On two new inequalities for Leray solutions of the Navier-Stokes equations in $\mathbb{R}^n$

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Abstract

We derive two asymptotic inequalities, not previously observed, that are valid for spatial derivatives (of arbitrary order) of Leray solutions to the incompressible Navier-Stokes equations in $\mathbb{R}^n$, $n \leq 4$. They are important to the decay investigation of solution derivatives.

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1. Introduction

In this note we derive two new inequalities for arbitrary Leray solutions of the incompressible Navier-Stokes equations (in dimension $2 \leq n \leq 4$), that is, global solutions $\boldsymbol{u}(\cdot,t) \in L^\infty((0,\infty), \boldsymbol{L}^2_\sigma(\mathbb{R}^n)) \cap L^2((0,\infty), \dot{\boldsymbol{H}}^1(\mathbb{R}^n)) \cap C_{\mathrm{w}}([0,\infty), \boldsymbol{L}^2(\mathbb{R}^n))$ of the fluid flow system

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \nu\,\Delta\boldsymbol{u}, \tag{1.1a}$$

$$\nabla\cdot\boldsymbol{u}(\cdot,t) = 0, \tag{1.1b}$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma(\mathbb{R}^n), \tag{1.1c}$$

that satisfy the strong energy inequality

$$\| \boldsymbol{u}(\cdot,t) \|^2_{L^2(\mathbb{R}^n)} + 2\,\nu \int_s^t \| D\boldsymbol{u}(\cdot,\tau) \|^2_{L^2(\mathbb{R}^n)} d\tau \;\le\; \| \boldsymbol{u}(\cdot,s) \|^2_{L^2(\mathbb{R}^n)}, \quad \forall\; t \ge s \tag{1.2}$$

for a.e. $s \ge 0$, including $s = 0$. (For the construction of Leray solutions, see e.g. [12, 15, 20, 21].) In (1.1) above, $\nu > 0$ is constant, $\boldsymbol{u} = \boldsymbol{u}(x,t)$ and $p = p(x,t)$ are the unknowns (the flow velocity and pressure, respectively), with condition (1.1c) satisfied in $\boldsymbol{L}^2(\mathbb{R}^n)$, i.e., $\| \boldsymbol{u}(\cdot,t) - \boldsymbol{u}_0 \|_{L^2(\mathbb{R}^n)} \to 0$ as $t \searrow 0$. In the present work, we assume $2 \le n \le 4$ throughout.

A well known property of Leray solutions is that they are eventually very regular: there is always some $t_* \ge 0$ such that one has $\boldsymbol{u} \in C^\infty(\mathbb{R}^n \times (t_*, \infty))$ and, moreover,

$$\boldsymbol{u}(\cdot,t) \in C((t_*,\infty), \boldsymbol{H}^m(\mathbb{R}^n)), \qquad \forall\; m \ge 0, \tag{1.3}$$

see e.g. [5, 9, 12, 19, 20].[1] It is also well established that $\lim\limits_{t\to 0} \| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = 0$ and, more generally,[2]

$$\lim_{t\to 0}\; t^{m/2} \| D^m \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = 0 \tag{1.4}$$

for every $m \ge 1$, and for all Leray solutions to the system (1.1) [1, 9, 13, 14, 16, 19]. With additional assumptions on the initial data, stronger decay properties of $\boldsymbol{u}(\cdot,t)$ are observed. A remarkable example are Leray solutions with exponential decay, i.e., $\| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(e^{-\alpha t})$ for some $\alpha > 0$, see [2, 17, 18] and the excellent review [4]. Like the familiar heat equation, their derivatives will then all decay at a similar rate, as the following result shows. (For the proof of Theorem A, see Section 3.) Thus, for such solutions, we have, for any m, k and large t: $\| D^m \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(e^{-\alpha t})$, $\| D_t^k D_x^m \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(e^{-\alpha t})$, $\| D_t^k D_x^m \boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} = O(e^{-\alpha t})$, $\| D^m p(\cdot,t) \|_{L^2} = O(e^{-2\alpha t})$, $\| D_t^k D_x^m p(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(e^{-2\alpha t})$, $\| D_t^k D_x^m p(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} = O(e^{-2\alpha t})$, and so on. These estimates have apparently not been obtained before.

Theorem A. *Let $n \le 4$, $\boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma(\mathbb{R}^n)$, and let $\boldsymbol{u}(\cdot,t)$ be any given Leray solution to the Navier-Stokes equations* (1.1). *Then we have, for every $\alpha > 0$:*

$$\limsup_{t\to\infty}\; e^{\alpha t} \| D^m \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \;\le\; K(m,\alpha)\, \nu^{-m/2} \limsup_{t\to\infty}\; e^{\alpha t} \| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \tag{1.5}$$

for every $m \ge 1$, where $K(m,\alpha) = 2^m\, \alpha^{m/2}$.

