$

2-5 Multiplication and Total Probability Rules

2-5.1 Multiplication Rule

$$P(A \cap B) = P(B \mid A)P(A) = P(A \mid B)P(B) \qquad (2\text{-}10)$$

Example: **$P(A|B) = .3$ and $P(B) = .6$, want to know what is $P(A\cap B)$ and $P(A'\cap B)$?**

Solution:

$P(A\cap B) = P(A|B)P(B)=.3\times.6=.18$

$P(A'\cap B)=P(A'|B)P(B)=(1-P(A|B))P(B)=(1-.3)\times.6=.42$

Or $P(A'\cap B) = P(B) - P(A\cap B) = .6-.18 =.42$

2-5.2 Total Probability Rule

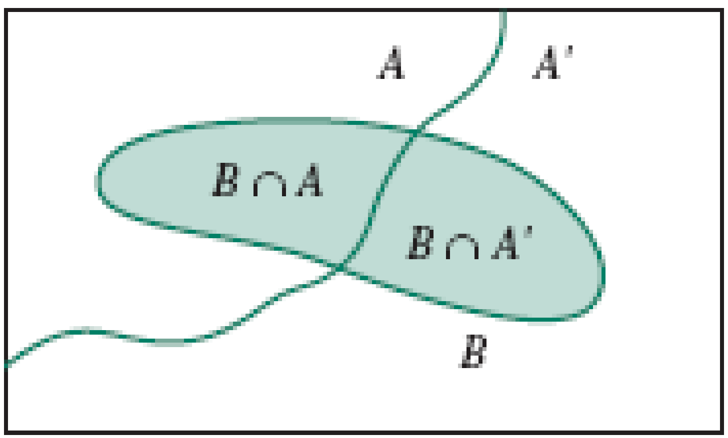
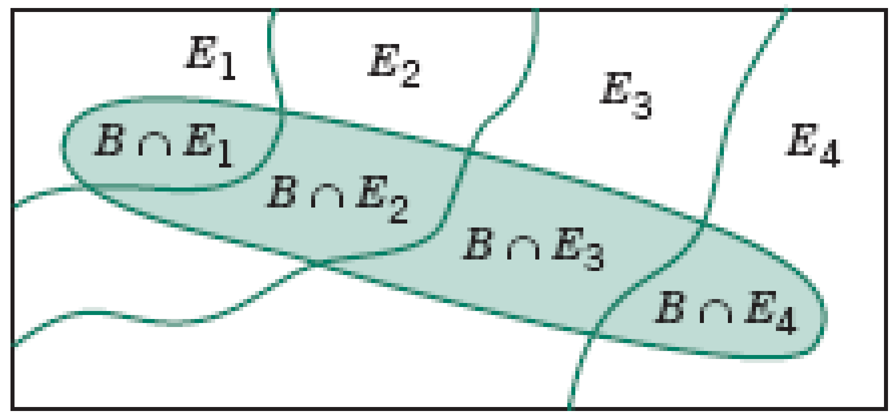


Figure 2-15 Partitioning an event into two mutually exclusive subsets.



$$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$$

Figure 2-16 Partitioning an event into several mutually exclusive subsets.

For any events A and B,

$$P(B) = P(B \cap A) + P(B \cap A') = P(B|A)P(A) + P(B|A')P(A') \qquad (2\text{-}11)$$

Example 2-27

Consider the contamination discussion at the start of this section. The information is summarized here

Probability of Failure	Level of Contamination	Probability of Level
0.1	High	0.2
0.005	Not High	0.8

Let F denote the event that the product fails, and let H denote the event that the chip is exposed to high levels of contamination. The requested probability is $P(F)$, and the information provided can be represented as

$$P(F \mid H) = 0.10 \quad \text{and} \quad P(F \mid H') = 0.005$$
$$P(H) = 0.20 \quad \text{and} \quad P(H') = 0.80$$

From Equation 2-11,

$$P(F) = 0.10(0.20) + 0.005(0.80) = 0.024$$

which can be interpreted as just the weighted average of the two probabilities of failure.

Practice: In the 2004 presidential election, exit polls from the critical state of ohio provided the following results, what is the probability a randomly selected respondent voted for Bush?

	Bush	**Kerry**
No college degree (62%)	50%	50%
College degree (38%)	53%	46%

Total Probability Rule (multiple events)

Assume $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive sets. Then

$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k) \\ &= P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k) \qquad \text{(2-12)} \end{aligned}$$

Practice: Assume the following probabilities for product failure subject to levels of contamination in semiconductor manufacturing in a particular production run. 20% of them are with high level contamination, within them, 10% are classified as product failure; 30% of them are with medium level contamination, within them, 1% are classified as product failure; and 50% of them are with low level contamination, within them, .1% of them are classified as product failure. A product is randomly selected, what is the probability that the selected product is classified as product failure?

