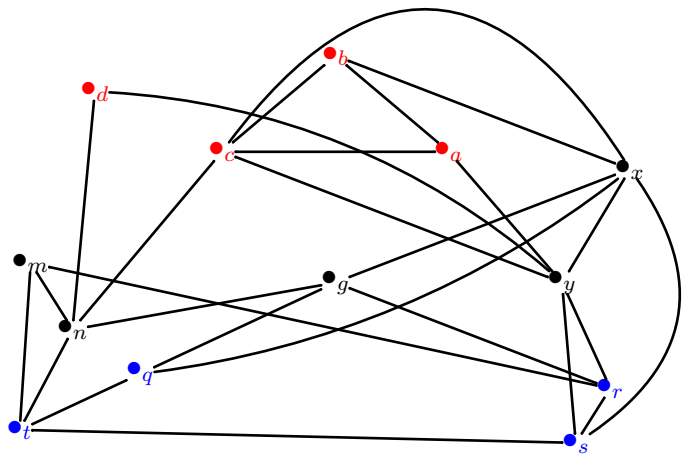
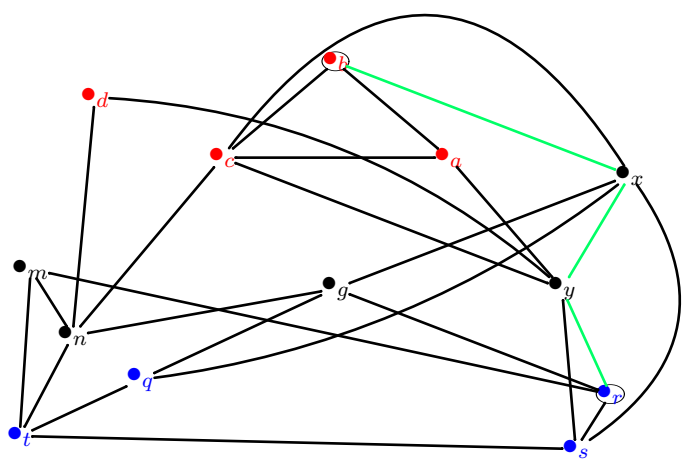
ON MAX FLOW, MIN CUT

Here is an example using the algorithm for finding disjoint paths that comes out of the first proof of Menger's theorem in Diestel's book.

Let us find as many disjoint paths from the set $A = \{a, b, c\}$ to $B = \{q, r, s, t\}$ as is possible. (No more than three are possible, since A has only three vertices to serve as endpoints.)

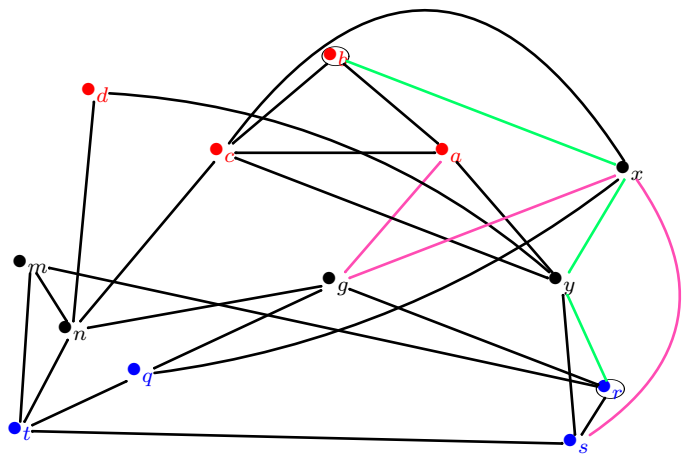


First we pick any path P_1 from a red to a blue vertex. I circled the endpoints since we want to keep these. (A blue endpoint may end up connected to a different red endpoint, but will remain an endpoint of some chosen path.)

