ISOMORPHISM

1. One to One Correspondence

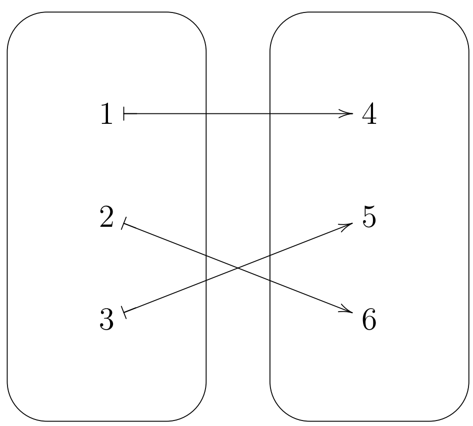
If f is a function from $\{1, 2, 3\}$ to $\{4, 5, 6\}$, we often summarize its domain and target sets by the notation

$$f : \{1, 2, 3\} \to \{4, 5, 6\}.$$

A particular instance of such a function can be described by listing the value f takes on each input, as in this example:

$$\begin{aligned} f(1) &= 4 \\ f(2) &= 5 \\ f(3) &= 6 \end{aligned}$$

A function between finite sets is easily pictured as a bunch of arrows. The function just described is drawn thus:



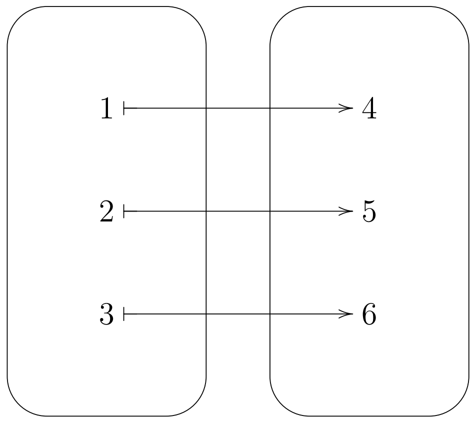
This is an example of a *one-to-one correspondance.* In particular, it is a *one-to-one correspondence between* $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. It is called this because exactly one element of A is sent to 4, exactly one element of A is sent to 5, and exactly one element of A is sent to 6.

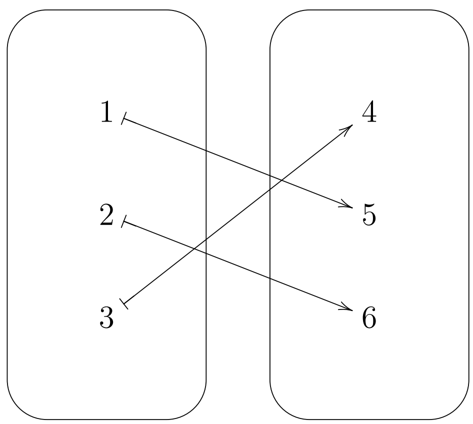
Definition 1. A function $f : X \to Y$ is a *one-to-one correspondence between* X *and* Y if, for each y in Y, there is exactly one solution in X to the equation

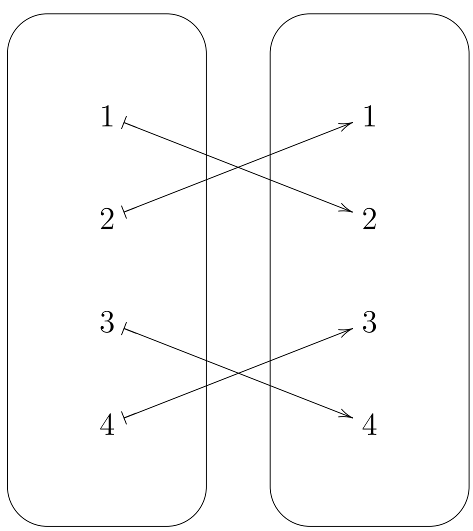
$$f(x) = y.$$

Other terminology is to say f is *bijective*, or that f is *one-to-one and onto*.

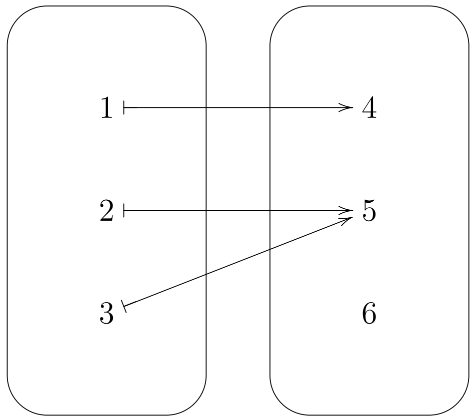
Here are some other one-to-one correspondences:

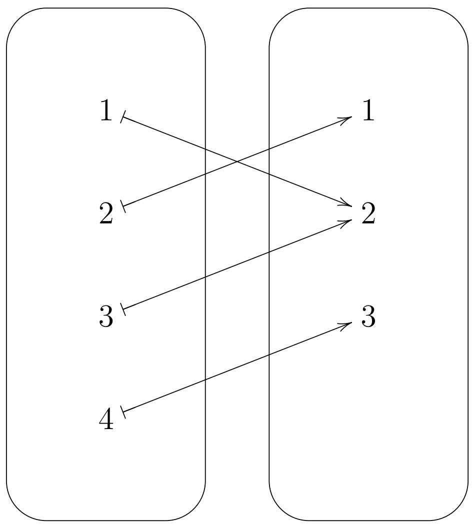


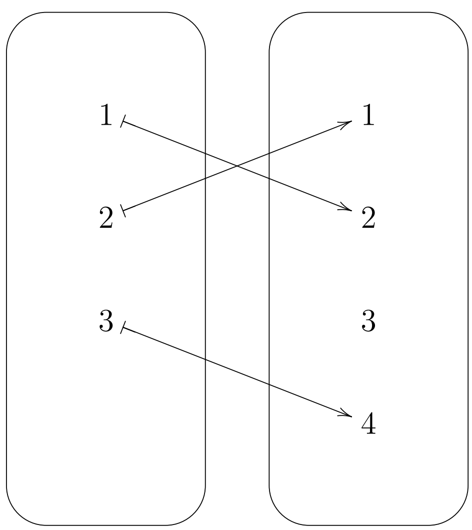




Here are some functions that are *not* one-to-one correspondences:





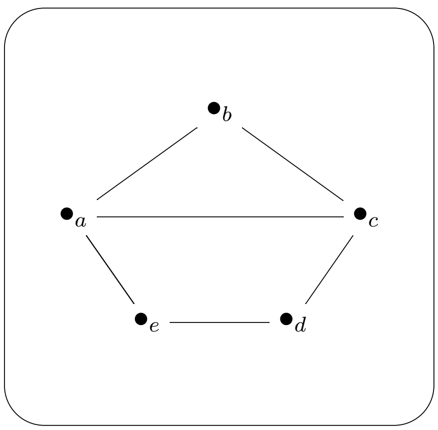


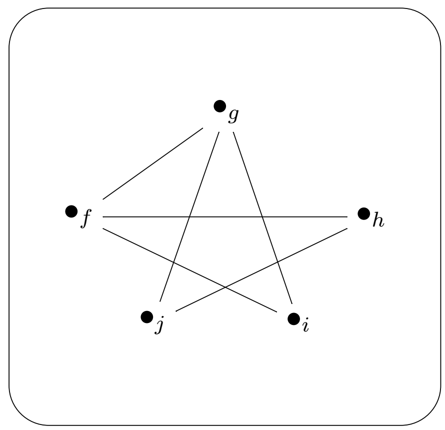
If two sets do not have the same number of elements, there can be no one-to-one correspondence between them. If there is a one-to-one correspondence between the two sets, then the have the same number of elements. (This is even true for infinite sets, because one-to-one corresponcences are used to define the "number" of elements in a infinite set.)

It is thought that the concept of a one-to-one correspondence is older than the concept of counting. At least this is asserted in a lecture by Katherine Boxat the University of Adelaide. I have no idea who this woman is.

2. Isomorphism of Graphs

Here are two graphs.





They have the same number of vertices, so we can define a one-to-one correspondence between the vertex sets,

$$\varphi : \{a, b, c, d, e\} \to \{f, g, h, i, j\}.$$

For example, we can define φ as

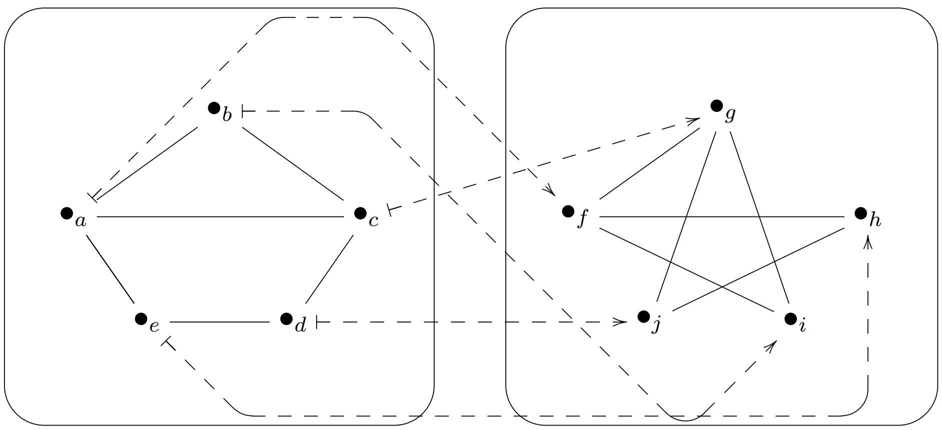
$$\begin{aligned} \varphi(a) &= f \\ \varphi(b) &= g \\ \varphi(c) &= h \\ \varphi(d) &= i \\ \varphi(e) &= j \end{aligned}$$

but this completely ignores the structure of the edges. For example, the first graph has an edge ab but the second does not have an edge

fg. If instead we define

$$\begin{aligned} \varphi(a) &= f \\ \varphi(b) &= i \\ \varphi(c) &= g \\ \varphi(d) &= j \\ \varphi(e) &= h \end{aligned}$$

the edges on the left correspond to edges on the right. If we use dotted lines to denote the action of the function φ, we have



This is hard to follow. An alternative is to relabel the vertices of one graph so that corresponding vertices get teh same name.