[1] Moreover, it is known that $t_* = 0$ if $n = 2$, $t_* \le 0.000\,465\; \nu^{-5} \| \boldsymbol{u}_0 \|^4_{L^2(\mathbb{R}^3)}$ if $n = 3$, and $t_* \le 0.002\,728\; \nu^{-3} \| \boldsymbol{u}_0 \|^2_{L^2(\mathbb{R}^4)}$ if $n = 4$: see e.g. [5], Theorem A.

[2] For the definition of $\| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)}$, $\| D^m \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)}$ and other similar norms, see (1.7).

A more usual situation is having only algebraic decay: $\| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha})$ with $\alpha \le n/4+1/2$, cf. [4, 8, 11, 16, 17, 22, 23]. For larger values of α, see [3, 4, 6, 7]. In this case, the following fundamental inequality (derived in Section 4) is of use.

Theorem B. *Let $n \le 4$, $\boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma(\mathbb{R}^n)$, and let $\boldsymbol{u}(\cdot,t)$ be any given Leray solution to the Navier-Stokes equations* (1.1). *Then we have, for every $\alpha \ge 0$:*

$$\limsup_{t\to\infty}\ t^{\alpha+m/2} \| D^m \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \le K(\alpha,m)\, \nu^{-m/2} \limsup_{t\to\infty}\ t^{\alpha} \| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \tag{1.6a}$$

for every $m \ge 1$, where

$$K(\alpha,m) = \min_{\delta>0} \Big\{ \delta^{-1/2} \prod_{j=0}^{m} \big(\alpha + j/2 + \delta\big)^{1/2} \Big\}. \tag{1.6b}$$

In particular, if $\| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha})$ for large t, then we will have, for any m, k and t big enough: $\| D_t^k D_x^m \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha-k-m/2})$, $\| D_t^k D_x^m \boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} = O(t^{-\alpha-k-m/2-n/4})$, $\| D_t^k D_x^m p(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(t^{-2\alpha-k-m/2-n/4})$, $\| D_t^k D_x^m p(\cdot,t) \|_{L^\infty} = O(t^{-2\alpha-k-m/2-n/2})$, etc. Thus, we quickly retrieve important results in [1, 14, 19].

Notation. As already shown, boldface letters are used for vector quantities, as in $\boldsymbol{u}(x,t) = (u_1(x,t), ..., u_n(x,t))$. Also, $\nabla p \equiv \nabla p(\cdot,t)$ denotes the spatial gradient of $p(\cdot,t)$; $D_j = \partial/\partial x_j$; $\nabla\cdot \boldsymbol{u} = D_1 u_1 + ... + D_n u_n$ is the (spatial) divergence of $\boldsymbol{u}(\cdot,t)$. $\boldsymbol{L}^2_\sigma(\mathbb{R}^n)$ denotes the space of solenoidal fields $\mathbf{v} = (v_1, ..., v_n) \in \boldsymbol{L}^2(\mathbb{R}^n) \equiv L^2(\mathbb{R}^n)^n$ with $\nabla\cdot \mathbf{v} = 0$ in the distributional sense; $\dot{\boldsymbol{H}}^1(\mathbb{R}^n) = \dot{H}^1(\mathbb{R}^n)^n$ with $\dot{H}^1(\mathbb{R}^n)$ being the homogeneous L^2 Sobolev space of order 1; $\boldsymbol{H}^m(\mathbb{R}^n) = H^m(\mathbb{R}^n)^n$, where $H^m(\mathbb{R}^n)$ is the space of functions $v \in L^2(\mathbb{R}^n)$ whose m-th order derivatives are also square integrable. $C_{\mathrm{w}}(I, \boldsymbol{L}^2(\mathbb{R}^n))$ denotes the set of mappings from a given interval $I \subseteq \mathbb{R}$ to $\boldsymbol{L}^2(\mathbb{R}^n)$ that are L^2-weakly continuous at each $t \in I$. $\| \cdot \|_{L^q(\mathbb{R}^n)}$, $1 \le q \le \infty$, are the standard norms of the Lebesgue spaces $L^q(\mathbb{R}^n)$, with the vector counterparts

$$\| \boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)} = \Big\{ \sum_{i=1}^{n} \int_{\mathbb{R}^n} | u_i(x,t) |^q \, dx \Big\}^{1/q} \tag{1.7a}$$

$$\| D^m \boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)} = \Big\{ \sum_{i,\, j_1,\dots,j_m=1}^{n} \int_{\mathbb{R}^n} | D_{j_1} \cdots D_{j_m} u_i(x,t) |^q \, dx \Big\}^{1/q} \tag{1.7b}$$

(also denoted by $\| D_x^m \boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)}$) if $1 \le q < \infty$; if $q = \infty$, then $\| \boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} = \max \big\{ \| u_i(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} \colon 1 \le i \le n \big\}$, $\| D\boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} = \max \big\{ \| D_j u_i(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} \colon 1 \le i, j \le n \big\}$, and so forth.