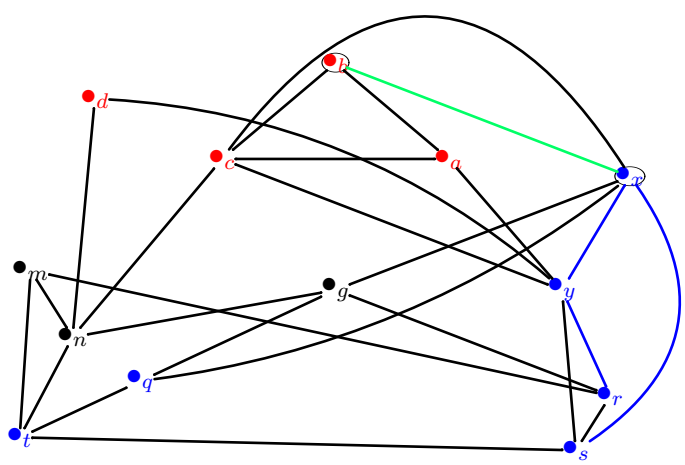


Now we want a second, disjoint path. To begin, we find any path (called R in the proof in the book) from any vertex in A to any vertex in B that does include r. (We can't use r as an endpoint or an intermediate

point.) Here is one, in pink:

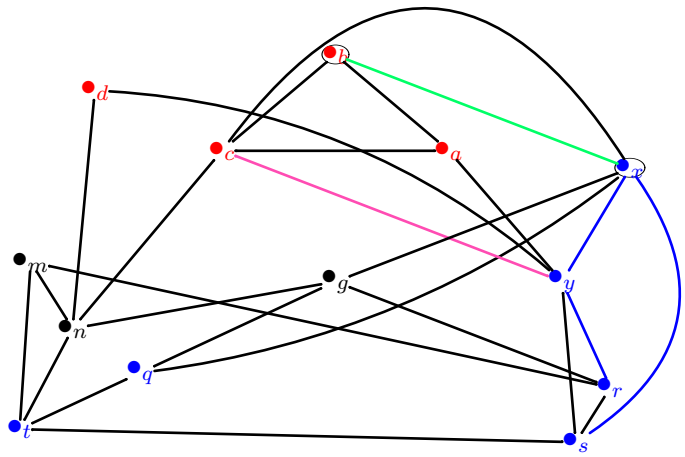


The last vertex on the pink path R that is also on the green path is x. So all the vertices between x and B on either the green or pink path become added to the blue set, temporarily. We shorten the green path, and mark its new endpoint, x. The edges after x in the pink and green path we turn to blue. These are called xP_1 and xR in the book.

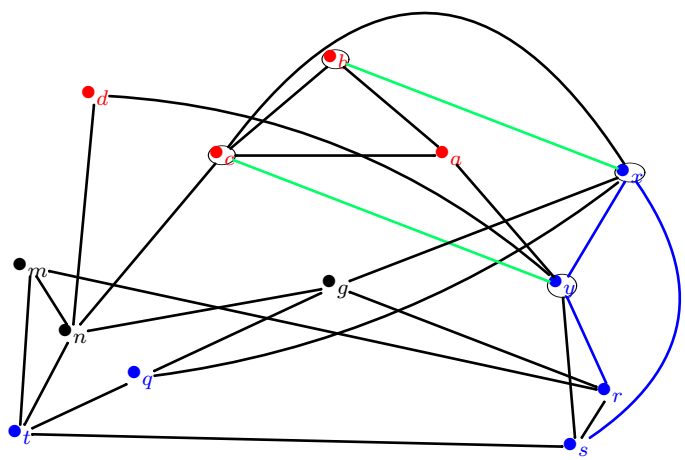


We now have an easier problem. Find a path disjoint from the shorter green path P_1x that goes from any red vertex to any of the blue vertices. (It is easier because there are so many more blue vertices.) The next step is to pick any path from any red vertex to any blue vertex that

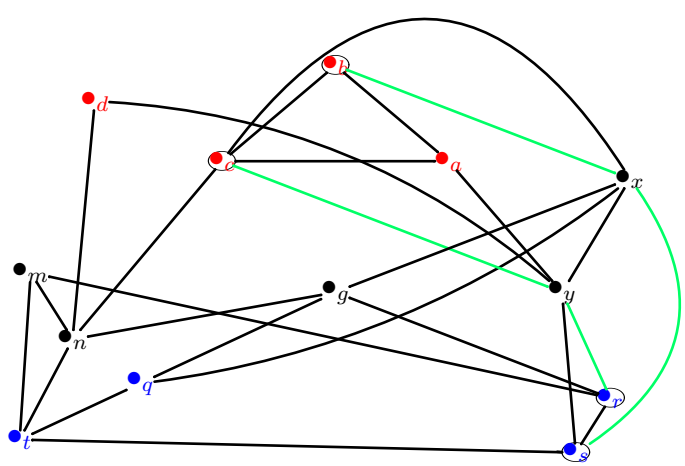
does not include x. Here is one, in pink.



