Lecture 29.

The "proof" of Menger's Theorem in the book is wrong.

Theorem. Suppose D is a connected digraph, and a and b are vertices such that $\exists$ a path from a to b. The minimum number of arcs needed to separate a from b equals the maximum number of arc-disjoint paths from a to b.

pf: If $\exists$ a-b-paths $P_1, \ldots, P_\ell$ that are arc-disjoint, then any cut-set of arcs must contain one arc from each P_i. $\therefore$ min arcs separating $\geq$ max. number of disjoint paths.

For generalizations, investigate "max-flow, min-cut" theorems + algorithms.

Need to show that if k is the minimum number of arcs that must be removed to disconnect a from b, then there are k arc-disjoint paths from a to b.

<u>Prop.</u> Given $B \subseteq V(D) \setminus \{a\}$*, and given a set $\mathcal{P}$ of arc-disjoint paths from a to B, if $|\mathcal{P}| < k$, there exists a set $\mathcal{P}'$ with $k+1$ arc-disjoint paths whose endpoints include the endpoints of $\mathcal{P}$.

(Menger's theorem follows using $B = \{b\}$.)

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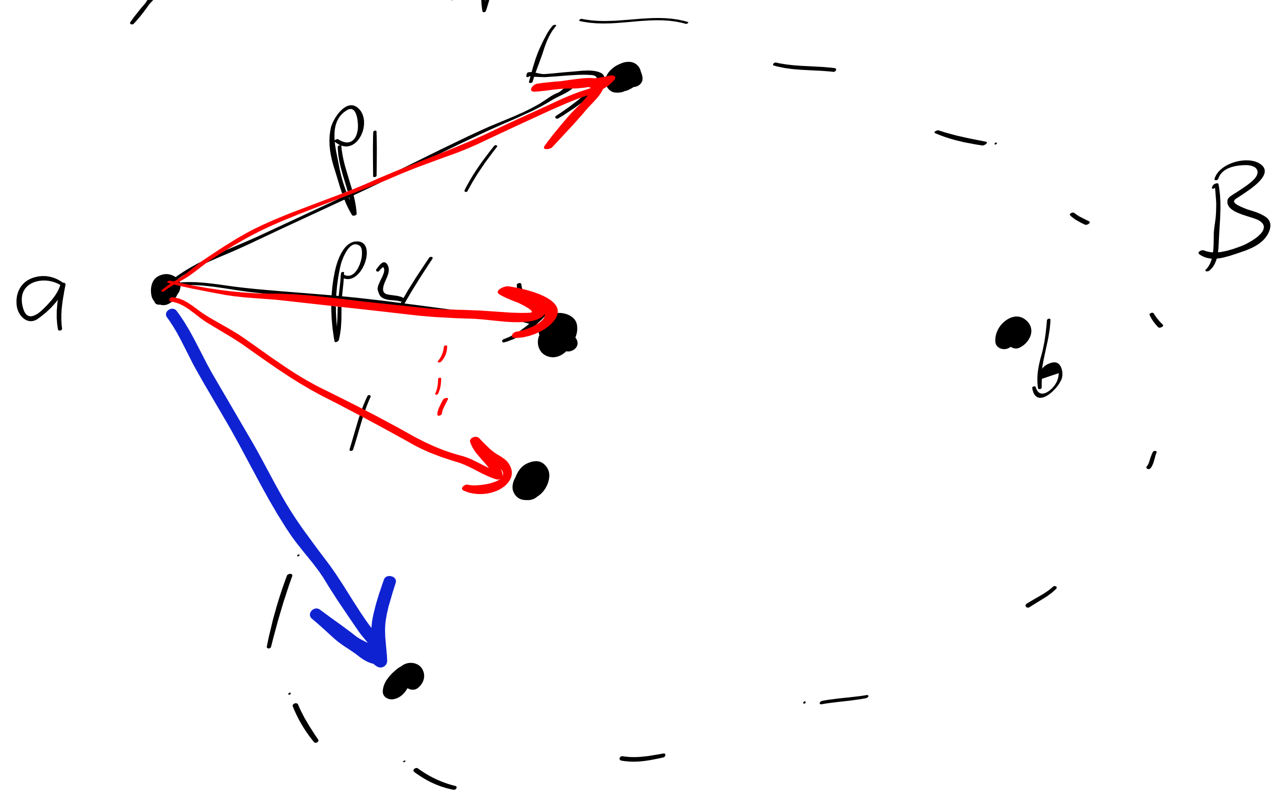
*set of all vertices, except a.