2. Preliminaries

We briefly recall some basic material that will be needed in the derivation of (1.5) and (1.6) in the next sections. In dimension $n = 2, 3$, pointwise values can be estimated in terms of H^2 norms: one has[3]

$$\| \mathrm{u} \|_{L^\infty(\mathbb{R}^2)} \leq \| \mathrm{u} \|_{L^2(\mathbb{R}^2)}^{1/2} \| D^2 \mathrm{u} \|_{L^2(\mathbb{R}^2)}^{1/2} \tag{2.1a}$$

for arbitrary $\mathrm{u} \in H^2(\mathbb{R}^2)$; likewise,

$$\| \mathrm{u} \|_{L^\infty(\mathbb{R}^3)} \leq \| \mathrm{u} \|_{L^2(\mathbb{R}^3)}^{1/4} \| D^2 \mathrm{u} \|_{L^2(\mathbb{R}^3)}^{3/4} \tag{2.1b}$$

for $\mathrm{u} \in H^2(\mathbb{R}^3)$. These can be shown by Fourier transform, which also gives

$$\| D\mathrm{u} \|_{L^2(\mathbb{R}^n)} \leq \| \mathrm{u} \|_{L^2(\mathbb{R}^n)}^{1/2} \| D^2 \mathrm{u} \|_{L^2(\mathbb{R}^n)}^{1/2} \tag{2.2a}$$

(for any n), or, more generally,

$$\| D^\ell \mathrm{u} \|_{L^2(\mathbb{R}^n)} \leq \| \mathrm{u} \|_{L^2(\mathbb{R}^n)}^{1-\theta} \| D^m \mathrm{u} \|_{L^2(\mathbb{R}^n)}^{\theta}, \qquad \theta = \frac{\ell}{m} \tag{2.2b}$$

for all $0 \leq \ell \leq m$. Combining (2.1) and (2.2), we get the following basic inequalities: for any $m \geq 1$, $0 \leq \ell \leq m-1$,

$$\| D^\ell \mathrm{u} \|_{L^\infty(\mathbb{R}^2)} \| D^{m-\ell} \mathrm{u} \|_{L^2(\mathbb{R}^2)} \leq \| \mathrm{u} \|_{L^2(\mathbb{R}^2)} \| D^{m+1} \mathrm{u} \|_{L^2(\mathbb{R}^2)}, \tag{2.3a}$$

$$\| D^\ell \mathrm{u} \|_{L^\infty(\mathbb{R}^3)} \| D^{m-\ell} \mathrm{u} \|_{L^2(\mathbb{R}^3)} \leq \| \mathrm{u} \|_{L^2(\mathbb{R}^3)}^{1/2} \| D\mathrm{u} \|_{L^2(\mathbb{R}^3)}^{1/2} \| D^{m+1} \mathrm{u} \|_{L^2(\mathbb{R}^3)}. \tag{2.3b}$$

In dimension $n = 4$, we begin instead with the fundamental Sobolev inequality

$$\| \mathrm{u} \|_{L^4(\mathbb{R}^4)} \leq \| D\mathrm{u} \|_{L^2(\mathbb{R}^4)} \tag{2.4}$$

(for arbitrary $\mathrm{u} \in \dot{H}^1(\mathbb{R}^4)$), and obtain, using (2.2) again,

$$\| D^\ell \mathrm{u} \|_{L^4(\mathbb{R}^4)} \| D^{m-\ell} \mathrm{u} \|_{L^2(\mathbb{R}^4)} \leq \| D\mathrm{u} \|_{L^2(\mathbb{R}^4)} \| D^{m+1} \mathrm{u} \|_{L^2(\mathbb{R}^4)} \tag{2.5}$$

for all $m \geq 0$, $0 \leq \ell \leq m$, which plays the role of (2.3) in this case. Finally, because of (2.3) and (2.5) above, we will need the following basic facts about Leray solutions of the Navier-Stokes system (1.1):

$$\lim_{t \to \infty} \| \boldsymbol{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} = 0 \tag{2.6}$$

[3] We actually have $\| \mathrm{u} \|_{L^\infty(\mathbb{R}^2)} \leq K \| \mathrm{u} \|_{L^2(\mathbb{R}^2)}^{1/2} \| D\mathrm{u} \|_{L^2(\mathbb{R}^2)}^{1/2}$ with $K = 1/2$, and similarly for (2.1b). All interpolation inequalities we use have constants $K \leq 1$, and so we simply take $K = 1$ throughout.