2-6 Independence

Definition (two events)

Two events are **independent** if any one of the following equivalent statements is true:

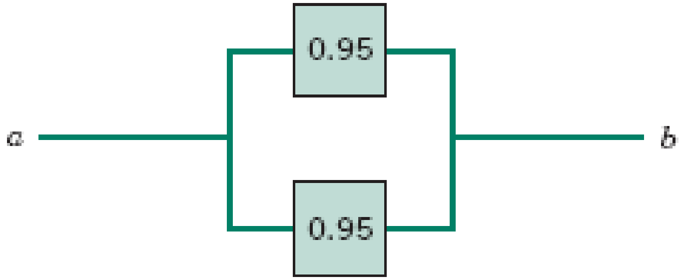
(1) $P(A|B) = P(A)$

(2) $P(B|A) = P(B)$

(3) $P(A \cap B) = P(A)P(B)$ (2-13)

Example 2-34

The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?



Let T and B denote the events that the top and bottom devices operate, respectively. There is a path if at least one device operates. The probability that the circuit operates is

$$P(T \text{ or } B) = 1 - P[(T \text{ or } B)'] = 1 - P(T' \text{ and } B')$$

a simple formula for the solution can be derived from the complements T' and B'. From the independence assumption,

$$P(T' \text{ and } B') = P(T')P(B') = (1 - 0.95)^2 = 0.05^2$$

so

$$P(T \text{ or } B) = 1 - 0.05^2 = 0.9975$$

Practice: 850 parts, 50 defective, two parts are selected without replacement. If the first part selected is defective, what is the probability that the second part selected is also defective? What is the probability that the second part selected is defective? Is the event first part selected is defective independent with the event that the second part selected is defective? Under what situation that these two events are independent?

Review:

Total probability rule

$$P(B) = P(B|A)P(A)+P(B|A')P(A')$$

2-7 Bayes' Theorem

Definition

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad \text{for} \quad P(B) > 0 \tag{2-15}$$

Bayes' Theorem

If $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive events and B is any event,

$$P(E_1|B) = \frac{P(B|E_1)P(E_1)}{P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k)} \quad (2\text{-}16)$$

for $P(B) > 0$

Example 2-37

Because a new medical procedure has been shown to be effective in the early detection of an illness, a medical screening of the population is proposed. The probability that the test correctly identifies someone with the illness as positive is 0.99, and the probability that the test correctly identifies someone without the illness as negative is 0.95. The incidence of the illness in the general population is 0.0001. You take the test, and the result is positive. What is the probability that you have the illness?

Let D denote the event that you have the illness, and let S denote the event that the test signals positive. The probability requested can be denoted as $P(D|S)$. The probability that the test correctly signals someone without the illness as negative is 0.95. Consequently, the probability of a positive test without the illness is

$$P(S|D') = 0.05$$

From Bayes' Theorem,

$$\begin{aligned} P(D|S) &= P(S|D)P(D)/[P(S|D)P(D) + P(S|D')P(D')] \\ &= 0.99(0.0001)/[0.99(0.0001) + 0.05(1 - 0.0001)] \\ &= 1/506 = 0.002 \end{aligned}$$

That is, the probability of now having the illness given a positive result from the test is only 0.002. Surprisingly, even though the test is effective, in the sense that $P(S|D)$ is high and $P(S|D')$ is low, because the incidence of the illness in the general population is low, the chances are quite small that you actually have the disease even if the test is positive.

Practice: A printer failure are associated with three types of problems, hardware problem with probability .1, software problem with probability .6 and other problems with probability .3. The probability of a printer failure given a hardware problem is .9; given a software problem is .2 and given any other type of problem is .5. What is the most likely cause of the problem?

2-8 Random Variables

Definition

A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.

A random variable is denoted by an uppercase letter such as X. After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as $x = 70$ milliamperes.

Example: Toss a coin two times. Let X={number of heads appear in each outcome}. X={0,1,2}. Possible values of r.v X is x=0, x=1 or x=2. P(X=0)=1/4, P(X=2)=1/4, P(X=1)=1/4+1/4=1/2

Toss 1	Toss 2	x	P(X=x)
H	H	2	1/4
H	T	1	1/4
T	H	1	1/4
T	T	0	1/4

Practice: In a semiconductor manufacturing process. Two wafers from a lot are tested. Each wafer is classified as pass or fail. Assume that the probability that a wafer passes the test is .8 and wafers are independent. Let X={number of wafers that pass}, what are the possible values of X? What are the probability of each x?

Definition

A **discrete** random variable is a random variable with a finite (or countably infinite) range.
A **continuous** random variable is a random variable with an interval (either finite or infinite) of real numbers for its range.

Examples of Random Variables

Examples of **continuous** random variables:

electrical current, length, pressure, temperature, time, voltage, weight

Examples of **discrete** random variables:

number of scratches on a surface, proportion of defective parts among 1000 tested, number of transmitted bits received in error