DETERMINANTS DEFINED BY ROW OPERATIONS

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1. Determinants Defined by Row operations

Determinants of square matrices are best understood in terms of row operations, in my opinion. Most books start by defining the determinant via formulas that are slow except on small matrices.

The determinant of A is a single number that can be calculated while doing Gaussian elimination. You already know 95% of what it takes to calculate a determinant. The extra 5% is keeping track of some "magic numbers's that you multiply at together to create another "magic number" called the determinant of A.

(How mathematicians came to discover these magic numbers is another topic.)

Definition 1.1. We define the *factor* of every row operation as follows:

Type	assumption	Row Operation	Factor
I	$j \neq k$	$\mathrm{Rj} \leftrightarrow \mathrm{Rk}$	-1
II	$\alpha \neq 0$	$\mathrm{Rj} \leftarrow \alpha \mathrm{Rj}$	$\frac{1}{\alpha}$
III	$j \neq k$	$\mathrm{Rj} \leftarrow \mathrm{Rj} + \beta \mathrm{Rk}$	1

Definition 1.2. We define the determinant $\det(A)$ of a square matrix as follows:

(a) The *determinant* of a singular matrix is 0.
(b) The *determinant* of the identity matrix is 1.
(c) If A is non-singular, then the *determinant of* A is the product of the factors of the row operations in a sequence of row operations that reduces A to the identity.

The notation we use is $\det(A)$ or $|A|$. Generally, one drops the braces on a matrix if using the $|A|$ notation, so

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \right).$$

You can calculate the determinant using any series of row operations that ends in I. We are skipping the reasoning that shows that the product of determinant factors comes out the same no matter what series of row operations you use.

Remark 1.3. If you want to also know A^{-1} then work with $[A \mid I]$. If you just want $\det(A)$, or A as a product of elementary matrices, or to know if A is invertible, then work with A. Same row ops, just longer or shorter rows.

Example 1.4. Find

$$\begin{vmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}.$$

Since

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{factor: } 2 \qquad \mathbf{R}1 \leftarrow \tfrac{1}{2}\mathbf{R}1 \qquad \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{factor: } 1 \qquad \mathbf{R}3 \leftarrow \mathbf{R}3 + \mathbf{R}1 \qquad \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\text{factor: } 2 \qquad \mathbf{R}3 \leftarrow \tfrac{1}{2}\mathbf{R}3 \qquad \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{factor: } 1 \qquad \mathbf{R}1 \leftarrow \mathbf{R}1 - \mathbf{R}3 \qquad \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we know

$$\det(A) = 2 \cdot 1 \cdot 2 \cdot 1 = 4$$

As a reality check, here is some Matlab output. I check the determinant of A as well as of the matrices that came up between A and I.

```
>> det([2 0 2; 0 1 0; -1 0 1])
ans =
4
>> det([1 0 1; 0 1 0; -1 0 1])
ans =
2
>> det([1 0 1; 0 1 0; 0 0 2])
ans =
2
>> det([1 0 1; 0 1 0; 0 0 1])
ans =
```

```
1
>> det([1 0 0; 0 1 0; 0 0 1])
ans =
1
>>
```

Example 1.5. Find

$$\begin{vmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{vmatrix}.$$

Since

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$$\text{factor: } 2 \qquad \mathbf{R}1 \leftarrow \tfrac{1}{2}\mathbf{R}1 \qquad \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$$\text{factor: } 1 \qquad \mathbf{R}3 \leftarrow \mathbf{R}3 + \mathbf{R}1 \qquad \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we find there only two pivots, so A is singular. Thus

$$\det(A) = 0.$$

Another reality check:

```
>> det([2 0 2; 0 1 0; -1 0 -1])
ans =
0
>> det([1 0 1; 0 1 0; -1 0 -1])
ans =
0
>> det([1 0 1; 0 1 0; 0 0 0])
ans =
0
>>
```

Example 1.6. Find

$$\det \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}.$$

Since

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$$

$$\begin{array}{lll} \text{factor: } 2 & \mathbf{R}1 \leftarrow \frac{1}{2}\mathbf{R}1 & \sim \begin{bmatrix} 1 & 2 \\ 1 & 6 \end{bmatrix} \\ \text{factor: } 1 & \mathbf{R}2 \leftarrow \mathbf{R}2 - \mathbf{R}1 & \sim \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \\ \text{factor: } 4 & \mathbf{R}2 \leftarrow \frac{1}{4}\mathbf{R}2 & \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ \text{factor: } 1 & \mathbf{R}1 \leftarrow \mathbf{R}1 - 2\mathbf{R}2 & \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

so

$$\det \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} = 2 \cdot 1 \cdot 4 \cdot 1 = 8.$$

2. Two-by-Two as a Special Case

For two-by-two matrices there is a fast formula for the determinant.

