MORE ON CASTING OUT

TERRY A. LORING

1. CASTING OUT TO TEST LINEAR INDEPENDENCE

A sore point in the terminology is that every linearly independent set is the basis of something. The reasoning is this. If \( b_1, b_2, \ldots, b_n \) form a linearly independant set in some \( V \) then:

1. any span is a subspace, and
2. any subspace is a vector space, so
3. the set

\[
\{b_1, b_2, \ldots, b_n\}
\]

is a basis of

\[
\text{span}(b_1, b_2, \ldots, b_n).
\]

We can rephrase the casting out technique. Working in any vector space \( V \):

**Input:** A finite set of vectors \( B = \{b_1, \ldots, b_m\} \) contained in \( V \).

**Output:** A subset \( B_m \) of \( B \) which is now a basis of \( \text{span}(b_1, b_2, \ldots, b_n) \).

If the algorithm runs without casting out any vectors, that can be a good thing.

- If none of the \( b_k \) are cast out, then the starting list \( b_1, \ldots, b_n \) was already linearly independent.

2. CASTING OUT TO TEST FOR BEING A BASIS

A special case of casting out is when you start with a set the spans \( V \). Then testing for linear indeendence is really testing to see if you have a basis of \( V \).

What does that tell us in the old situation where \( V = \mathbb{R}^n \)?

- If \( b_1, \ldots, b_n \) span \( \mathbb{R}^n \) and none of the \( b_k \) are cast out, then the starting list \( b_1, \ldots, b_n \) was already a basis of \( \mathbb{R}^n \).

In the accelerated version of casting out, the trick was to put the \( n \) vectors together as columns in an \( n \)-by-\( n \) matrix. Any columns that end up after two reduction to lack a pivot tell us to cast out a vector. So nothing cast out means no missing pivots.

Wait. That’s the test for invertibility! Moreover, \( n \)-vectors cannot be a basis for a proper subspace of \( \mathbb{R}^n \), only of \( \mathbb{R}^n \) itself. So we could have said

- If \( b_1, \ldots, b_n \) are any \( n \)-vectors in \( \mathbb{R}^n \) and none of the \( b_k \) are cast out, then the starting list \( b_1, \ldots, b_n \) was already a basis of \( \mathbb{R}^n \).
Theorem 1. If $b_1, \ldots, b_n$ are any $n$-vectors in $\mathbb{R}^n$, then

$$
\begin{bmatrix}
| & | & | \\
| b_1 \ b_2 \ \cdots \ b_n |
| & | & |
\end{bmatrix}
$$

is invertible if and only if $b_1, \ldots, b_n$ is a basis for $\mathbb{R}^n$. 