Emergent topology from finite volume topological insulators

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The Haldane Chern insulator

In two-dimensional momentum space,

$$H(\boldsymbol{k}) = \left(t_1 \sum_j \cos(\boldsymbol{k} \cdot \boldsymbol{a}_j)\right) \sigma_x - \left(t_1 \sum_j \sin(\boldsymbol{k} \cdot \boldsymbol{a}_j)\right) \sigma_y + \left(M + 2t_2 \sum_j \sin(\boldsymbol{k} \cdot \boldsymbol{b}_j)\right) \sigma_z,$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

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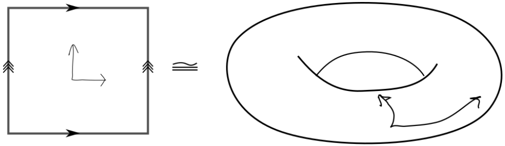
This is essentially

$$\mathbb{T}^2 \to \mathrm{Ham}(1, \mathbb{C}^2)$$

where $\mathrm{Ham}(1, \mathbb{C}^2)$ is the space of all two-by-two “insulating” Hamiltonians with one negative eigenvalue.



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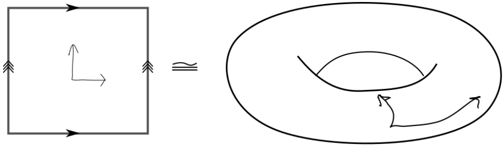
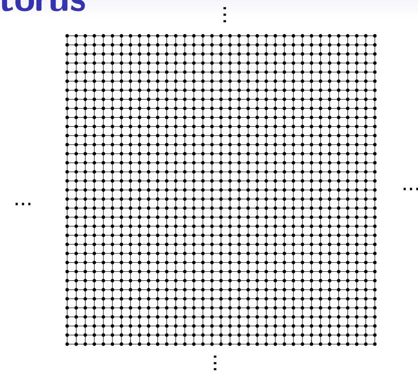
Mathematically, the torus is the Pontryagin dual of $\mathbb{Z}^2$,

$$\mathbb{T}^2 \cong \hom(\mathbb{Z}^2, \mathbb{T})$$



Momentum torus

Basic model of free fermions, H periodic on $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{2k}$.



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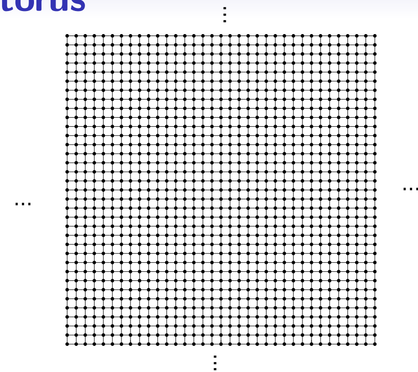

$$\mathrm{Ham}(k, \mathbb{C}^{2k}) = \left\{ A \in \boldsymbol{M}_{2k}(\mathbb{C}) \,\middle|\, A^\dagger = A,\ 0 \notin \sigma(A),\ \mathrm{sig}(A) = 0 \right\}$$

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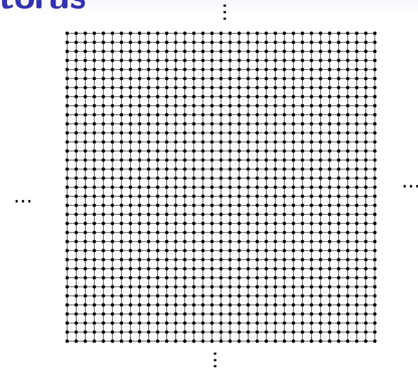

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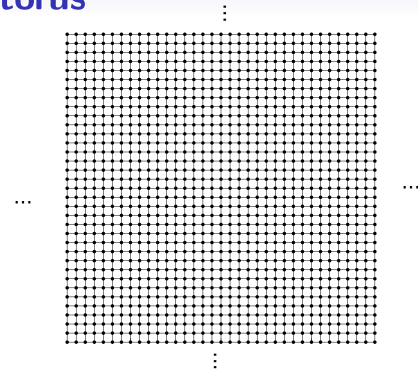

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$$\left[\mathbb{T}^2, \mathrm{Ham}(k, \mathbb{C}^{2k})\right] \approx \widetilde{K}^0(\mathbb{T}^2) \cong \mathbb{Z}$$

