1. Determinants Defined by Row Operations

Recall that the approach we’ve taken to determinants is that they are a measure of the row operations needed to reduce a matrix to $I$. Behind the scenes in that approach is are these two lemmas.

**Lemma 1.1.** If $A$ is noninvertible, then it cannot be expressed as a product of elementary matrices and its determinant is zero.

**Lemma 1.2.** If $A$ is invertible, then it is possible to write

$$A = E_1 E_2 \cdots E_m$$

where the $E_j$ are elementary, and

$$\det(A) = \det(E_1) \det(E_2) \cdots \det(E_m).$$

Recall there are easy formulas for the determinants and inverses of elementary matrices. However, it is generally more efficient to work with the row operations, and not the elementary matrices. An exception is in the proofs of the formulas in the next section.

(The key thing is to know the formulas, not those proofs.)

2. Determinants of Products, Inverses and Transposes.

**Theorem 2.1.** Suppose $A$ and $B$ are $n$-by-$n$ matrices.

(a) The determinant of a product is the product of the determinants

$$\det(AB) = \det(A) \det(B).$$

(b) The determinant of a transpose is the determinant of the original matrix

$$\det(A^T) = \det(A).$$

(c) If a matrix is nonsingular, the determinant of its inverse is the reciprocal of its determinant

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

or

$$\det(A^{-1}) = (\det(A))^{-1}.$$
It is worth noting how much easier finding products and inverses becomes after a determinant has turned the matrices into numbers.

**Example 2.2.** Compute

\[
\det \left( A^{-1}BA \right)
\]

where

\[
A = \begin{bmatrix}
1 & 2 & 1 \\
0 & 3 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
3 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}.
\]

If you do this “as written,” you need to invert a matrix and compute two products, then take a determinant. Using the rules from the last theorem, and what we know about diagonal matrices, we find

\[
\det \left( A^{-1}BA \right) = \det \left( A^{-1} \right) \det \left( B \right) \det \left( A \right)
\]

\[
= \frac{1}{\det \left( A \right)} \det \left( B \right) \det \left( A \right)
\]

\[
= \det \left( B \right)
\]

\[
= \det \left( \begin{bmatrix}
3 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 2 \\
\end{bmatrix} \right)
\]

\[
= 3 \cdot 5 \cdot 2
\]

\[
= 30.
\]

This is not just a “made up” example. We will see that products like \( A^{-1}BA \) are critical in linear algebra.

### 3. Column Operations

The main thing to know about column operations is that column ops cannot be used in the algorithms you’ve learned for inversion and solving systems.

You can use them for determinants. This follows from the fact that transposes don’t effect determinants.

For example, to compute

\[
\det \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 2 & 3 \\
\end{bmatrix}
\]

you might use

\[
R_3 \leftarrow R_3 - R_1
\]

\[
R_3 \leftarrow \frac{1}{2} R_3
\]

\[
R_3 \leftarrow R_3 - R_2
\]
to conclude
\[
\begin{vmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 2 & 3
\end{vmatrix}
= \begin{vmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 2 & 2
\end{vmatrix}
= \begin{vmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 2 & 2
\end{vmatrix}
= 2.
\]

You could instead use
\[
\mathbf{C}_3 \leftarrow \mathbf{C}_3 - \mathbf{C}_1
\]
to conclude
\[
\begin{vmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 2 & 3
\end{vmatrix}
= \begin{vmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 2 & 2
\end{vmatrix}
= 2.
\]

E-mail address: loring@math.unm.edu

University of New Mexico