Proof By induction on $|V(D) \setminus B|$.*

Base case: $|V(D) \setminus B| = 1$. This can only happen when $B = V(D) \setminus \{a\}$.



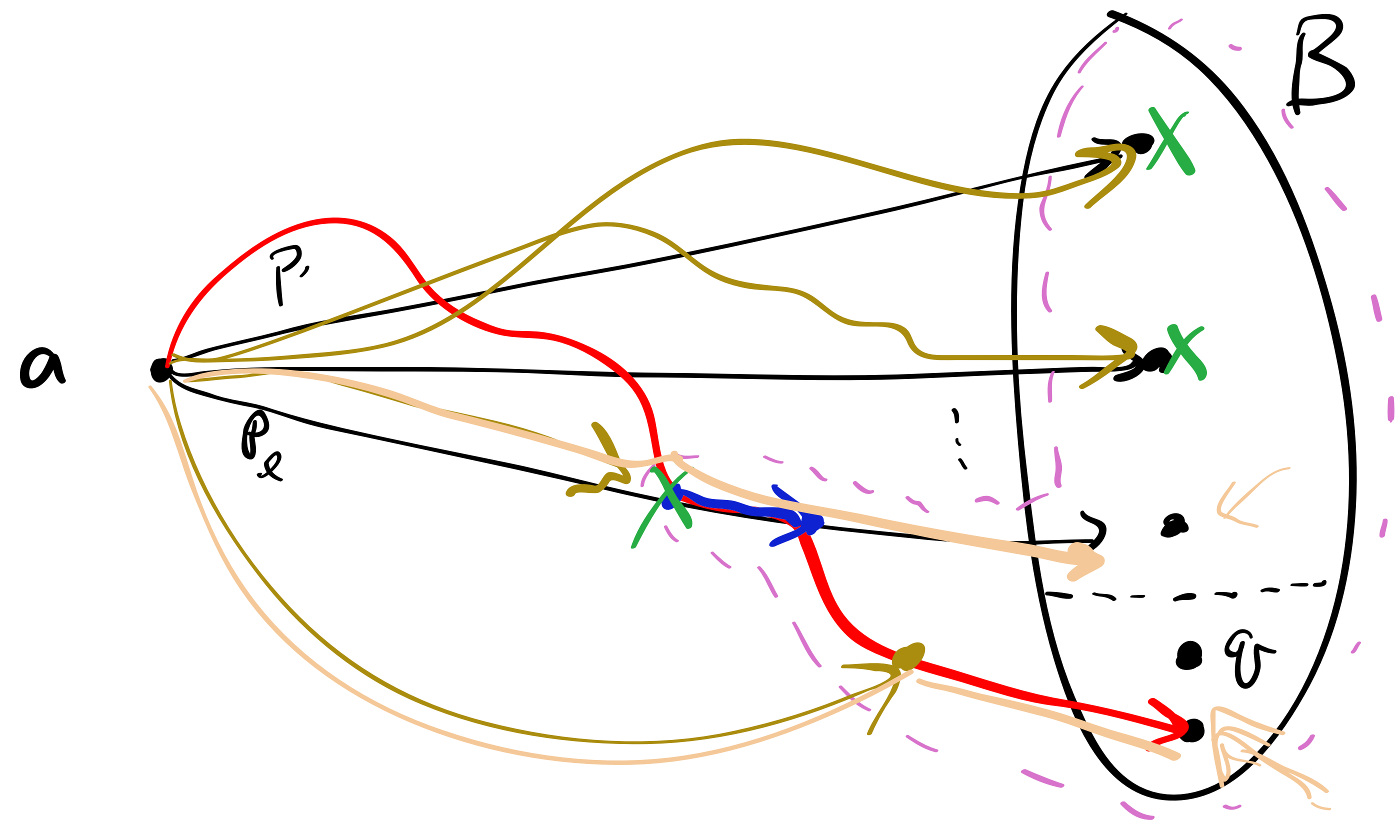
k is min a-b cutset size.