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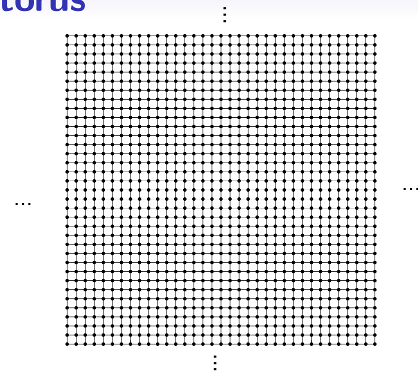

Spectrally flattened, Fourier transformed

$$\mathbb{T}^2 \to \mathrm{Gr}(k, \mathbb{C}^{2k})$$

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$$\mathrm{Gr}(k, \mathbb{C}^{2k}) = \left\{ A \in \boldsymbol{M}_{2k}(\mathbb{C}) \,\middle|\, A^\dagger = A,\ A^2 = A,\ \mathrm{rank}(A) = k \right\}$$

Breaking the momentum torus

1. Finite area
2. Open boundary conditions
3. Boundary between two phases
4. Quasicrystals

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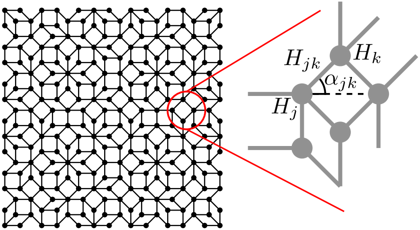
A few of these can be handled with periodic boundary conditions (flux torus/twisted boundary conditions, Bott index).

Quasicrystalline Chern insulator

Aperiodic Ammann-Beenker tiling.



"$p_x + ip_y$" tight binding model

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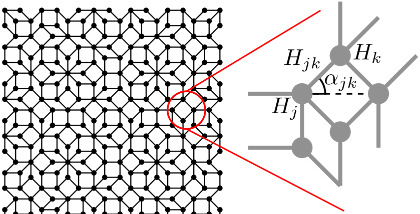

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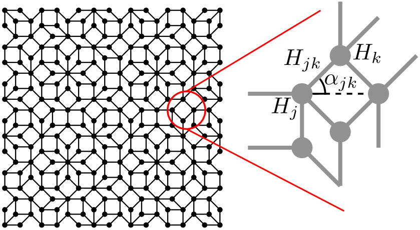

For Chern number -1:

$$\mu = 1,\ t = 1,\ \Delta = 2.$$

For Chern number 0:

$$\mu = 1,\ t = \frac{1}{3},\ \Delta = 2.$$

"$p_x + ip_y$" tight binding model

H_{QC}:

$$H_j = -\mu\sigma_z$$

$$H_{jk} = -t\sigma_z - \frac{i}{2}\Delta\sigma_x \cos(\alpha_{jk}) - \frac{i}{2}\Delta\sigma_y \sin(\alpha_{jk})$$

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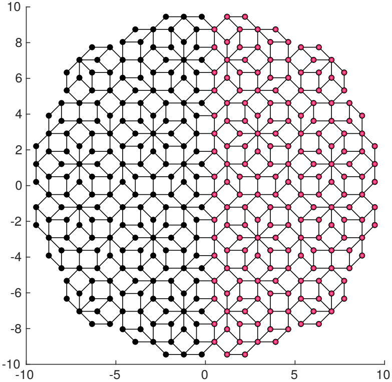
Gapped and ungapped by location



Set constants for Chern number -1 on the left (black vertices).

Set constants for Chern number 0 on the right (red vertices).

The units indicated define position operators X and Y. Using Dirichlet boundary conditions (just compress).

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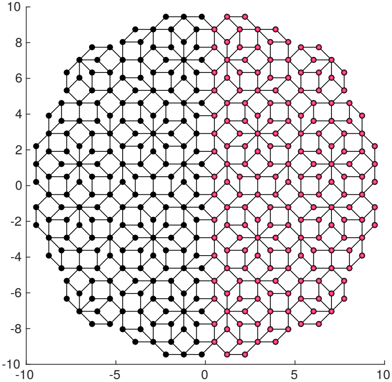

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Kitaev: How can we described gapped and gapless using the same Hilbert space?