Lemma 2.1. *For any real numbers a, b, c, d,*

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Rather than give a formal proof, I just ask at you recall from class how we used row operations to invert

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(with some special assumptions like $a \neq 0$.) The row ops were

$$\begin{array}{ll} \text{factor: a} & \mathbf{R}1 \leftarrow \frac{1}{a}\mathbf{R}1 \\ \text{factor: 1} & \mathbf{R}2 \leftarrow \mathbf{R}2 - c\mathbf{R}1 \\ \text{factor: } \frac{ad-bc}{a} & \mathbf{R}1 \leftarrow \frac{a}{ad-bc}\mathbf{R}1 \\ \text{factor: 1} & \mathbf{R}1 \leftarrow \mathbf{R}2 - \frac{b}{a}\mathbf{R}1 \end{array}$$

and the factors multiply to $ad - bc$.

3. Elementary Matrices as a Special Case.

Since elementary matrices are barely different from I, they are easy to deal with. As with their inverses, I recommend that you memorize their determinants.

Lemma 3.1.

(a) *An elementary matrix of type I has determinant* -1.

(b) *An elementary matrix of type II that has non-unit diagonal element* α *has determinant* α.
(c) *An elementary matrix of type III determinant* 1.

Rather than prove this, I offer some examples.

Example 3.2. Find

$$\det \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\mathbf{R}1 \leftarrow \mathbf{R}1 - 2\mathbf{R}3$ takes this back to I, we have

$$\det \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

Example 3.3. Find

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}.$$

Since $\mathbf{R}1 \leftrightarrow \mathbf{R}3$ takes this to I, we have

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1.$$

Example 3.4. Find

$$\begin{vmatrix} 1 & 0 \\ 0 & \alpha \end{vmatrix}$$

where $\alpha \neq 0$.

We can use the special 2-by-2 formula:

$$\begin{vmatrix} 1 & 0 \\ 0 & \alpha \end{vmatrix} = 1\alpha - 0 \cdot 0 = \alpha.$$

4. Upper Triangular as a Special Case

If a matrix is upper triangular,

$$A = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ & & \ddots & \\ 0 & 0 & & d_n \end{bmatrix},$$

then the row ops needed are a bunch of type I to make it diagonal, all with factor 1, followed by

$$\begin{array}{ll} \text{factor: } d_1 & \mathbf{R}1 \leftarrow \frac{1}{d_1}\mathbf{R}1 \\ \text{factor: } d_2 & \mathbf{R}2 \leftarrow \frac{1}{d_2}\mathbf{R}2 \\ & \vdots \\ \text{factor: } d_n & \mathbf{R}1 \leftarrow \frac{1}{d_n}\mathbf{R}1 \end{array}$$

and we get another easy formula:

Lemma 4.1. *The determinant of an upper (or lower) triangular matrix is the product of its diagonal elements:*

$$\begin{vmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ & & \ddots & \\ 0 & 0 & & d_n \end{vmatrix} = d_1 d_2 \cdots d_n.$$

5. Speeding up the Algorithm

If you don't need the inverse of A, you can just do row operations until you get to a matrix where you know how to compute the determinant.

Theorem 5.1. *If a sequence of elementary row operations reduces A to B, and if the factors of these row operations are $r_1, \ldots, r_n$, then*

$$\det(A) = r_1 r_2 \ldots r_n \det(B).$$

A good algorithm (especial if you are a computer) is to row reduce to weak row echelon form. If there are enough pivots to tell you A is invertible, the determinant of A is

the product of the factors of the row ops used times the product of pivots.

Example 5.2. Calculate the the determinant of

$$A = \begin{bmatrix} 2 & 4 & 0 & 4 \\ 0 & 5 & 1 & 0 \\ 2 & 9 & 0 & 4 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

Following this method, we find

$$\begin{bmatrix} 2 & 4 & 0 & 4 \\ 0 & 5 & 1 & 0 \\ 2 & 9 & 0 & 4 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\text{factor: } 2 \qquad \mathbf{R1} \leftarrow \tfrac{1}{2}\mathbf{R}1 \qquad \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 5 & 1 & 0 \\ 2 & 9 & 0 & 4 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} \text{factor: } 1 & \mathbf{R}3 \leftarrow \mathbf{R}3 - 2\mathbf{R1} \\ \text{factor: } 1 & \mathbf{R}4 \leftarrow \mathbf{R}4 - \mathbf{R1} \end{matrix} \qquad \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 5 & 1 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\text{factor: } 1 \qquad \mathbf{R}3 \leftarrow \mathbf{R}3 - \mathbf{R2} \qquad \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and so

$$\det(A) = (2 \cdot 1 \cdot 1 \cdot 1)\,(1 \cdot 5(-1)(-1)) = 10.$$

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