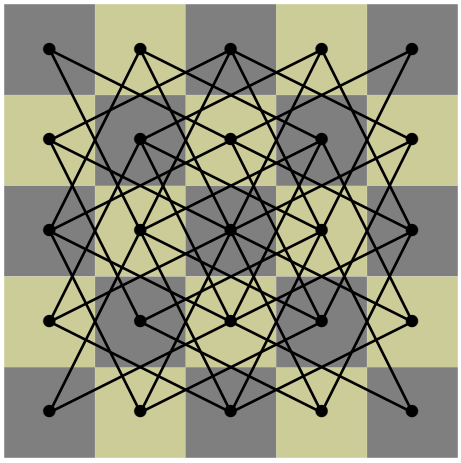
SMALL KNIGHT'S TOUR PROBLEMS

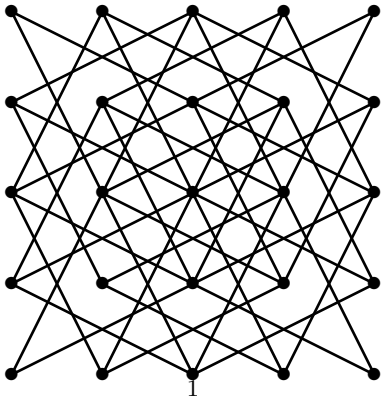
1. Five by Five

Here are all the legal moves for a knight on a five by five board.

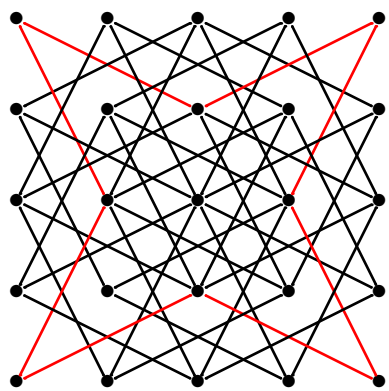


The puzzle is to find a closed Knight's tour. Some folks cheat and allow a tour that does not return to the start.

The colored board is not helpful, so let's just look at the graph. What we want to know is whether or not the graph is Hameltonian.



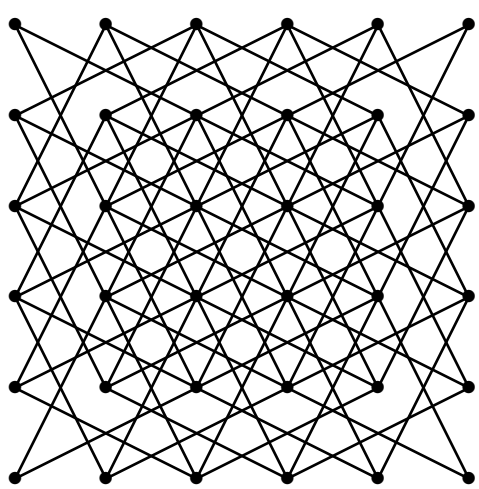
Suppose there is a Hameltonian cycle. Since the corner vertices are all of degree 2, the cycle would have to include both the edges incident to the corner vertices. So we highlight them in red.



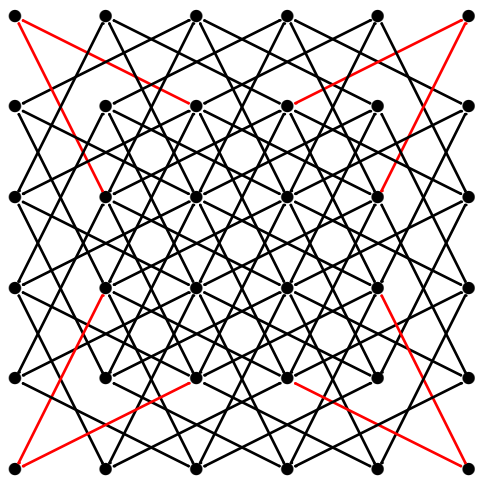
These edges form a cycle, and there is no way to expand a cycle to a bigger cycle without first removing some edges. Since the edges were forced on us, we get a contradiction. Therefore there is no Hameltonian cycle. A five by five knight's tour is not possible.

2. Six by Six

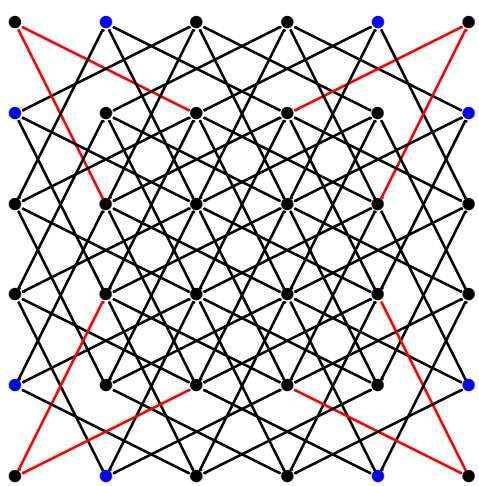
Moving to six by six, we a confronted with this graph.



Again the corners have degree two, so any Hameltonian cycle will need to include the eight edges shown in red.



There are eight vertices of degree three, shown in blue.