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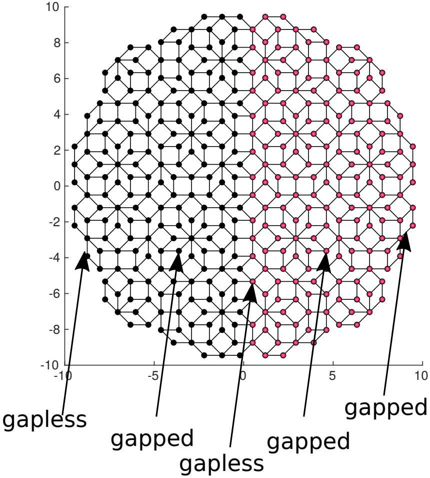

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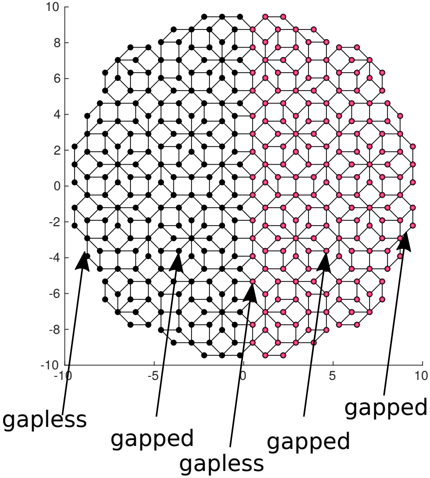

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Kitaev, A. "K-theoretic classification of free-fermion Hamiltonians." West Coast Operator Algebra Seminar, Albuquerque, 2011.

Topology from joint spectrum

Finite-area model summarized by three Hermitian matrices: X, Y, H.

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Joint approximate eigenvectors: $\|\boldsymbol{v}\| = 1$ and $\lambda_j \in \mathbb{R}$ with

$$\left(\|X\boldsymbol{v} - \lambda_1\boldsymbol{v}\|^2 + \|Y\boldsymbol{v} - \lambda_2\boldsymbol{v}\|^2 + \|H\boldsymbol{v} - \lambda_3\boldsymbol{v}\|^2\right)^{\frac{1}{2}}$$

small. Look for local minima?

Topology from joint spectrum

If we set

$$Q_{\boldsymbol{\lambda}}(X, Y, H) = (X - \lambda_1)^2 + (Y - \lambda_2)^2 + (H - \lambda_3)^2$$

then

$$\min_{\|\boldsymbol{v}\|=1} \left(\|X\boldsymbol{v} - \lambda_1 \boldsymbol{v}\|^2 + \|Y\boldsymbol{v} - \lambda_2 \boldsymbol{v}\|^2 + \|H\boldsymbol{v} - \lambda_3 \boldsymbol{v}\|^2\right)^{\frac{1}{2}} = (\sigma_{\min}(Q_{\boldsymbol{\lambda}}(X, Y, H))^{\frac{1}{2}} .$$

Notation: $\sigma_{\min}(B)$ is the smallest singular value of a matrix.

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Def. The quadratic spectrum of a triple (X,Y,H) of Hermitian matrices is the set

$$\Lambda^{Q}(X,Y,H) = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^3 \;\middle|\; \sigma_{\min}(Q_{\boldsymbol{\lambda}}(X,Y,H) = 0 \right\}$$

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A partial fix:

Def. The quadratic pseudospectrum of a triple (X, Y, H) of Hermitian matrices is based on the function

$$\boldsymbol{\lambda} \mapsto \left(\sigma_{\min}(Q_{\boldsymbol{\lambda}}(X, Y, H))\right)^{\frac{1}{2}}$$

so $$\Lambda^Q_\epsilon(X, Y, H) = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^3 \,\middle|\, \left(\sigma_{\min}(Q_{\boldsymbol{\lambda}}(X, Y, H))\right)^{\frac{1}{2}} \leq \epsilon \right\}$$

Clifford joint spectrum

Define "the localizer"

$$L_{\boldsymbol{\lambda}}(X, Y, H) = (X - \lambda_1) \otimes \sigma_x + (Y - \lambda_2) \otimes \sigma_y + (H - \lambda_3) \otimes \sigma_z$$

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A “sphere” emerges

Separate Hilbert space for bulk and boundary:

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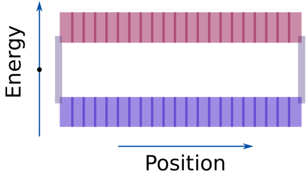
Same Hilbert space, bulk and boundary (slice at fixed-y), $\Lambda_\epsilon(X, Y, H)$:



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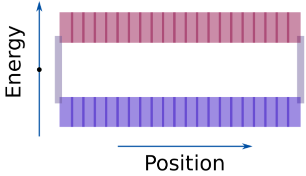
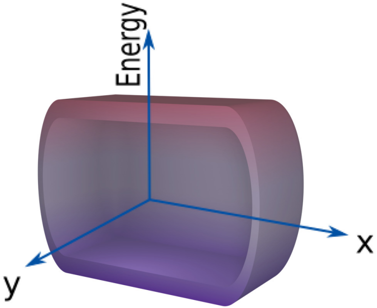
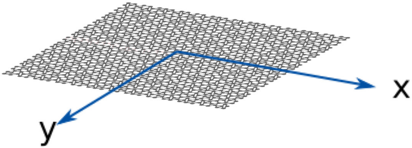


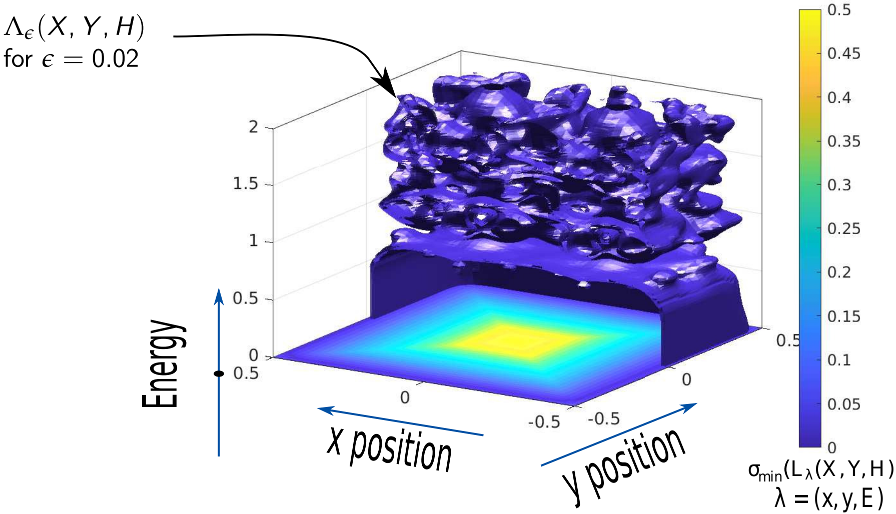




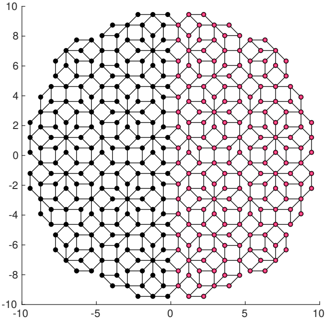
Square sample with quasiperiodic Chern insulator on all of sample.

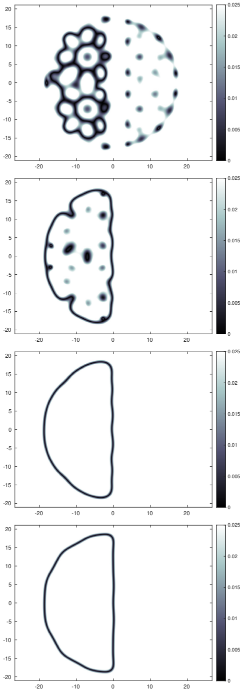
A "sphere" emerges

Square sample with quasiperiodic Chern insulator everywhere.



Chern insulator on the left, trivial insulator on the right.





$E = 1.0$

$E = 0.7$

$E = 0.6$

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K-theory

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$$L(x, y, z) = \begin{bmatrix} z & (x+5) - iy \\ (x+5) + iy & -z \end{bmatrix} \in \boldsymbol{M}_2(C(M)).$$

For conventional picture of K-theory: spectrally flatten; take a formal difference.

