HOMEWORK #8

Problem 1. Consider the following matrix that depends on parameters \(r\) and \(s\) that can vary of the real numbers:

\[
A = \begin{bmatrix}
3r^3 & r^2 & rt \\
3r^2 & r + t - 1 & t + 1 \\
6r^2 & 2r & 2rt + t - 2
\end{bmatrix}
\]

For which values of the parameters is this invertible?

Problem 2. Let

\[
A = \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Find the eigenvalues (with or without finding the eigenvectors) for the following matrices.

(1) \(A\)
(2) \(B\)
(3) \(ABA^{-1}\)
(4) \(BAB^{-1}\)
(5) \(A^T\)

Problem 3. Use the method of Example 4.29 (in the second edition... this is diagonalizing \(A = WDW^{-1}\) and using \(A^n = WD^nW^{-1}....\)) to compute the following.

What is \(A^{-8}\) when

\[
A = \begin{bmatrix}
4 & -3 \\
-1 & 2
\end{bmatrix}.
\]

Problem 4. Consider

\[
A = \begin{bmatrix}
3 & 1 & -2 \\
2 & 2 & -2 \\
2 & 1 & -1
\end{bmatrix}.
\]

It is hard to find the characteristic polynomial here as there are no zeros and \(\det(A - \lambda I)\) has a \(\lambda\) in every row and column. Let me calculate and factor the characteristic polynomial. I start with negating all three rows so the intermediate polynomials look
better.

\[
\begin{vmatrix}
3 - \lambda & 1 & -2 \\
2 & 2 - \lambda & -2 \\
2 & 1 & -1 - \lambda
\end{vmatrix} = -
\begin{vmatrix}
\lambda - 3 & -1 & 2 \\
-2 & \lambda - 2 & 2 \\
-2 & -1 & \lambda + 1
\end{vmatrix}
\]

\[
= - \left( (\lambda - 3) \begin{vmatrix} \lambda - 2 & 2 \\ -1 & \lambda + 1 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ -2 & \lambda + 1 \end{vmatrix} + \begin{vmatrix} -2 & \lambda - 2 \\ -2 & -1 \end{vmatrix} \right)
\]

\[
= - ((\lambda - 3) ((\lambda - 2)(\lambda + 1) + 2) + (-2(\lambda + 1) + 4) + 2 (2 + 2(\lambda - 2)))
\]

\[
= - ((\lambda - 3) (\lambda^2 - \lambda) - 2 (\lambda - 1) + 4 (\lambda - 1))
\]

\[
= - (\lambda - 1) ((\lambda - 3)\lambda - 2 - 4)
\]

\[
= - (\lambda - 1) (\lambda^2 - 3\lambda - 6)
\]

\[
= - (\lambda - 1) (\lambda - 1) (\lambda - 2)
\]

\[
= - (\lambda - 1)^2 (\lambda - 2)
\]

(1) What are the eigenvalues of \(A\), and for each eigenvalue, what are the algebraic and geometric multiplicities?

(2) What are the eigenvalues of \(A^{-2}\), and for each eigenvalue, what are the algebraic and geometric multiplicities?

**Problem 5.** Suppose \(A\) is a square matrix so that

\[A^3 = A^2 + I.\]

Following the argument in exercise 4.3.34 (in the second edition), find formulas for the following. For full credit, give your answers are polynomials in \(A\) of degree at most 2.

(1) \(A^{-1}\)

(2) \(A^4\)

(3) \(A^5\)

(4) \(A^{-2}\)