Lecture 36. Details regarding counting colorings of graphs.

\[ G = K_k \]

\[ P_G(n) = n(n-1) \ldots (n-k) \]

\[ P_G(k) = k! \]

\[ K_3 \]

3 colors

6 colorings

3 \[ \rightarrow \] B, 3 \[ \rightarrow \] P, 3 \[ \rightarrow \] R

\[ 1 \] way to color \( K_3 \): Give each vertex its own color.
There is a symmetry 
(\cong \text{ of } G \text{ with } G)

\begin{align*}
    a &\rightarrow b \\
    b &\rightarrow a \\
    c &\rightarrow c \\
    d &\rightarrow d
\end{align*}

\[ P^*_G(n) = n(n-1)^2(n-2) \]
\[ \chi(G) = 3, \quad P^*_G(3) = 12. \]

Consider colorings "equivalent" if a symmetry of G takes one coloring to the other.
This G has 6 non-equivalent colorings.
One way to color:

Put a color in middle,
Put a color on end,
Repeat on one remain vertex,
Use third color for the last.
This 7-cycle has at least two "truly different" ways to be colored with 3 colors.

One color only is used once.

Every color appears at least twice.