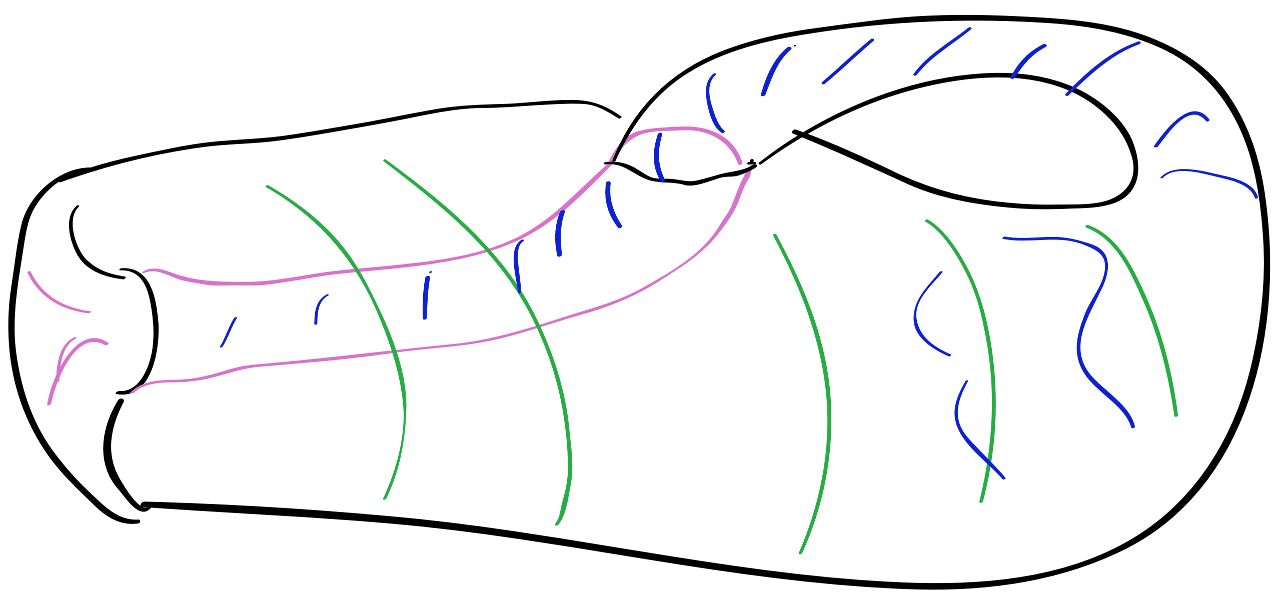
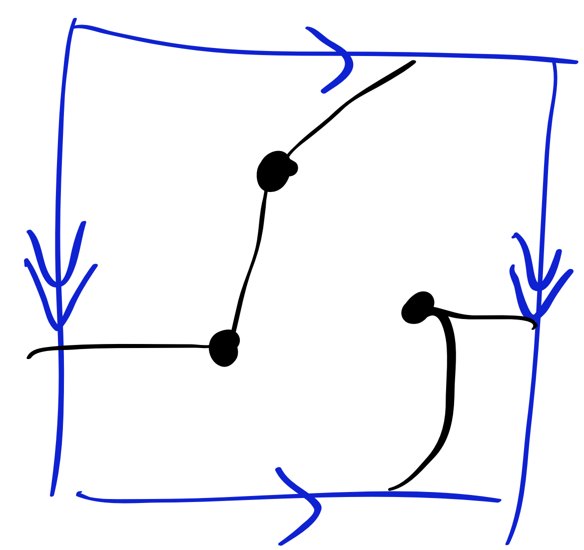
Lecture 32. HW#8, ch11: 3, 13, 19(b), 29(a)

(19(a) for extra credit)

Graphs on a torus.
(For extra credit, also draw such graphs on a Klein Bottle.)*



K_3



---

* if possible!

Proof of Eulers Theorem.