This time, the pink path happens to be disjoint from the green path. (I'm picking these pink paths as much at random as I can.) So we elevate it to a green path.

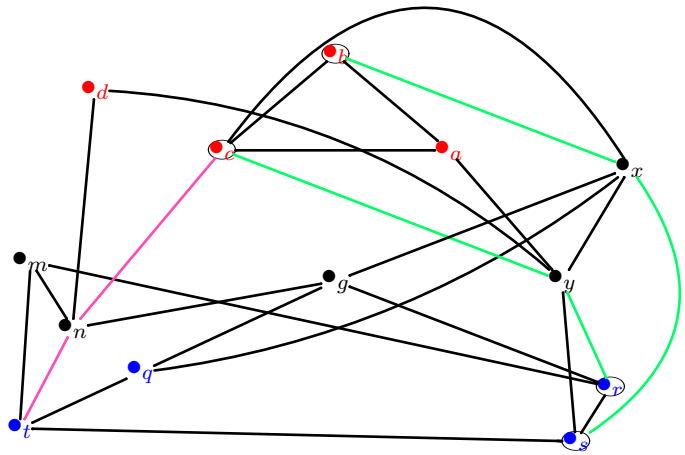


Now we extend the two blue paths back to the original blue by using different parts of the blue edges.

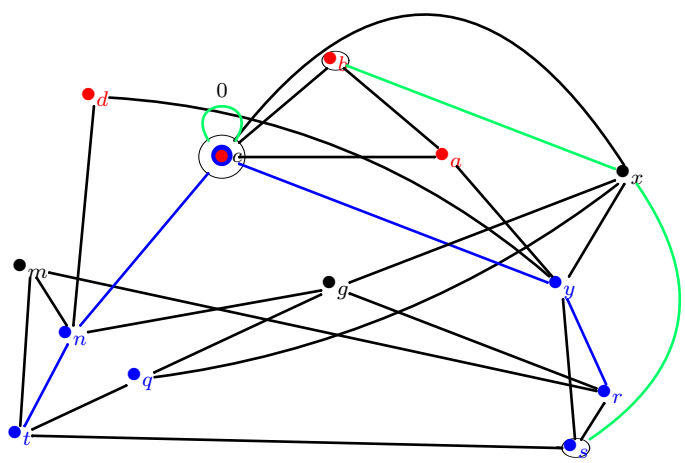


We call these green paths P_1' and P_2'. Notice that now b is connected by a green path to s. Originally, b was connected to r. What is important is that the endpoint sets $\{a, b\}$ and $\{r, s\}$ contain the original endpoint sets $\{b\}$ and $\{r\}$.

Now we want a third path, disjoint from the two green paths. The first step is to pick a pink path from any red vertex to any blue vertex that does not include vertices from the blue-endpoint set $E_1 = \{r, s\}$ Here is one such.

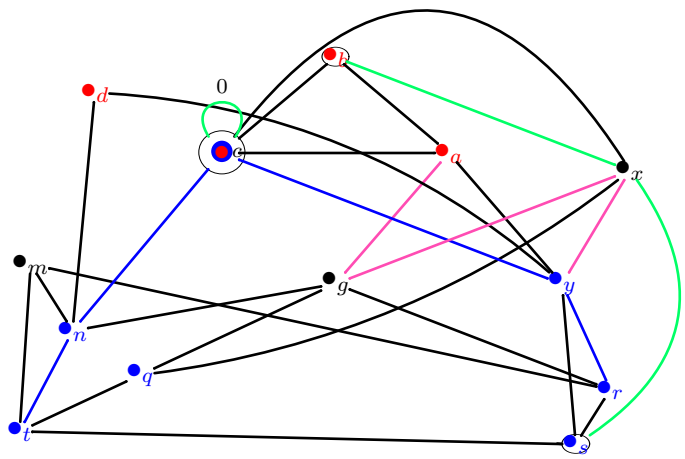


The last vertex on the pink path that is in a green path is c. So we add c, n and y to B. This means I need to mark c both red and blue, as it is in A and the new B : We shorten the green c-r path to end at c, giving us a path of length zero. I mark this with a green loop with a zero on it to remind us it is not a loop in the original graph. This time, all of the vertices on the pink and the green c-r path get turned blue. At this stage the blue-endpoint set is $E_2 = \{c, s\}$