(actually, (2.6) will be needed only if $n = 2$), and

$$\lim_{t\to\infty} \| D\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = 0, \tag{2.7}$$

which will be required if $n = 3, 4$. The following elementary proofs of (2.6) and (2.7) (taken from [10, 11, 23]) are noteworthy: because (by (1.2)) $\int_0^\infty \| D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}\, dt$ is finite and $\| D\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}$ is monotone for large t (see e.g. [5], THEOREM B), it follows that $t\, \| D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \to 0$ as $t \to \infty$. This not only shows (2.7) above but, using some simple heat kernel estimates, yields (2.6) too (for details, see [10, 11, 23]).

3. Proof of Theorem A

Let $\boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma(\mathbb{R}^n)$, and let $\boldsymbol{u}(\cdot,t)$ be any given Leray solution of the Navier-Stokes system (1.1) such that we have, for some $\alpha > 0$,

$$\limsup_{t\to\infty}\, e^{\alpha t} \| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} =: \lambda_0(\alpha) < \infty. \tag{3.1}$$

We now prove (1.5), for every m. As in [10], the strategy is to use induction in m. The argument is similar for $n = 2, 3, 4$, so we will show the details for $n = 3$ only. Let $t_* \geq 0$ be chosen so that (1.3) holds, and let, for convenience:

$$f(t) := e^{2\beta t}, \qquad g(t) := e^{2(\beta-\alpha)t} \tag{3.2}$$

(for some $\beta > \alpha$ that we will choose later). Given $0 < \epsilon < 1$, we begin with $m = 1$. First, taking the dot product of (1.1a) with $f(t)\, \boldsymbol{u}(x,t)$ and integrating the result on $\mathbb{R}^3 \times [\, t_0, t\,]$, for $t \geq t_0 > t_*$, we get, because of (1.1b),

$$\begin{aligned} f(t)\, \| \boldsymbol{u}(\cdot,t) \|^2_{L^2(\mathbb{R}^3)} + 2\nu \int_{t_0}^t f(\tau)\, \| D\boldsymbol{u}(\cdot,\tau) \|^2_{L^2(\mathbb{R}^3)} d\tau \\ = f(t_0)\, \| \boldsymbol{u}(\cdot,t_0) \|^2_{L^2(\mathbb{R}^3)} + 2\beta \int_{t_0}^t f(\tau)\, \| \boldsymbol{u}(\cdot,\tau) \|^2_{L^2(\mathbb{R}^3)} d\tau. \end{aligned}$$

By (3.1), we can then choose $t_0 > t_*$ sufficiently large so that

$$\begin{aligned} f(t)\, \| \boldsymbol{u}(\cdot,t) \|^2_{L^2(\mathbb{R}^3)} + 2\nu \int_{t_0}^t f(\tau)\, \| D\boldsymbol{u}(\cdot,\tau) \|^2_{L^2(\mathbb{R}^3)} d\tau \leq \\ \leq g(t_0)\, (\lambda_0(\alpha)+\epsilon)^2 + 2\beta \int_{t_0}^t g(\tau)\, (\lambda_0(\alpha)+\epsilon)^2\, d\tau \\ \leq g(t_0)\, (\lambda_0(\alpha)+\epsilon)^2 + \frac{\beta}{\beta-\alpha}\, (\lambda_0(\alpha)+\epsilon)^2 \big\{ g(t) - g(t_0) \big\} \\ \leq \frac{\beta}{\beta-\alpha}\, (\lambda_0(\alpha)+\epsilon)^2\, e^{2(\beta-\alpha)t} \end{aligned}$$

for all $t \geq t_0$. This gives, in particular,

$$\int_{t_0}^{t} f(\tau) \, \| \, D\boldsymbol{u}(\cdot,\tau) \, \|^2_{L^2(\mathbb{R}^3)} d\tau \;\leq\; \frac{1}{2\,\nu} \, \frac{\beta}{\beta-\alpha} \, (\lambda_0(\alpha)+\epsilon)^2 \, e^{2(\beta-\alpha)t} \tag{3.3}$$

