NSF Program MCTP
Minicourse on Modeling
Summer 2009

Jens Lorenz
June 25, 2009

Department of Mathematics and Statistics,
UNM, Albuquerque, NM 87131

Contents

1 Introduction 3
2 A Simple Evolution on Two Time–Scales 4
3 Matlab Programs and Output 6
4 Matlab Programs and Output, Continued 22
5 General Remarks on Modeling 28
   5.1 Weather and Climate Predictions 28
   5.2 Small Scale and Large Scale Models for Fluids 29
   5.3 Rough Estimates on Numerical Weather Prediction 30
6 Further Remarks 31
   6.1 Differential Equations, Randomness, and Elementary Probability 31
   6.2 A Fibonacci Cartoon 31
   6.3 Path Integrals and Heat Engines 32
   6.4 Discussion 32
7 Deterministic ODEs and Difference Equations 33
   7.1 Exponential Growth 33
   7.2 Logistic Growth 33
   7.3 Discrete–Time Growth Models 34
   7.4 Stability of Fixed Points of ODEs; Hysteresis 37
   7.5 Projects 42
# The Gambler’s Ruin Problem

8.1 The Game .......................... 44
8.2 Some Questions ........................ 44
8.3 The Transition Matrix .................. 45
8.4 Interpretation of Powers of $P$ ................. 46
8.5 Probability Density Vectors .............. 46
8.6 Realizations .......................... 48
8.7 Probability of Ruin and Probability of Winning 49
8.8 Project: Expected Time for Game .......... 52

# Stochastic Model of a Simple Birth Process

9.1 The Forward Kolmogorov Equations .......... 53
9.2 Solution of the Forward Kolmogorov Equations 54
9.3 The Sum of the $p_j(t)$ .................. 56
9.4 The Expected Value of $X_t$ ................ 58
9.5 The Variance of $X_t$ ...................... 59
9.6 Interevent Times and Stochastic Realizations 62

# Introduction to Kinetic Theory

10.1 Daniel Bernoulli: Pressure, Volume, and Particle Velocity .... 68
10.2 Maxwell’s Velocity Distribution ............. 70
10.3 Optimal Heat Engines: Carnot’s Cycle .......... 75

# Projects On Deterministic Differential and Difference Equations

11.1 Fibonacci Sequence .................... 80
11.2 Radiocarbon Dating ...................... 80
11.3 Logistic Growth ........................ 81
11.4 The Delayed Logistic Map ................ 81
11.5 Hysteresis ............................ 81

# Projects on Gambler’s Ruin

12.1 Numerical Simulations .................... 83
12.2 Derivation of Formulas ................... 83
12.3 Expected Time for Game .................. 84

# Projects on the Simple Birth Process

13.1 Carbon Dating ........................ 86
13.2 Gambler’s Ruin ........................ 86
13.3 Gambler’s Winning ....................... 87
13.4 Expected Time for Game .................. 88
1 Introduction

The aim of the course is to give the students a basic understanding of:

a) initial value problems for ODEs

\[ \frac{du}{dt} = f(u, \lambda), \quad u(0) = u_0, \]

where \( \lambda \) is a parameter. We discuss a simple example where the solution varies on two time scales and an example illustrating the hysteresis phenomenon.

b) difference equations

\[ u_{n+1} = \Phi(u_n). \]

We use the logistic differential equation and a corresponding delayed logistic map to show that time discretization can change the qualitative properties of the evolution.

c) random evolution

We use the gambler’s ruin problem and a simple stochastic birth process to introduce concepts from probability theory. The concepts are applied to analyze the evolution and to run numerical simulations.

For all three classes of problems, the students are asked to write Matlab codes.

As a general introduction, we mention the weather/climate system as well as micro/macro models for fluids. For the weather system we know detailed equations, but the weather system is too sensitive to be predictable over longer periods of time. It seems that we do not have a rational process to arrive at important averaged quantities, and simple relations between them, which would allow us to predict the climate reliably.

The situation is easier for micro/macro models of fluids since in this case we know good equations at both scales. Still, it is no easy task to connect the two levels rationally. To get a first understanding of this interesting subject, which involves probabilistic arguments, we give a short introduction to the kinetic theory of gases and to thermodynamics.

It seems that one faces a similar situation in economics, for micro and macro economics. The models work on different scales, and it is a challenge to connect them rationally with statistical arguments.
2 A Simple Evolution on Two Time–Scales

We denote the unknown function by \( u = u(t) \) and think of the independent variable \( t \) as time.

