# NULL SPACE AND NULLITY

## 1. An Example

Recall that a system is homogeneous if it is of the form

$$A\mathbf{x} = \mathbf{0}.$$

The solution set here goes by the name "the null space of $A$," or $N(A)$. We can speed up the row operations a little if we notice that when doing row operations on

$$[A|\,\mathbf{0}]$$

the last column never changes. We can do operation just on $A$, as long as we remember when converting rows back to equations to put zeros "$= 0$" on the right.

**Example 1.** Find the null space of $A$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 4 & 2 & 0 & 0 & 3 \\ 1 & 1 & 1 & -2 & 1 \\ 2 & 2 & 0 & 0 & 2 \\ 1 & 1 & 2 & -4 & 1 \end{bmatrix},$$

and find a basis for this null space.

We need to solve

$$A\mathbf{x} = \mathbf{0}.$$

Using just $A$, we use row ops to find reduced row echelon form:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 4 & 2 & 0 & 0 & 3 \\ 1 & 1 & 1 & -2 & 1 \\ 2 & 2 & 0 & 0 & 2 \\ 1 & 1 & 2 & -4 & 1 \end{bmatrix}$$

$$\Big\downarrow \begin{array}{c} \mathrm{R3} - 4\mathrm{R1} \to \mathrm{R3} \\ \mathrm{R4} - \mathrm{R1} \to \mathrm{R4} \\ \mathrm{R5} - 2\mathrm{R1} \to \mathrm{R5} \\ \mathrm{R6} - \mathrm{R1} \to \mathrm{R6} \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -4 & 0 \end{bmatrix}$$

$$\Big\downarrow \mathrm{R2} \leftrightarrow \mathrm{R3}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -4 & 0 \end{bmatrix}$$

$$\Big\downarrow \begin{array}{c} \mathrm{R4} - \mathrm{R3} \to \mathrm{R4} \\ \mathrm{R6} - 2\mathrm{R3} \to \mathrm{R6} \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -4 & 0 \end{bmatrix}$$

$$\Big\downarrow \begin{array}{c} \mathrm{R4} - \mathrm{R3} \to \mathrm{R4} \\ \mathrm{R6} - 2\mathrm{R3} \to \mathrm{R6} \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Big\downarrow -\tfrac{1}{2}\mathrm{R2} \to \mathrm{R2}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Big\downarrow \mathrm{R1} - \mathrm{R2} \to \mathrm{R1}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so we have the equations

$$\begin{array}{lccccccccc} & x_1 & & & & & + & \frac{1}{2}x_5 & = & 0 \\ (1) & & x_2 & & & & + & \frac{1}{2}x_5 & = & 0 \\ & & & x_3 & - & 2x_4 & & & = & 0 \end{array}$$

There are two free variables; we set $x_4 = r$ and $x_5 = s$ and find that $N(A)$ is the set of all $\mathbf{x}$ where

$$\mathbf{x} = \begin{bmatrix} -\frac{1}{2}s \\ -\frac{1}{2}s \\ 2r \\ r \\ s \end{bmatrix}.$$

To find a basis, we exand this formula to

$$\mathbf{x} = r \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This shows that $N(A)$ is spanned by two vectors:

$$N(A) = \text{span}\left( \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

In fact, these two vectors are linearly independent; to see this, we assume

$$x \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Amoung the five equations are these two:

$$\begin{aligned} 1x + 0y &= 0 \\ 0x + 1y &= 0 \end{aligned}$$

and so the only solution is $(x, y) = (0, 0)$. In conclusion,

$$\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for $N(A)$.

## 2. Null Space vs Nullity

Sometimes we only want to know how big the solution set is to

$$A\mathbf{x} = \mathbf{0}.$$

**Definition 1.** The *nullity* of a matrix $A$ is the dimension of its null space:

$$\text{nullity}(A) = \dim(N(A)).$$

It is easier to find the nullity than to find the null space. This is because

The number of free variables (in the solved equations) equals the nullity of $A$.

## 3. Finding the Null Space and Basis

There is a general method to find a basis for a null space.

(a) Use only row operations and get simplified equations for the null space like (1).
(b) Set all of the free variables to $0$, except one which you set to $1$, and calculate x. This is one element of the basis.
(c) Repeat (b) for each free variable. (So once per non-pivot row.)

**Example 2.** Find a basis for $N(A)$ where

$$A = \begin{bmatrix} 2 & 2 & -4 & 3 & -9 & 1 \\ 1 & 1 & -2 & -3 & 0 & 2 \\ 3 & 3 & -6 & -3 & -6 & 2 \end{bmatrix}.$$

Row ops, all the way to reduced row echelon:

$$\begin{bmatrix} 2 & 2 & -4 & 3 & -9 & 1 \\ 1 & 1 & -2 & -3 & 0 & 2 \\ 3 & 3 & -6 & -3 & -6 & 2 \end{bmatrix}$$

$\downarrow \mathrm{R1} \leftrightarrow \mathrm{R2}$

$$\begin{bmatrix} 1 & 1 & -2 & -3 & 0 & 2 \\ 2 & 2 & -4 & 3 & -9 & 1 \\ 3 & 3 & -6 & -3 & -6 & 2 \end{bmatrix}$$