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$$L_{(-5,0,0)}(X, Y, H) = \begin{bmatrix} H & (X+5) - iY \\ (X+5) + iY & -H \end{bmatrix} \in \boldsymbol{M}_{2N}(\mathbb{C})$$

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Where this sits in $K_0(\boldsymbol{M}_N(\mathbb{C})) \cong \mathbb{Z}$ can be done on a computer,

$$\left[L_{(-5,0,0)}(X, Y, H) \right] \mapsto \frac{1}{2}\mathrm{sig}\left(L_{(-5,0,0)}(X, Y, H) \right)$$

A Local Index

We obtain a local index for a finite system, which can be centered at any point not in $\Lambda(X, Y, H)$,

$$\mathrm{ind}_{\lambda}(X, Y, H) = \frac{1}{2}\mathrm{Sig}\left(L_{\lambda}\left(X, Y, H\right)\right)$$

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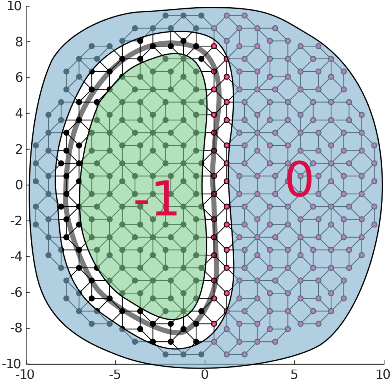

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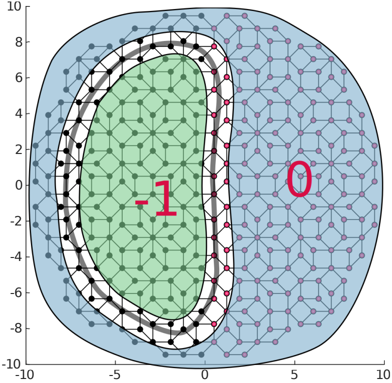
$$\mathrm{ind}_{\lambda}(X, Y, H) = \frac{1}{2}\mathrm{Sig}\left(L_{\lambda}\left(X, Y, H\right)\right)$$

$\sigma_{\min}\left(L_{\lambda}\left(X, Y, H\right)\right)$ large means more protection by the local index.

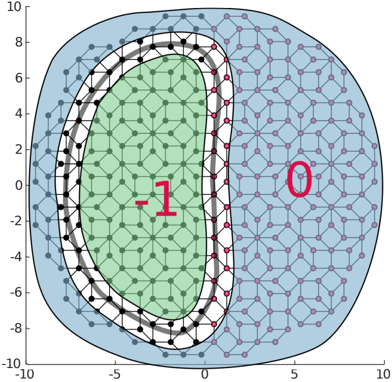


Other local K -theory markers:

1. Kitaev (2006)
2. Bianco and Resta (2011)
3. Li and Mong (2019)

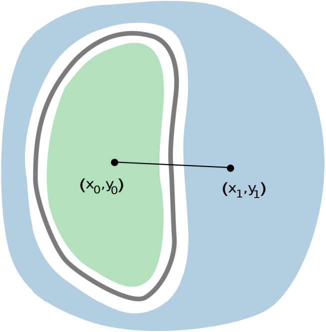
Quantifying topological protection of bulk points

$$\|\Delta H\| < \sigma_{\min}(L_\lambda(X, Y, H)) \implies \mathrm{ind}_\lambda(X, Y, H) = \mathrm{ind}_\lambda(X, Y, H + \Delta H)$$



Quantifying protection of boundary states

Assume $\operatorname{ind}_{(x_0,y_0,0)}(X,Y,H)$ does not equal $\operatorname{ind}_{(x_1,y_1,0)}(X,Y,H)$.

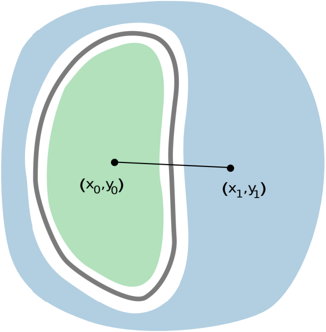


Quantifying protection of boundary states

Assume $\text{ind}_{(x_0,y_0,0)}(X, Y, H)$ does not equal $\text{ind}_{(x_1,y_1,0)}(X, Y, H)$.