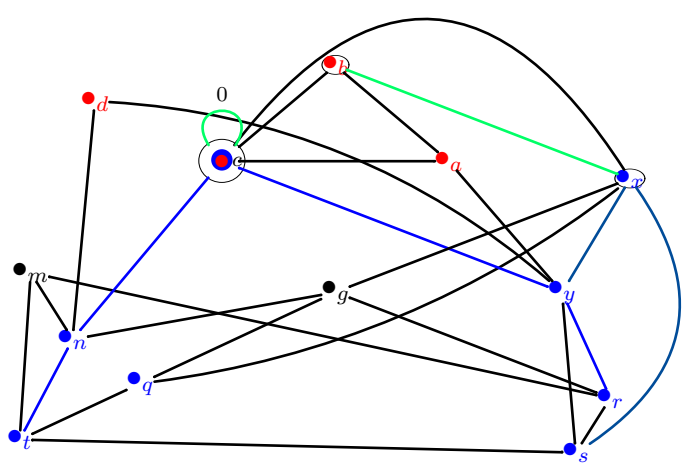


Now we have an easier problem. Find a third path, disjoint from the two green paths, from a red to a blue vertex. Again, we first find any path for a red vertex to any blue vertex that avoids the endpoints c

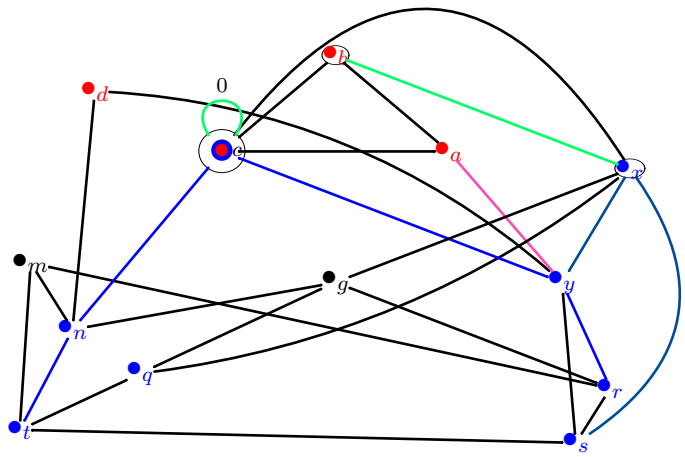
and s. Here is one, in pink:



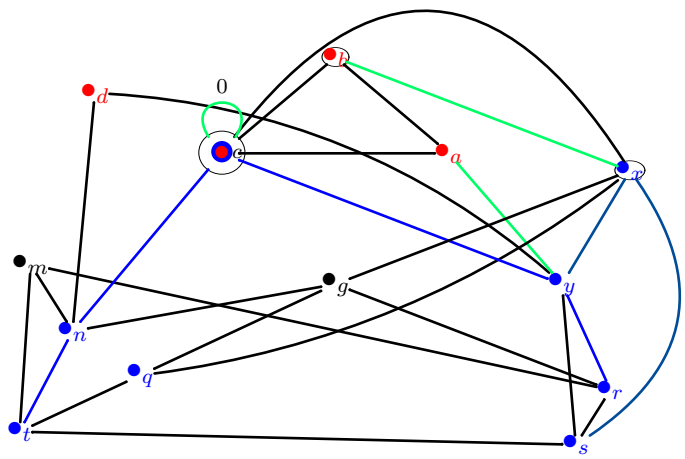
The last vertex on this path that is in a green path is x. We shorten the green b-s path to end at x and turn to blue x and everything on that green path and the pink path that past x. We replace s by x to get a new blue-endpoint set $E_3 = \{c, x\}$.