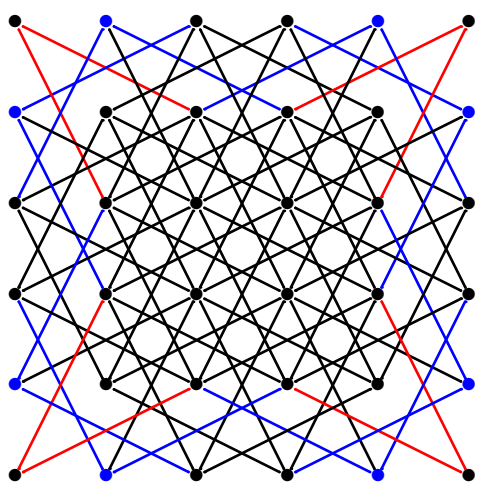


A Hameltonian cycle must include two of the three edges at each blue vertex. Put another way, any cycle must omit one of the three edges at each of eight blue vertices. We potentially must consider all $3^8 = 6561$ ways of making these choices. Fortunately, there is a lot of symmetry (rotate by 90, 180,270 degrees and reflection) in the graph, so there are not that many.

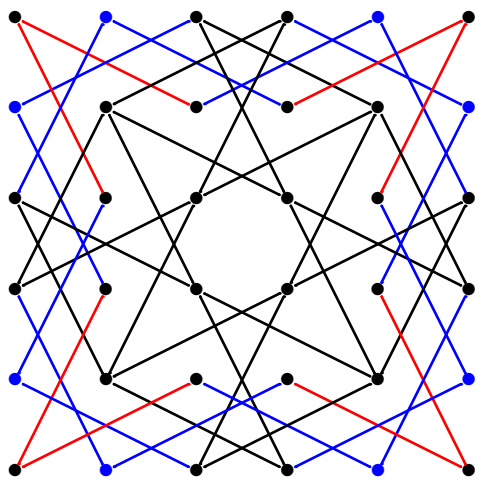
Before we get too serious about showing you cannot make a Hameltonian cycle, let's be optomistic and figure that one of these choices works out. Moreover, let's start my making a symmetric choice, since it seems reasonable that if a solution exists, there is a symmtric solution.

The first thing I tried was to exlude at each blue vertex the edge that extends the deepest inside the square. After all, there many more squares on the edge or in one from the edge than there are square farther in, so it makes some sense to try to take care of the outside first.

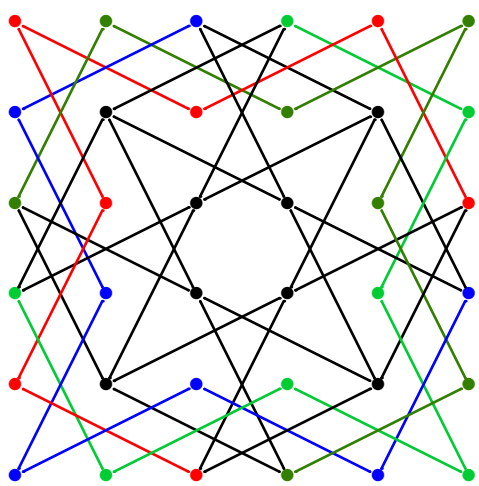
So we first try to use the edges shown in blue.



If we use the red and blue edges, there are many vertices that now are incident two edges in our partial cycle. We cannot use any more of the edges incident to these vertices, so let's remove them.

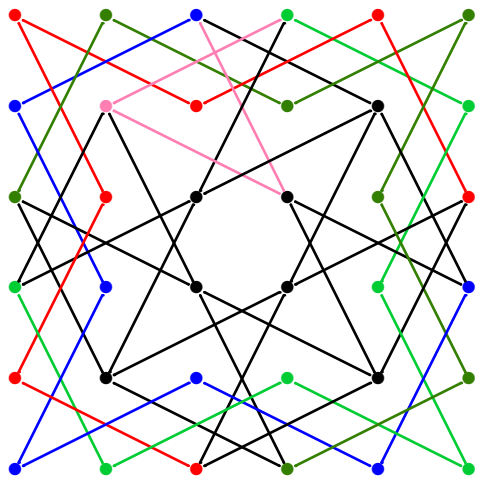


With this coloring it is hard to see what we have. In fact, the selected edges form four disjoint chains, which we highlight with four colors.

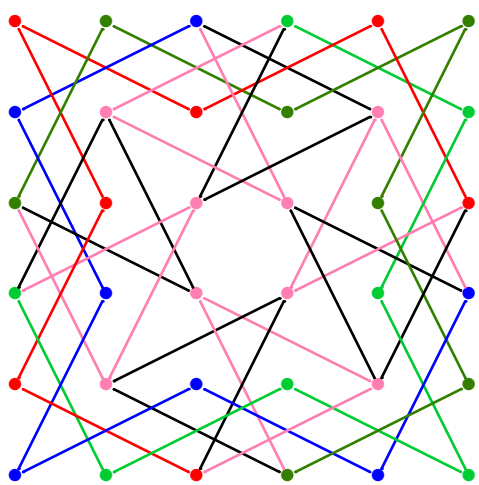


We need to connect four of the pairs of colors to make a cycle. There are eight vertices where we are uncommited, shown in black. To keep up the symmetry, we want to use two of the black vertices to connect two colors, or three knight moves.

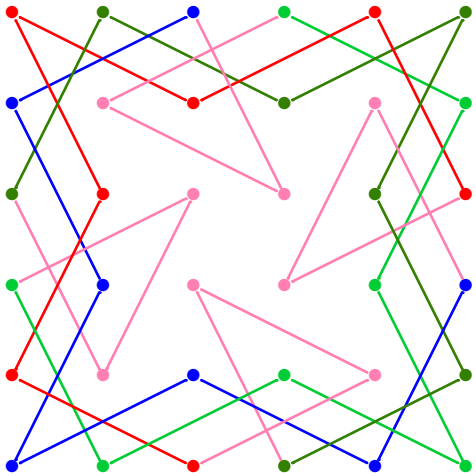
We can add three moves in the following way to connnect the blue and the light green.



Do that three more times.



Remove the unused lines so we can see the cycle.



Now put the chessboard underneath.