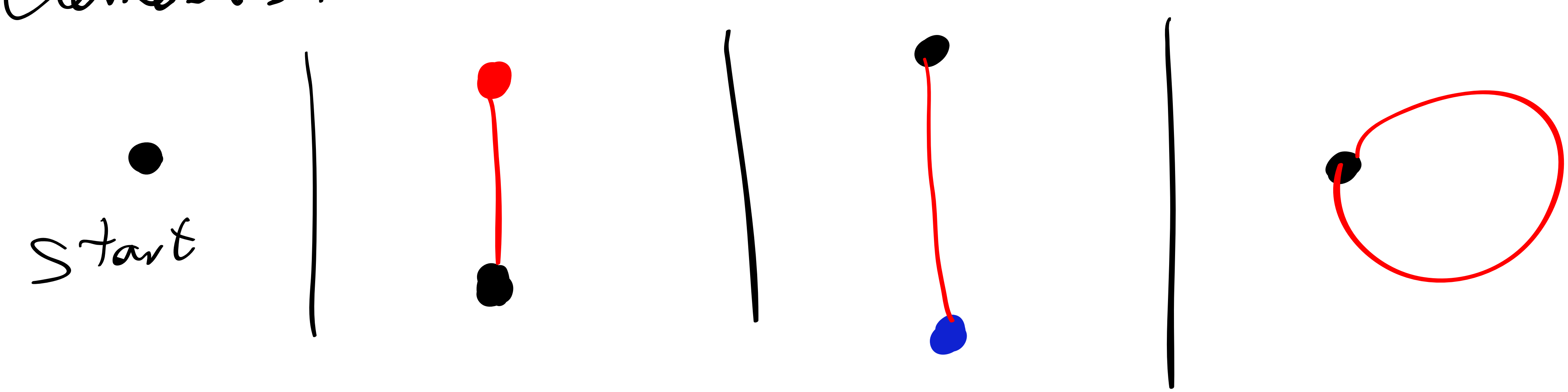
n = # of vertices

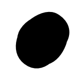
m = # of edges

f = # of faces in planar drawing.

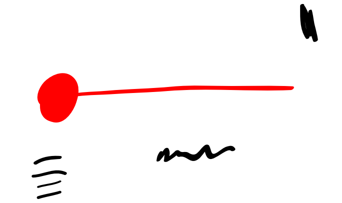
Claim: $n - m + f = 2$.

We saw any planar drawing of a graph can be built atop K_1 (•) adding only 3 elements:





Here $n - m + f = 1 - 0 + 1 = 2$ ✓

What is the effect of adding a "  "

$n_0 - m_0 + f_0 = 2$

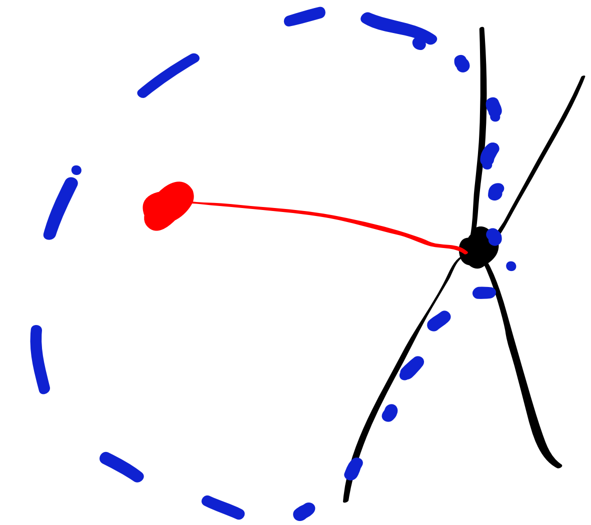
$1 - 1 + 0 = 0$

$n - m + f = 2$

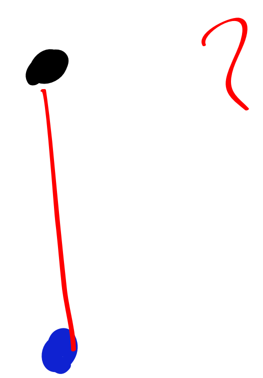
Adds 1 to n.