$\downarrow \begin{array}{l} \mathrm{R2} - 2\mathrm{R1} \to \mathrm{R2} \\ \mathrm{R3} - 3\mathrm{R1} \to \mathrm{R3} \end{array}$

$$\begin{bmatrix} 1 & 1 & -2 & -3 & 0 & 2 \\ 0 & 0 & 0 & 9 & -9 & -3 \\ 0 & 0 & 0 & 6 & -6 & -4 \end{bmatrix}$$

$\downarrow \begin{array}{l} \frac{1}{3}\mathrm{R2} \to \mathrm{R2} \\ \frac{1}{3}\mathrm{R3} \to \mathrm{R3} \end{array}$

$$\begin{bmatrix} 1 & 1 & -2 & -3 & 0 & 2 \\ 0 & 0 & 0 & 3 & -3 & -1 \\ 0 & 0 & 0 & 3 & -3 & -2 \end{bmatrix}$$

$\downarrow \mathrm{R3} - \mathrm{R2} \to \mathrm{R3}$

$$\begin{bmatrix} 1 & 1 & -2 & -3 & 0 & 2 \\ 0 & 0 & 0 & 3 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$\downarrow -\mathrm{R3} \to \mathrm{R3}$

$$\begin{bmatrix} 1 & 1 & -2 & -3 & 0 & 2 \\ 0 & 0 & 0 & 3 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\downarrow \begin{array}{l} \mathrm{R1} - 2\mathrm{R3} \to \mathrm{R1} \\ \mathrm{R2} + \mathrm{R3} \to \mathrm{R2} \end{array}$

$$\begin{bmatrix} 1 & 1 & -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\downarrow \mathrm{R1} + \mathrm{R2} \to \mathrm{R1}$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\downarrow \frac{1}{3}\mathrm{R2} \to \mathrm{R2}$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So we have three free variables, and the equations become

$$\begin{array}{rcrcrcrcrcrcl} x_1 & + & x_2 & - & 2x_3 & & & - & 3x_5 & & & = & 0 \\ & & & & & & x_4 & - & x_5 & & & = & 0 \\ & & & & & & & & & & x_6 & = & 0 \end{array}$$

and so the null space is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -r + 2s + 3t \\ r \\ s \\ t \\ t \\ 0 \end{bmatrix}.$$

Set $r = 1$, $s = 0$ and $t = 0$ and we get

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Set $r = 0$, $s = 1$ and $t = 0$ and we get

$$\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Set $r = 0$, $s = 0$ and $t = 1$ and we get

$$\begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

A basis is

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Of course, the nullity of $A$ is $3$, but we can find that with less work if we don't care about a precise description of the null space.

## 4. Finding Just the Nullity

To find the nullity of $A$ :

(a) Use row and column operations to get to row echelon form.
(b) The number of non-pivot rows is the nullity of $A$.

This is similar to the situation with determinants. For now, you must trust me that this makes sense. We'll understand later why column ops can alter the null space but can't change its dimension.

**Example 3.** Find the nullity of $A$ where $A$ is again the matrix

$$A = \begin{bmatrix} 2 & 2 & -4 & 3 & -9 & 1 \\ 1 & 1 & -2 & -3 & 0 & 2 \\ 3 & 3 & -6 & -3 & -6 & 2 \end{bmatrix}.$$

Now we can also do column operations, and can stop as soon as we see where the pivots will be.

$$\begin{bmatrix} 2 & 2 & -4 & 3 & -9 & 1 \\ 1 & 1 & -2 & -3 & 0 & 2 \\ 3 & 3 & -6 & -3 & -6 & 2 \end{bmatrix}$$

$\downarrow$ R2 − 2R1 → R2
R3 − 2R1 → R3

$$\begin{bmatrix} 2 & 2 & -4 & 3 & -9 & 1 \\ -3 & -3 & 6 & -9 & 18 & 0 \\ -1 & -1 & 2 & -9 & 12 & 0 \end{bmatrix}$$

$\downarrow$C1 $\leftrightarrow$ C6

$$\begin{bmatrix} 1 & 2 & -4 & 3 & -9 & 2 \\ 0 & -3 & 6 & -9 & 18 & -3 \\ 0 & -1 & 2 & -9 & 12 & -1 \end{bmatrix}$$

$\downarrow$R2 − 3R3 → R2

$$\begin{bmatrix} 1 & 2 & -4 & 3 & -9 & 2 \\ 0 & 0 & 0 & 18 & -18 & 0 \\ 0 & -1 & 2 & -9 & 12 & -1 \end{bmatrix}$$

$\downarrow$R2 $\leftrightarrow$ R3

$$\begin{bmatrix} 1 & 2 & -4 & 3 & -9 & 2 \\ 0 & -1 & 2 & -9 & 12 & -1 \\ 0 & 0 & 0 & 18 & -18 & 0 \end{bmatrix}$$

This is enough for us to see there are three non-pivot rows, so

$$\text{nullity}(A) = 0.$$

Notice that the pivots don't end up in the same locations as before, so we have lost track of $N(A)$.

There is often no advantage of being able to do column and row operations together; it is too complicated to figure out what to do next. In specific calculations, in in the theory of rank, it tells us a lot.