$\deg(a) \geq k$. We can add path (of one arc).

* I use $|S|$ for the number of elements in a set

Assume Prop is true when restricted to sets (for "B") of sizes $|V(D)|-1, |V(D)|-2, \ldots, |V(D)|-n$.

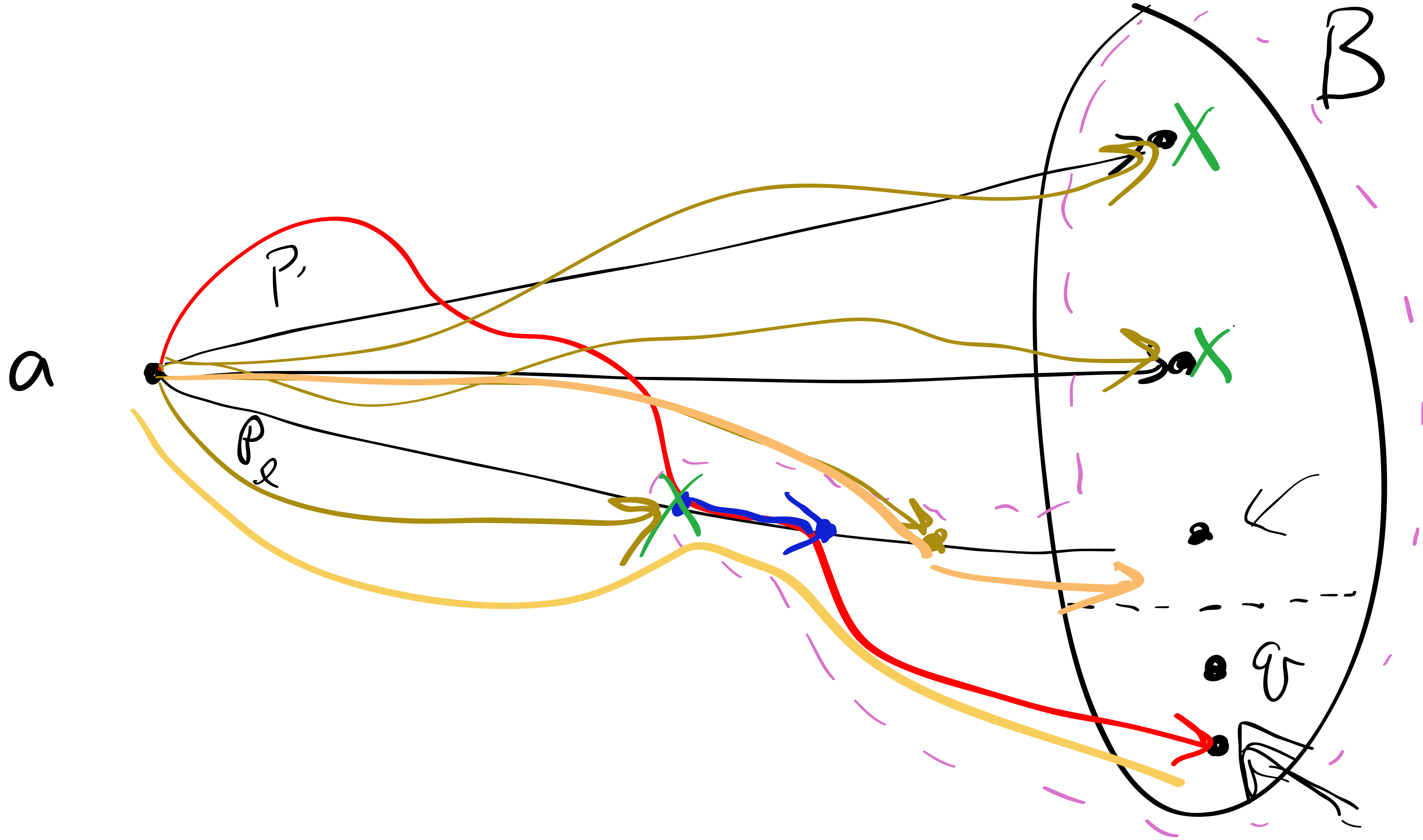
Suppose B is such a set with $|V(D)|-n-1$.