Adds 1 to m.

Adds 0 to f.



does not divide any face.

What about adding  ?

Adds 0 to n.

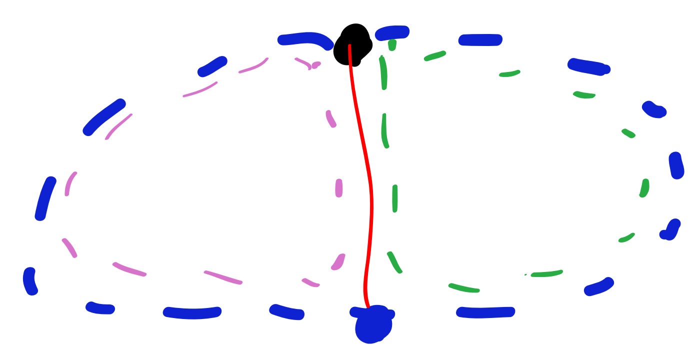
Adds 1 to m.

Adds 1 to f.

$n_0 - m_0 + f_0 = 2$

$0 - 1 + 1 = 0$

$n - m + f = 2$

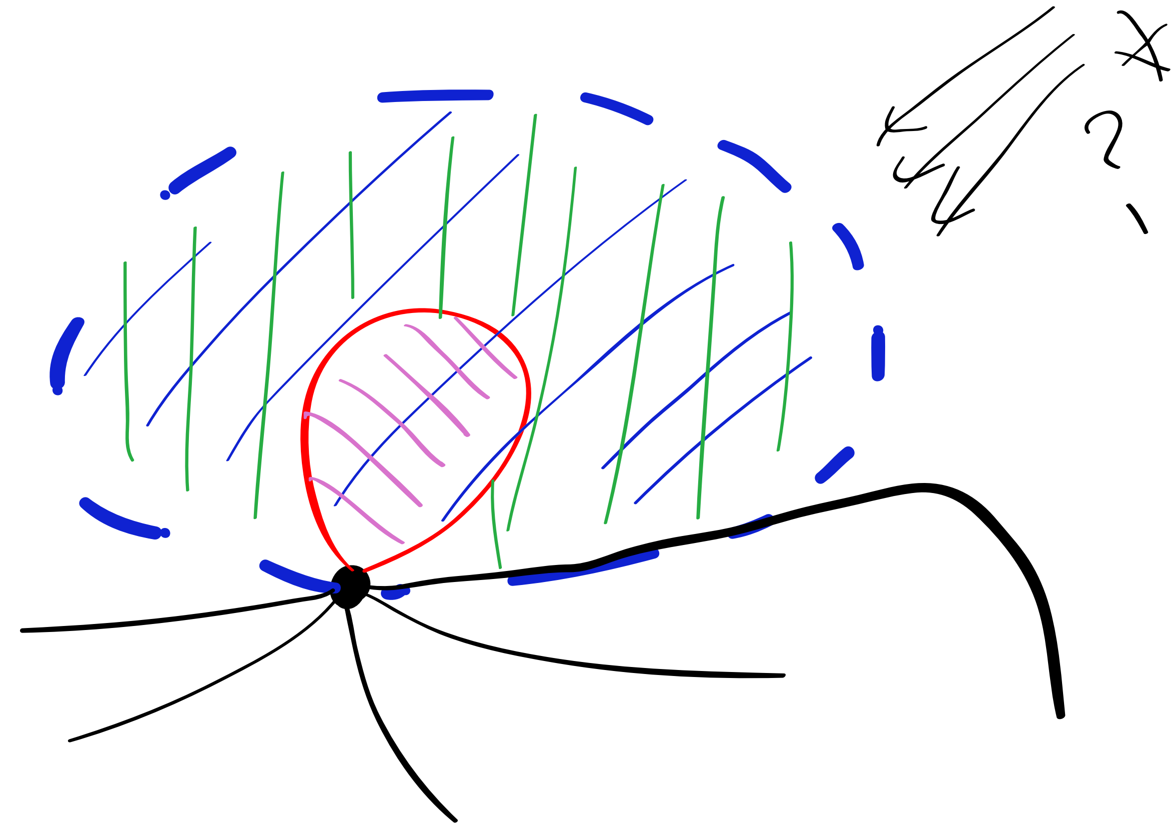


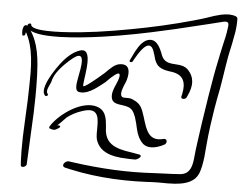
A Face is divided

Adding a loop.

