Lecture 19  Euler Circuits

Thm: If $G$ is connected and has no vertices of odd degree, then $G$ has an Euler Circuit.

pf by induction?

Claim: Thm is true for all such $G$ with $n$ edges.

Base Case: $n=0$.

Path = "". ✔
Assume this true for $n = 1, \ldots, k-1$. Suppose $G$ has $n = k$ edges, connected, no odd vertices. $G_1 = G \setminus \{e\}$, for some random edge. $G_1$ has $k-1$ edges, so then applies.

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$G \quad \rightarrow \quad G_1$
How can I derive a "smaller" graph from $G$, and keep "connected", "no odd vertices!" subtracting " $\bullet\rightarrow\bullet$ " failed

Remove " $\bullet\rightarrow\bullet$ 

or $C_k$, $\triangle$
If $G$ has no odd vertices,

then there is a disjoint union of circuits in $G$ that contains each edge once.

pf: By strong induction on the number of edges.

$h=0$:

Done with zero cycles.
Assume that holds for graphs with \( n = 0, 1, \ldots, k-1 \) edges, \( k > 1 \).

Suppose we're given \( G \), has \( k \) edges and no odd vertices, \( k > 1 \), so pick an edge \( e \).

\( G \) contains: \( e \)

\[ \text{loop case} \rightarrow (G - \{e\}) \text{ has} \]

Same components as \( G \).

By \( \text{\textcircled{1}} \), have circuits, disjoint, union including all edges. "Spleia" into a circuit.
H has \( v \), odd degree.

\( \# \) of odd vertices is even, always.

So \( \exists \) vertex of odd degree.

\( \therefore G \) has vertex of odd degree \( \Rightarrow \) \( G \) can't have a bridge. So \( e \) is in a cycle.
Now remove the cycle, call the remainder $G$. The purple circuits exist since $\tilde{G}$ has fewer than $k$ edges. Alternate blue and purple to finish off the component containing $e$. page 7