Also assume

$$\|\Delta H\| < \sigma_{\min}(L_{(x_j,y_j,0)}(X, Y, H)).$$



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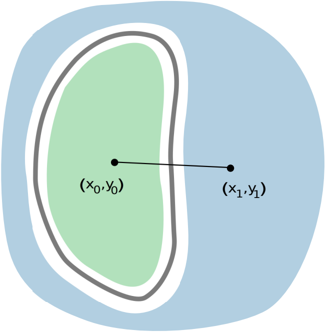


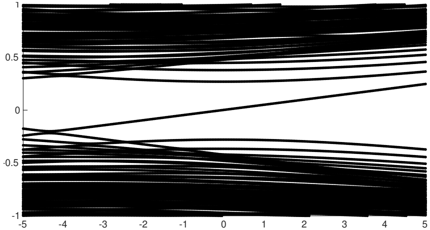
Also assume

$$\|\Delta H\| < \sigma_{\min}(L_{(x_j,y_j,0)}(X,Y,H)).$$

This means

$$L_{(x_t,y_t,0)}(X,Y,H+\Delta H)$$

has an eigenvalue cross from positive to negative.



Quantifying protection of boundary states

Assume $\mathrm{ind}_{(x_0,y_0,0)}(X,Y,H)$ does not equal $\mathrm{ind}_{(x_1,y_1,0)}(X,Y,H)$.

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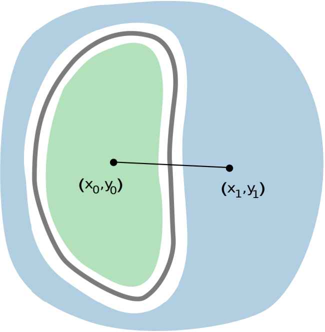
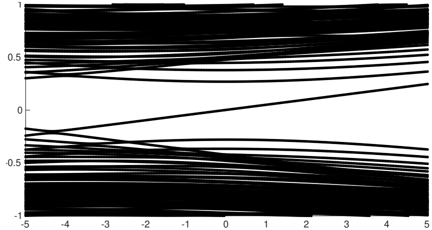


This means

$$L_{(x_t,y_t,0)}(X,Y,H+\Delta H)$$

has an eigenvalue cross from positive to negative.

Thus there is a point μ on the line with $\mu \in \Lambda(X,Y,H)$.



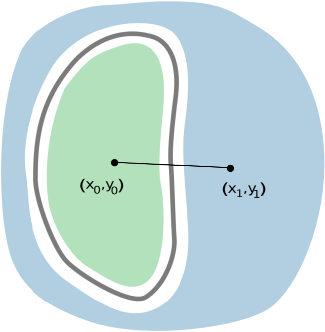
Quantifying protection of boundary states

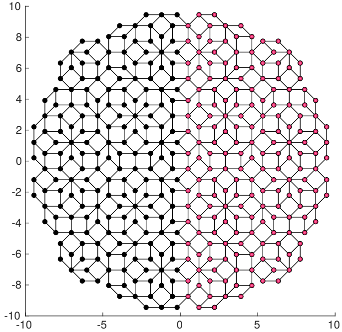
Assume
$$\operatorname{ind}_{(x_0,y_0,0)}(X,Y,H) \neq \operatorname{ind}_{(x_1,y_1,0)}(X,Y,H).$$

Also assume, for $j = 0, 1$,
$$\|\Delta H\| < \sigma_{\min}(L_{(x_j,y_j,0)}(X,Y,H)).$$

We have proven there is a unit vector $\boldsymbol{v}$ with
$$\left(\|X\boldsymbol{v} - x_t\boldsymbol{v}\|^2 + \|Y\boldsymbol{v} - y_t\boldsymbol{v}\|^2 + \|H\boldsymbol{v} - 0\boldsymbol{v}\|^2\right)^{\frac{1}{2}}$$

less than some specific bound.





Other places localizer has been used

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- 1D systems, class BDI.

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- Weak topological insulators in 2D, class D.

Other places localizer has been used

- 1D systems, class BDI.
- Weak topological insulators in 2D, class D.
- Disordered semimetals.

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- Schulz-Baldes, Hermann, and Tom Stoiber. "Invariants of disordered semimetals via the spectral localizer." EPL (Europhysics Letters) (2021).

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Thank you