for all $t \geq t_0$. Now, we proceed as follows. Differentiating the equation (1.1a) with respect to x_ℓ, taking the dot product with $(f(t)-f(t_0))\, D_\ell \boldsymbol{u}(x,t)$, and integrating the result on $\mathbb{R}^3 \times [\, t_0, t\,]$, $t \geq t_0$, we get, summing over $1 \leq \ell \leq 3$,

$$\begin{aligned}
&(f(t)-f(t_0)) \, \| \, D\boldsymbol{u}(\cdot,t) \, \|^2_{L^2(\mathbb{R}^3)} + \, 2\,\nu \int_{t_0}^{t} (f(\tau)-f(t_0)) \, \| \, D^2\boldsymbol{u}(\cdot,\tau) \, \|^2_{L^2(\mathbb{R}^3)} d\tau \\
&\qquad \leq \; 2\,\beta \int_{t_0}^{t} f(\tau) \, \| \, D\boldsymbol{u}(\cdot,\tau) \, \|^2_{L^2(\mathbb{R}^3)} d\tau \; + \\
&\qquad\quad K_1 \int_{t_0}^{t} (f(\tau)-f(t_0)) \, \| \, \boldsymbol{u}(\cdot,\tau) \, \|_{L^\infty(\mathbb{R}^3)} \| \, D\boldsymbol{u}(\cdot,\tau) \, \|_{L^2(\mathbb{R}^3)} \| \, D^2\boldsymbol{u}(\cdot,\tau) \, \|_{L^2(\mathbb{R}^3)} d\tau \\
&\qquad \leq \; 2\,\beta \int_{t_0}^{t} f(\tau) \, \| \, D\boldsymbol{u}(\cdot,\tau) \, \|^2_{L^2(\mathbb{R}^3)} d\tau \; + \\
&\qquad\quad K_1 \int_{t_0}^{t} (f(\tau)-f(t_0)) \, \| \, \boldsymbol{u}(\cdot,\tau) \, \|^{1/2}_{L^2(\mathbb{R}^3)} \| \, D\boldsymbol{u}(\cdot,\tau) \, \|^{1/2}_{L^2(\mathbb{R}^3)} \| \, D^2\boldsymbol{u}(\cdot,\tau) \, \|^2_{L^2(\mathbb{R}^3)} d\tau
\end{aligned}$$

by (1.1b) and (2.3b), where $K_1 = 2\sqrt{3}$. Therefore, by (2.7) and (3.3), and increasing t_0 if necessary, we get

$$\begin{aligned}
&(f(t)-f(t_0)) \, \| \, D\boldsymbol{u}(\cdot,t) \, \|^2_{L^2(\mathbb{R}^3)} + \, (2-\epsilon)\,\nu \int_{t_0}^{t} (f(\tau)-f(t_0)) \, \| \, D^2\boldsymbol{u}(\cdot,\tau) \, \|^2_{L^2(\mathbb{R}^3)} d\tau \\
&\qquad\qquad\qquad \leq \; \frac{\beta}{\nu} \, \frac{\beta}{\beta-\alpha} \, (\lambda_0(\alpha)+\epsilon)^2 \, e^{2(\beta-\alpha)t}
\end{aligned}$$

for all $t \geq t_0$. Writing $f(t)-f(t_0) = f(t)\big(1-e^{-2\beta(t-t_0)}\big)$ and taking $t_1 > t_0$ large enough that $e^{-2\beta(t_1-t_0)} < \epsilon$, this gives

$$f(t) \, \| \, D\boldsymbol{u}(\cdot,t) \, \|^2_{L^2(\mathbb{R}^3)} \;\leq\; A_1 \, (\lambda_0(\alpha)+\epsilon)^2 g(t), \qquad A_1 := \; \frac{1}{1-\epsilon} \, \frac{\beta^2}{\beta-\alpha} \, \nu^{-1} \tag{3.4}$$

for all $t \geq t_1$. We now proceed by induction: having obtained that, for some $m \geq 1$,

$$f(t) \, \| \, D^m\boldsymbol{u}(\cdot,t) \, \|^2_{L^2(\mathbb{R}^3)} \;\leq\; A_m \, (\lambda_0(\alpha)+\epsilon)^2 \, g(t), \qquad \forall \; t \geq t_m \tag{3.5a}$$