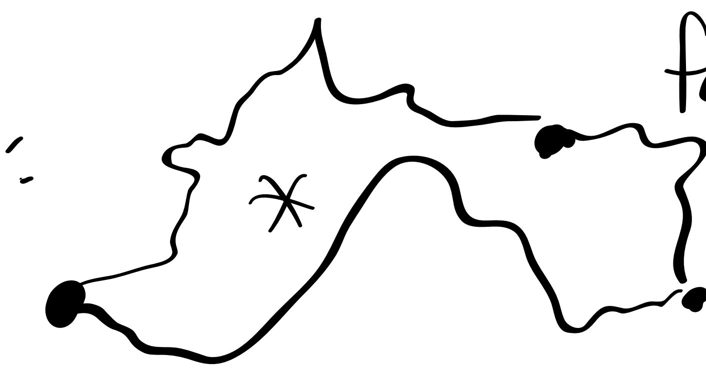
Adds 0 to n
Adds 1 to m
Adds 1 to f.

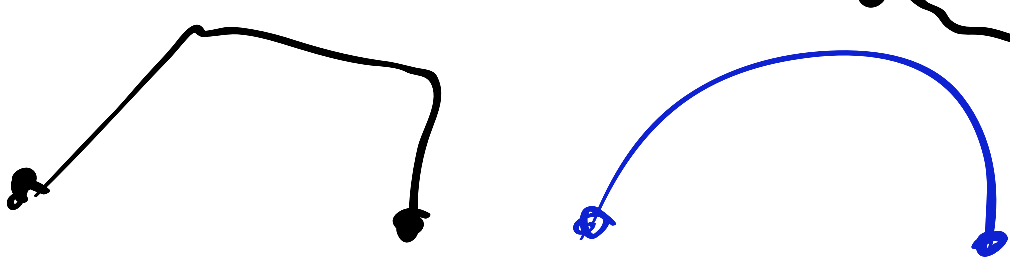
$$\begin{array}{rcccccl} & n_0 & - & m_0 & + & f_0 & = 2 \\ + & 0 & - & 1 & + & 1 & = 0 \\ \hline & n & - & m & + & f & = 2 \end{array}$$ ✓.



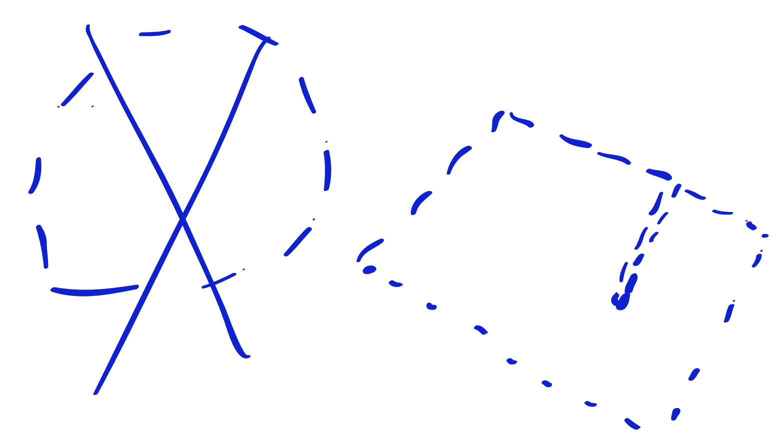


Non-rigor comes from:





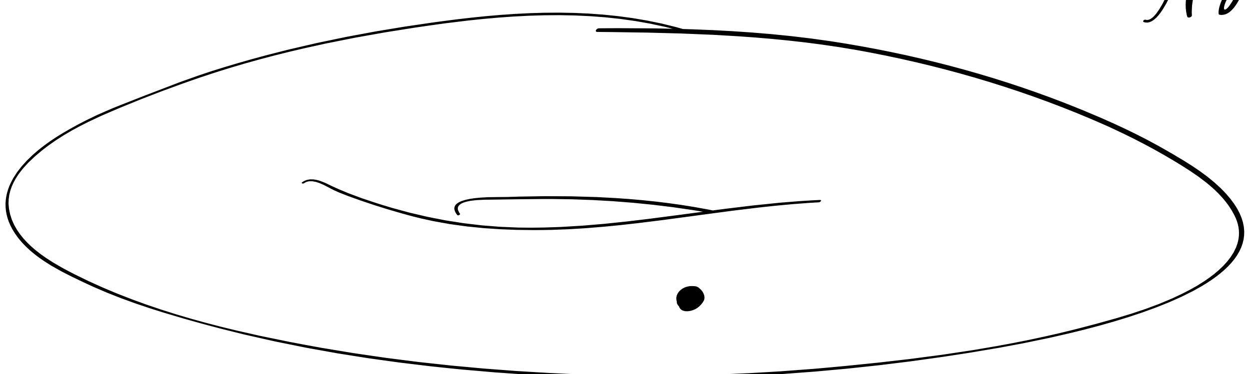
faces could be like this ☹

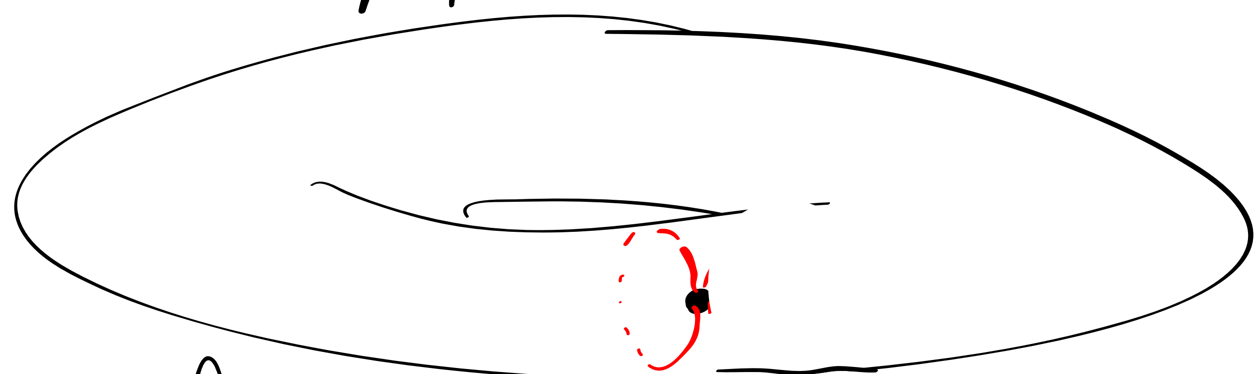


Proof was valid for graphs drawn on a torus or a klein bottle.

Except - we need a new base case.