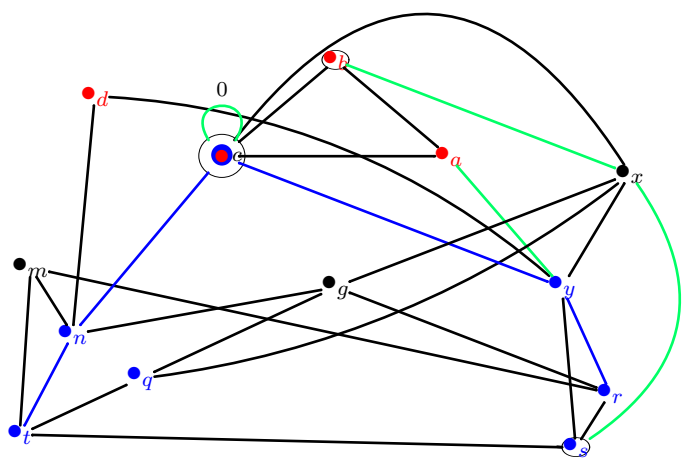
Once more, we randomly find a pink path from any red vertex to any blue vertex, avoiding the current blue-endpoint set.



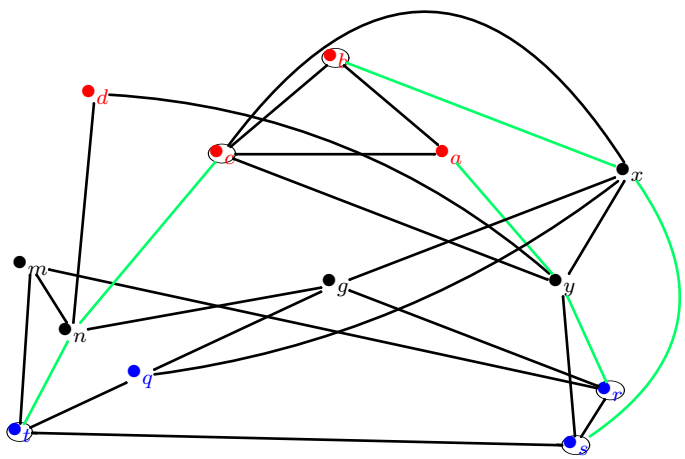
This is independent the two green paths, so we elevate it to a green path:



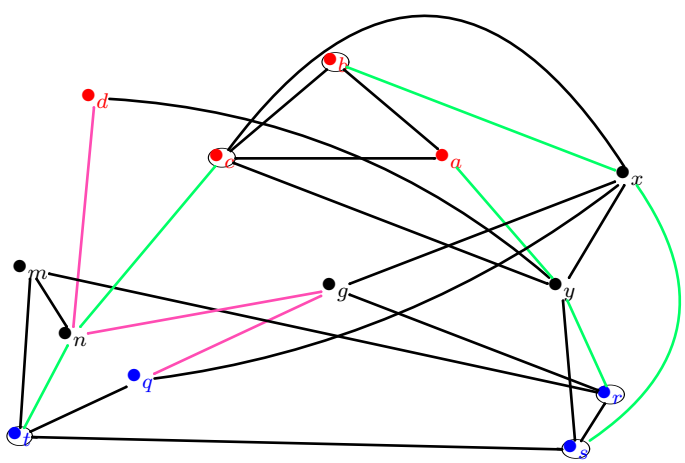
The previous endpoint set was $E_2 = \{c, s\}$ so we extend one green path along one of the dark-blue edges:



The blue-endpoint set before E_2 was $E_1 = \{r, s\}$. We extend the length-0 path along one fork of the blue and extend the a-y path along the other.

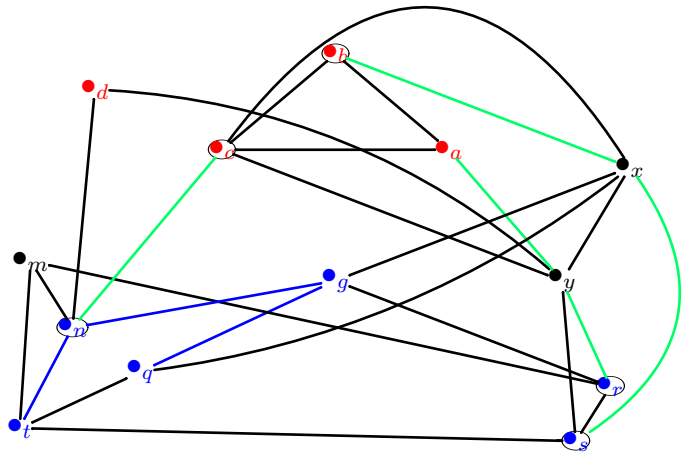


Now we try for a fourth path. Our blue-endpoint set is $F_1 = \{r, s, t\}$. We start by looking for any path from a red vertex to the one remaining blue vertex, q, while avoiding r, s and t. Here is one, in pink.

