0.1. **Why we do this.** We have good reasons for finding tangent lines. It’s not just making graphs more impressive.

A term like

\[(x + 1)^3\]

can be some work to deal with, and

\[(x + 1)^{100}\]

is a lot worse. Often we are willing to give up exact answers if we can use a simpler expression. Tangent lines will often do this for us.

We know already the shape of the graph of \(f(x) = x^n\), for \(n = 2, 3, 4, \ldots\). To the right of the \(y\)-axis it bends upwards. Therefore the tangent line at \(x = 1\) will be below the curve for \(x \geq 0\), except it will touch the curve at \(x = 1\).

For \(n = 2\) and \(n = 4\):

![Graph](image1)

![Graph](image2)
For $n = 5$:

What is that tangent line?

$$f(x) = x^n \quad f'(x) = nx^{n-1}$$

$$f(1) = 1 \quad f'(1) = n$$

so we solve

$$n = \frac{y - 1}{x - 1}$$

and get

$$y = 1 + n(x - 1).$$

Tangent line $y = L(x)$ is

$$L(x) = 1 + n(x - 1).$$

For $x \geq 0$, this is below or equal to $f(x)$ and so we conclude $f(x) \geq L(x)$ which says

(1) \quad $$x^n \geq 1 + n(x - 1) \quad \text{if } x \geq 0.$$

We just proved a very useful estimate:

**Lemma 1.** For any $n = 1, 2, 3, 4, \ldots$,

$$(1 + z)^n \geq 1 + nz \quad \text{if } z \geq -1.$$

**Proof.** You get from (1) to this by substituting $z = x - 1$. The restriction that $x$ is at least zero translates to the restriction $z \geq -1$. \qed
If you let \( w = -z \) this becomes
\[
(1 - w)^n \geq 1 - nw \quad \text{(if } w \leq 1) \nonumber.
\]
So a different way to say the same this is this:

**Lemma 2.** For any \( n = 1, 2, 3, 4, \ldots \),
\[
(1 - z)^n \geq 1 - nz \quad \text{(if } z \leq 1) \nonumber.
\]
(In class, I worked this out using the tangent line at \(-1\), but that is more work.)

0.2. **The estimates, a recap.** Using facts about derivatives, tangent lines, and what we believe about the curves \( y = x^n \), we get some useful estimates on powers of sums. We were a bit vague about the “bending up” business, but that will be made very precise using second derivatives next week.

The estimates are, for \( n = 1, 2, 3, 4, \ldots \) and \( z \) a real number,
\[
(1 + z)^n \geq 1 + nz \quad \text{(if } z \geq -1) \nonumber.
\]

and
\[
(1 - z)^n \geq 1 - nz \quad \text{(if } z \leq 1) \nonumber.
\]

If you want these to be close estimates, you will want to stick with something like \(-\frac{1}{2} < z < \frac{1}{2}\). For our work with \( e \), it will suffice to know that the estimates is a lower estimate.

0.3. **How I graphed that.** I don’t want to give a Matlab lesson here, but it is always a good idea to give examples. Here is one of the *.m files:

For \( n = 5 \) and the tangent at \( x = 1 \):
\[
n=5 \\
x = -1.5:.001:1.5; \\
y = x.^n; \\
L = n*(x-1) + 1; \\
plot(x,y,x,L); \\
grid on \\
axis equal \\
axis([-1.5 1.5 -3 3]) \\
URL: http://www.math.unm.edu/~loring \\
E-mail address: loring@math.unm.edu
\]

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