(for some $A_m > 0$, and for some $t_m > t_*$ sufficiently large), we proceed as follows to estimate $f(t) \, \| \, D^{m+1}\boldsymbol{u}(\cdot,t) \, \|^2_{L^2(\mathbb{R}^3)}$: increasing t_m if necessary, we obtain, as before,

$$\begin{aligned}
&f(t) \, \| \, D^m\boldsymbol{u}(\cdot,t) \, \|^2_{L^2(\mathbb{R}^3)} + \, (2-\epsilon)\,\nu \int_{t_m}^{t} f(\tau) \, \| \, D^{m+1}\boldsymbol{u}(\cdot,\tau) \, \|^2_{L^2(\mathbb{R}^3)} d\tau \;\leq \\
&\qquad\qquad \leq \; f(t_m) \, \| \, D^m\boldsymbol{u}(\cdot,t_m) \, \|^2_{L^2(\mathbb{R}^3)} + \, 2\,\beta \int_{t_m}^{t} f(\tau) \, \| \, D^m\boldsymbol{u}(\cdot,\tau) \, \|^2_{L^2(\mathbb{R}^3)} \, d\tau
\end{aligned}$$

for all $t \geq t_m$. Recalling (3.5a), this then gives

$$
\begin{aligned}
f(t) \,\| D^m \boldsymbol{u}(\cdot,t) \|^2_{L^2(\mathbb{R}^3)} \,+\, (2-\epsilon)\,\nu \int_{t_m}^t f(\tau) \,\| D^{m+1} \boldsymbol{u}(\cdot,\tau) \|^2_{L^2(\mathbb{R}^3)} d\tau \;\leq \\
\leq\; A_m\,(\lambda_0(\alpha)+\epsilon)^2\, g(t_m) \;+\; 2\,\beta \int_{t_m}^t A_m\,(\lambda_0(\alpha)+\epsilon)^2\, g(\tau)\, d\tau \quad [\text{ by } (3.5a)] \\
=\; A_m\,(\lambda_0(\alpha)+\epsilon)^2\, g(t_m) \;+\; \frac{\beta}{\beta-\alpha}\, A_m\,(\lambda_0(\alpha)+\epsilon)^2 \big(g(t)-g(t_m)\big) \\
\leq\; \frac{\beta}{\beta-\alpha}\, A_m\,(\lambda_0(\alpha)+\epsilon)^2\, g(t)
\end{aligned}
$$

for all $t \geq t_m$, so that we have

$$
\int_{t_m}^t f(\tau) \,\| D^{m+1} \boldsymbol{u}(\cdot,\tau) \|^2_{L^2(\mathbb{R}^3)} d\tau \;\leq\; \frac{1}{2-\epsilon}\, \frac{\beta}{\beta-\alpha}\, \nu^{-1} A_m\,(\lambda_0(\alpha)+\epsilon)^2\, g(t) \qquad (3.5b)
$$

for all $t \geq t_m$. Since, by (2.3b) and (2.7) again, we get (increasing t_m if necessary):

$$
\begin{aligned}
(f(t)-f(t_m)) \,\| D^{m+1} \boldsymbol{u}(\cdot,t) \|^2_{L^2(\mathbb{R}^3)} \,+\, (2-\epsilon)\,\nu \int_{t_m}^t (f(\tau)-f(t_m)) \,\| D^{m+2} \boldsymbol{u}(\cdot,\tau) \|^2_{L^2(\mathbb{R}^3)} \\
\leq\; 2\,\beta \int_{t_m}^t f(\tau) \,\| D^{m+1} \boldsymbol{u}(\cdot,\tau) \|^2_{L^2(\mathbb{R}^3)}\, d\tau
\end{aligned}
$$

for all $t \geq t_m$, we then obtain, by (3.5b),

$$
\begin{aligned}
(f(t)-f(t_m)) \,\| D^{m+1} \boldsymbol{u}(\cdot,t) \|^2_{L^2(\mathbb{R}^3)} \,+\, (2-\epsilon)\,\nu \int_{t_m}^t (f(\tau)-f(t_m)) \,\| D^{m+2} \boldsymbol{u}(\cdot,\tau) \|^2_{L^2(\mathbb{R}^3)} \\
\leq\; \frac{2}{2-\epsilon}\, \frac{\beta^2}{\beta-\alpha}\, \nu^{-1} A_m\,(\lambda_0(\alpha)+\epsilon)^2\, g(t)
\end{aligned}
$$

for all $t \geq t_m$. Writing $(f(t)-f(t_m)) = f(t)\,(1-e^{-2\beta(t-t_m)})$ and taking $t_{m+1} > t_m$ such that $e^{-2\beta(t_{m+1}-t_m)} < \epsilon$, this gives

$$
f(t) \,\| D^{m+1} \boldsymbol{u}(\cdot,t) \|^2_{L^2(\mathbb{R}^3)} \;\leq\; \frac{1}{1-\epsilon}\, \frac{2}{2-\epsilon}\, \frac{\beta^2}{\beta-\alpha}\, \nu^{-1} A_m\,(\lambda_0(\alpha)+\epsilon)^2\, g(t) \qquad (3.6)
$$