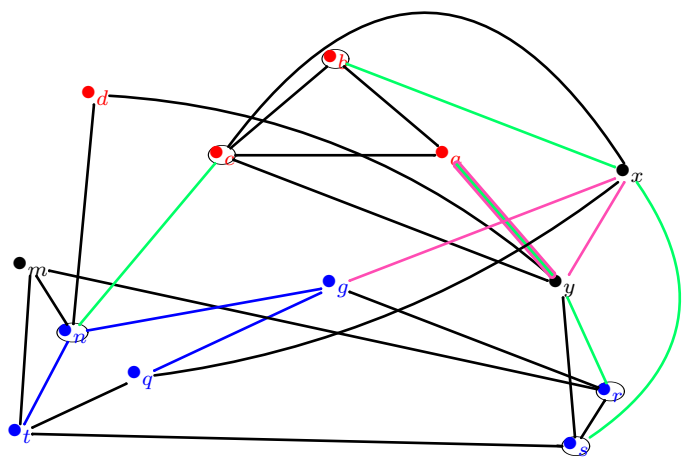


The last vertex on the pink path that is also in a green path is n. So we ignore the pink path before n and turn the green and pink path to

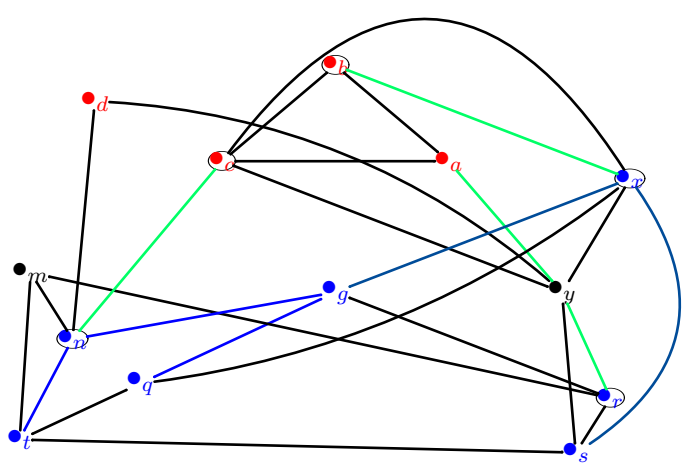
blue after n. Our new blue-endpoint set is $F_2 = \{r, s, n\}$.



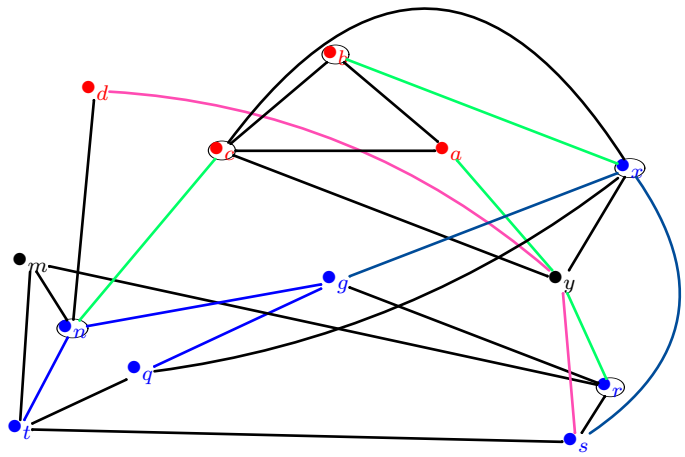
The next step is to find a path from the uncircled blue vertices back to any red vertex, avoiding r s and n. Here is one, (a, y, x, g), where one edge is in a green path and the new pink path.



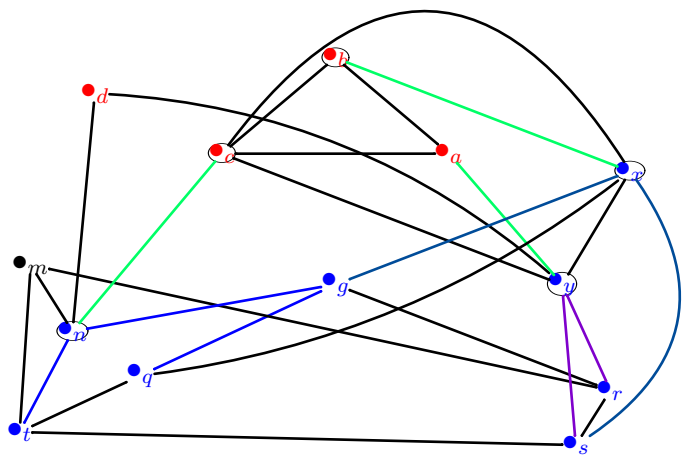
The last vertex on the pink path that is also on a green path is x. Recoloring as always, our new blue-endpoint set is $F_3 = \{r, x, n\}$.