$\ell < k$.

Find any path a to q. If it's disjoint from $P_1, \ldots, P_\ell$, we lucked out and are done.

Otherwise, we apply induction hypothesis with B' as shown.

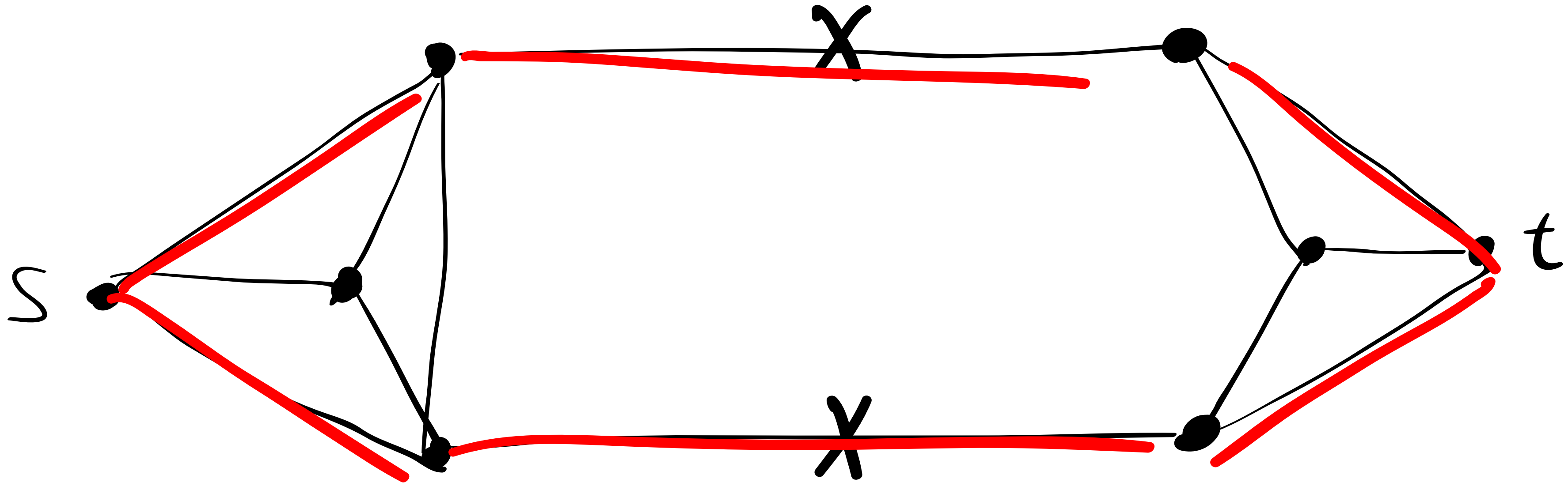


And, if the extra path ends on P_ℓ ...

Menger's Theorem: There is a version for edges in a graph and vertices in a graph.

For G connected: if you find an s-element edge cut-set of G, and between all pairs you find r disjoint paths, then

$$r \leq \lambda(G) \leq s.$$



∃ a 2-element s-t cut-set.

∃ 2 edge disjoint s-t paths.

∴ Any s-t cut set has ≥ 2 edges, and there are no edge-disjoint sets of paths from s to t with more than 2 paths in them.