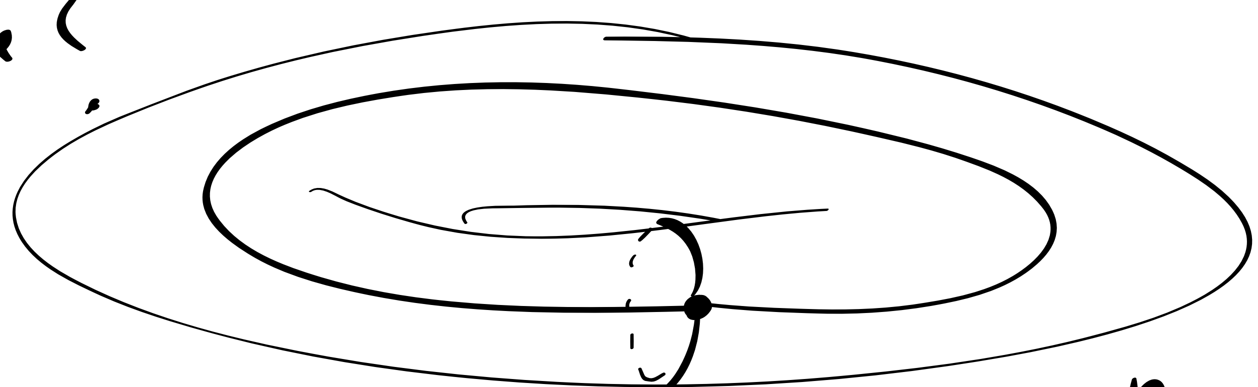
Add a loop?





No new face!

Base Case?

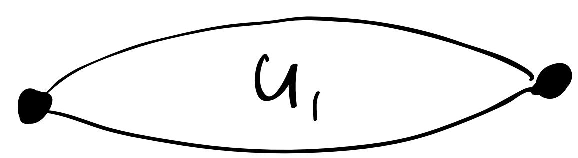


$n - m + f = 1 - 2 + 1 = 0$

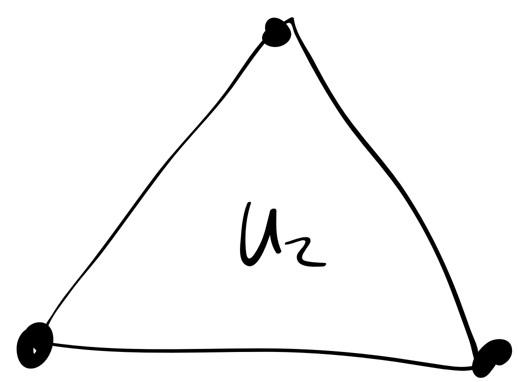
Euler on a Torus: Always get $n - m + f = 0$.

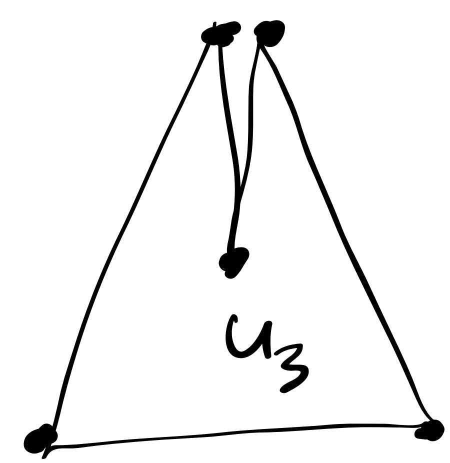
Corollary. Suppose G is a simple connected planar graph. Then $m \leq 3n-6$.

In a simple graph, faces don't look like this:



Can have





Will set:

$\deg(u_1) = 2$

$\deg(u_2) = 3$

$\deg(u_3) = 5.$

The degree of a face u is the drawing of a graph is the number of edges it touches.

pf: In the drawing of a simple graph, $\deg(u) \geq 3$, for all faces u.

Face-shaking lemma:

$$\sum_{u \text{ a face}} \deg(u) = 2m.$$

$$\therefore \sum_{u \text{ a face}} \deg(u) = \sum_{v \text{ a vertex}} \deg(v)$$

$$3 \cdot f = \sum_{u \text{ a face}} 3 \leq \sum_{u \text{ a face}} \deg(u) = 2m$$

$$\Big(\underbrace{3+3+\cdots+3}_{f} = 3f \Big).$$

$$\therefore \quad f \leq \tfrac{2}{3} m.$$

$$(+) \quad -n+m-f \leq -2$$

$$m-n \leq \frac{2}{3} m - 2$$

$$\frac{1}{3} m - n \leq -2$$

$$m - 3n \leq -6$$

$$\Longrightarrow m \leq 3n-6.$$