for all $t \geq t_{m+1}$. That is, by induction in m, we have shown that, for every $m \geq 1$:

$$
f(t) \,\| D^m \boldsymbol{u}(\cdot,t) \|^2_{L^2(\mathbb{R}^3)} \;\leq\; A_m\,(\lambda_0(\alpha)+\epsilon)^2\, g(t) \qquad \forall\; t \geq t_m \qquad (3.7a)
$$

for $t_m > t_*$ sufficiently large, and where the constant $A_m > 0$ satisfies the recurrence

$$
A_m \;=\; \frac{1}{1-\epsilon}\, \frac{2}{2-\epsilon}\, \frac{\beta^2}{\beta-\alpha}\, \nu^{-1} A_{m-1} \qquad (3.7b)
$$

for all $m \geq 2$, with $A_1 > 0$ given in (3.4) above. This gives

$$A_m \;=\; \Big(\frac{1}{1-\epsilon}\Big)^m \Big(\frac{2}{2-\epsilon}\Big)^{m-1} \Big(\frac{\beta^2}{\beta-\alpha}\Big)^m \nu^{-m} \tag{3.8}$$

for all $m \geq 1$. Choosing $\beta = 2\,\alpha$, we then obtain, by (3.7a) and (3.8),

$$\limsup_{t\to\infty}\; e^{2\alpha t}\,\|\,D^m \boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^3)} \;\leq\; \Big(\frac{1}{1-\epsilon}\Big)^m \Big(\frac{2}{2-\epsilon}\Big)^{m-1} 4^m\,\alpha^m\,\nu^{-m}\,(\lambda_0(\alpha)+\epsilon)^2$$

for every $m \geq 1$, where $\epsilon \in (0,1)$ is arbitrary. Letting $\epsilon \to 0$, this gives the result.

□

4. Proof of Theorem B

We now show (1.6). Again, since the proofs for $n = 2,3,4$ are entirely similar, we will present the details for one case only — say, $n = 4$. Let then $\boldsymbol{u}(\cdot,t)$ be any given Leray solution to (1.1), in $\mathbb{R}^4$, such that we have, for some $\alpha \geq 0$,

$$\limsup_{t\to\infty}\; t^{\alpha}\,\|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^4)} =:\; \lambda_0(\alpha) \;<\; \infty. \tag{4.1}$$

Let $\delta > 0$, $0 < \epsilon < 2$ be given, and let t_* be the solution's regularity time as defined in (1.3). Taking the dot product of (1.1a) with $(t-t_0)^{2\alpha+\delta}\boldsymbol{u}(x,t)$ and integrating the result on $\mathbb{R}^4 \times [\,t_0, t\,]$, for $t \geq t_0 > t_*$, we obtain, because of (1.1b),

$$\begin{aligned}(t-t_0)^{2\alpha+\delta}\,\|\,\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^4)} \;+\; 2\,\nu\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\,\|\,D\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau \\ =\; (2\alpha+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta-1}\,\|\,\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau\end{aligned}$$

for $t \geq t_0 > t_*$. This promptly gives, by (4.1), taking $t_0 > t_*$ sufficiently large, that

$$\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\,\|\,D\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau \;\leq\; \frac{1}{2\nu}\,\frac{2\alpha+\delta}{\delta}\,(\lambda_0(\alpha)+\epsilon)^2\,(t-t_0)^{\delta} \tag{4.2}$$

for all $t \geq t_0$. Next, for $m = 1$, we similarly have

$$\begin{aligned}(t-t_0)^{2\alpha+1+\delta}\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^4)} \;+\; 2\,\nu\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\,\|\,D^2\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau \\ \leq\; (2\alpha+1+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\,\|\,D\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau \;+ \\ K_1\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\,\|\,\boldsymbol{u}(\cdot,\tau)\,\|_{L^4(\mathbb{R}^4)}\|\,D\boldsymbol{u}(\cdot,\tau)\,\|_{L^4(\mathbb{R}^4)}\|\,D^2\boldsymbol{u}(\cdot,\tau)\,\|_{L^2(\mathbb{R}^4)}d\tau\end{aligned}$$