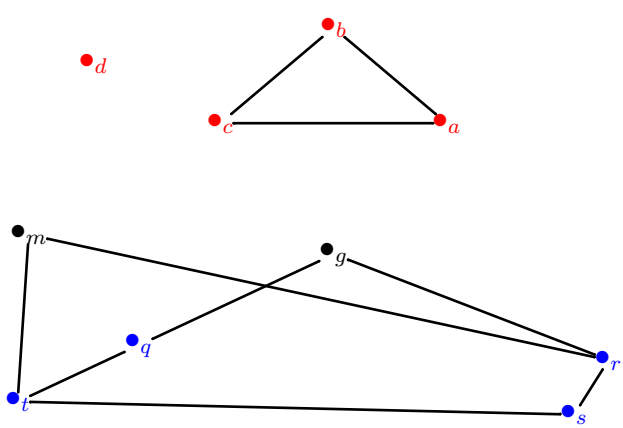
There is a path from d to s, avoiding the circled vertices:



The last vertex on the pink path that is is a green path is y. Our new blue-endpoint set is $F_4 = \{y, x, n\}$.

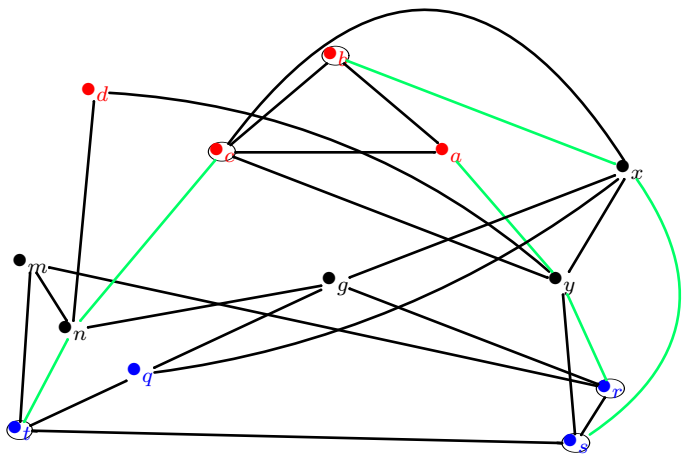


Now things come to a halt, because *there is not path from a red to a blue vertex that does not include one of* y x *or* n since every path out Indeed, $\{n, x, y\}$ separates $\{a, b, c, d\}$ from $\{g, q, r, s, t, x, y\}$. This means $\{n, x, y\}$ A from the original B. Here is $G - \{n, x, y\}$:

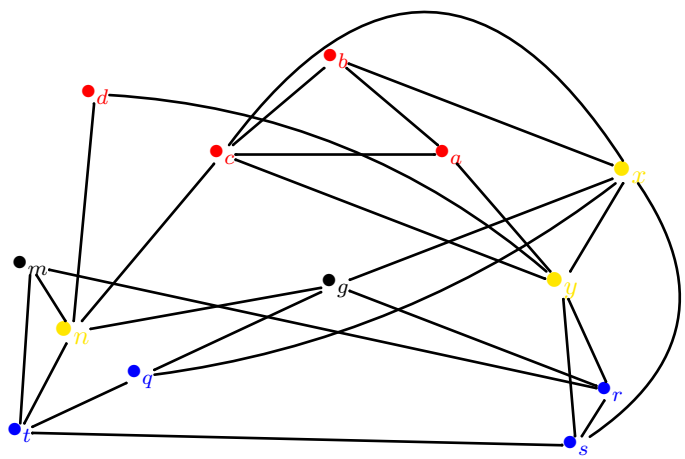


Finally, we can say that it takes a minimum of three vertices to separate A from B and the most disjoint paths from A to B is also three.

Here are three disjoint A-B paths:



Here are three vertices that separte A from B :



As you would expect, each green path goes through exactly one yellow vertex.