(where $K_1 = 8\sqrt{2}$), which gives, by (2.5):

$$(t-t_0)^{2\alpha+1+\delta}\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^4)}\;+\;2\,\nu\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\,\|\,D^2\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau$$
$$\leq\;(2\alpha+1+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\,\|\,D\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau\;+$$
$$K_1\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\,\|\,D\boldsymbol{u}(\cdot,\tau)\,\|_{L^2(\mathbb{R}^4)}\|\,D^2\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau$$

for $t\geq t_0$. By (2.7) and (4.2), we then get (increasing t_0 if necessary)

$$(t-t_0)^{2\alpha+1}\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^4)}\;\leq\;\frac{1}{2\,\nu}\,(2\alpha+1+\delta)\,\frac{2\alpha+\delta}{\delta}\,(\lambda_0(\alpha)+\epsilon)^2 \tag{4.3a}$$

and

$$\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\,\|\,D^2\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau\;\leq\;\frac{(2\alpha+1+\delta)\,(2\alpha+\delta)}{\delta\cdot[\,(2-\epsilon)\nu\,]^2}\,(\lambda_0(\alpha)+\epsilon)^2\,(t-t_0)^{\delta} \tag{4.3b}$$

for all $t\geq t_0$. Proceeding in this way ($m=2,3,...$), we obtain, at the mth step:

$$(t-t_0)^{2\alpha+m+\delta}\,\|\,D^m\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^4)}\;+\;2\,\nu\int_{t_0}^{t}(\tau-t_0)^{2\alpha+m+\delta}\,\|\,D^{m+1}\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau$$
$$\leq\;(2\alpha+m+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+m-1+\delta}\,\|\,D^m\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau\;+$$
$$K_m\int_{t_0}^{t}(\tau-t_0)^{2\alpha+m+\delta}\,\|\,D^{m+1}\boldsymbol{u}(\cdot,\tau)\,\|_{L^2(\mathbb{R}^4)}\sum_{\ell=0}^{[\,m/2\,]}\|\,D^{\ell}\boldsymbol{u}(\cdot,\tau)\,\|_{L^4(\mathbb{R}^4)}\|\,D^{m-\ell}\boldsymbol{u}(\cdot,\tau)\,\|_{L^4}d\tau$$

for $t\geq t_0$, and some constant $K_m>0$, where $[\,m/2\,]$ denotes the integer part of $m/2$. This gives, by (2.5):

$$\begin{aligned}(t-t_0)^{2\alpha+m+\delta}\,\|\,D^m\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^4)}\;+\;2\,\nu\int_{t_0}^{t}(\tau-t_0)^{2\alpha+m+\delta}\,\|\,D^{m+1}\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau\\ \leq\;(2\alpha+m+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+m-1+\delta}\,\|\,D^m\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau\;+\\ \Big(1+\Big[\,\frac{m}{2}\,\Big]\Big)\cdot K_m\int_{t_0}^{t}(\tau-t_0)^{2\alpha+m+\delta}\,\|\,D\boldsymbol{u}(\cdot,\tau)\,\|_{L^2(\mathbb{R}^4)}\|\,D^{m+1}\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau.\end{aligned} \tag{4.4}$$

At this stage, we would have already obtained from the previous steps that

$$(t-t_0)^{2\alpha+k}\,\|\,D^k\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^4)}\leq\;\frac{1}{\delta\cdot[\,(2-\epsilon)\nu\,]^k}\,\Big\{\prod_{j=0}^{k}(2\alpha+j+\delta)\Big\}\,(\lambda_0(\alpha)+\epsilon)^2 \tag{4.5a}$$

and

$$\begin{aligned}\int_{t_0}^{t}(\tau-t_0)^{2\alpha+k+\delta}\,\|\,D^{k+1}\boldsymbol{u}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^4)}d\tau\;\leq\;\frac{\delta^{-1}}{[\,(2-\epsilon)\nu\,]^{k+1}}\,\Big\{\prod_{j=0}^{k}(2\alpha+j+\delta)\Big\}\\ \times\,(\lambda_0(\alpha)+\epsilon)^2\cdot(t-t_0)^{\delta}\end{aligned} \tag{4.5b}$$

for all $t \geq t_0$, and each $0 \leq k < m$. By (2.7) and (4.4), and increasing t_0 if necessary, we would then obtain (4.5) for $k = m$ as well, completing the induction step.

The argument above established that, for each $m \geq 1$, we have

$$(t - t_0)^{2\alpha+m} \| D^m \boldsymbol{u}(\cdot, t) \|^2_{L^2(\mathbb{R}^4)} \leq \frac{1}{\delta \cdot [(2-\epsilon)\nu]^m} \left\{ \prod_{j=0}^{m} (2\alpha + j + \delta) \right\} (\lambda_0(\alpha) + \epsilon)^2$$

for all t sufficiently large. Since $\delta > 0$, $0 < \epsilon < 2$ are arbitrary, this gives the result. □

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