Recall that the initial–value problem

\[
    u' = iu, \quad u(0) = 1 ,
\]

is solved by

\[
    u(t) = e^{it} = \cos t + i \sin t .
\]

Next consider the initial–value problem

\[
    u' = \frac{i}{\varepsilon} u, \quad u(0) = 1, \quad 0 < \varepsilon << 1 ,
\]

with solution

\[
    u(t) = e^{it/\varepsilon} = \cos(t/\varepsilon) + i \sin(t/\varepsilon) . \tag{2.1}
\]

If we consider the solution for \( 0 \leq t \leq 2\pi \), say, then the argument

\[
    \tau = t/\varepsilon
\]

changes from 0 to \( 2\pi/\varepsilon \). If \( \varepsilon = 10^{-4} \), say, then the solution \( u(t) \) goes through \( 10^4 \) oscillations as \( t \) goes from 0 to \( 2\pi \).

Now consider the forced equation

\[
    u' = \frac{i}{\varepsilon} u - \frac{i}{\varepsilon} \sin t, \quad u(0) = 1 . \tag{2.2}
\]

Note that the forcing term

\[
    f(t) = -\frac{i}{\varepsilon} \sin t
\]

goes through one oscillation for \( 0 \leq t \leq 2\pi \). One says that the forcing varies on the slow time scale whereas the solution of the homogeneous problem,

\[
    u_{\text{hom}}(t) = e^{it/\varepsilon} = \cos(t/\varepsilon) + i \sin(t/\varepsilon)
\]

varies on the fast time scale. We expect the solution of (2.2) to vary on both time scales.

One can solve (2.2) in closed form:

\[
    u(t) = e^{it/\varepsilon} - \frac{i}{\varepsilon} \int_0^t e^{i(t-s)/\varepsilon} \sin s \, ds . \tag{2.3}
\]

However, it is interesting to go through a process which yields approximate solutions.

We rewrite (2.2) as

\[
    \varepsilon u' = iu - i \sin t, \quad u(0) = 1 . \tag{2.4}
\]
If we put $\varepsilon = 0$ and ignore the initial condition, we obtain the reduced equation

$$0 = iu - i \sin t$$

with reduced solution

$$u_0(t) = \sin t .$$

In (2.4) let

$$u(t) = v(t) + \sin t$$

with a new unknown function $v(t)$. Obtain

$$\varepsilon(v' + \cos t) = i(v + \sin t) - i \sin t, \quad v(0) = 1 ,$$

or

$$\varepsilon v' = iv - \varepsilon \cos t, \quad v(0) = 1 . \quad (2.5)$$

If we compare with (2.4), we see that the forcing is reduced by order $\varepsilon$.

If we ignore the forcing term $-\varepsilon \cos t$ in (2.5), we obtain the solution

$$v_0(t) = \cos(t/\varepsilon) + i \sin(t/\varepsilon) .$$

This suggests that, to leading order, the solution of the given problem (2.2) is

$$u^{(0)}(t) = \sin t + \cos(t/\varepsilon) + i \sin(t/\varepsilon) .$$

In fact, one can prove that

$$u(t) = u^{(0)}(t) + O(\varepsilon)$$

where $u(t)$ denotes the exact solution of (2.2).

**Projects:**

1. **Analysis:** Use (2.3) and integration by parts to show that

$$u(t) = e^{it/\varepsilon} \left( 1 + \frac{\varepsilon}{i} \right) + \sin t - \frac{\varepsilon}{i} \cos t + O(\varepsilon^2) .$$

2. **Matlab:** Take $\varepsilon = 0.02$, for example. Plot the real part of the leading order approximation,

$$\Re u^{(0)}(t) = \sin(t) + \cos(t/\varepsilon), \quad 0 \leq t \leq 2\pi ,$$

together with the slowly varying function $\sin t$. 
3 Matlab Programs and Output

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%prog1.m
% plotting the sine function
\texttt{t=linspace(0,2*pi,101);}
\texttt{U=sin(t);}
\texttt{plot(t,U)}
\texttt{xlabel('time t')}
\texttt{ylabel('U(t)')}  
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

The output of prog1.m is shown in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Output of prog1.m}
\end{figure}

The Matlab command
\texttt{\textit{print -deps fig1}}
gen\text{}er\text{}a\text{}t\text{}e\text{}s\text{}e\text{} the\text{} file\text{} fig1.eps.\text{} With\text{}\texttt{\textit{evince fig1.eps}}\text{} you\text{} can\text{} see\text{} the\text{} figure\text{} again.\text{} The\text{} file\text{} \textit{fig1.eps} can\text{} be\text{} incorporated\text{} into\text{} a latex file.
%prog2.m
% plotting a function which varies on two time scales
t=linspace(0,2*pi,3001);
eps=0.02;
u=sin(t)+cos(t/eps);
plot(t,u)
xlabel('time t')
ylabel('u(t)')

The output of prog2.m is shown in Figure 2.
% prog3.m
% plotting two functions
% 
t=linspace(0,2*pi,3001);
U=sin(t);
u=sin(t)+cos(t/eps);
plot(t,U,t,u)
xlabel('time t')
ylabel('U(t) and u(t)')

The output of prog3.m is shown in Figure 3.
%progode1.m
% numerical approximation of the solution of the
% initial value problem
% u'=u(1-u), u(0)=0.01
% using ode45
f=@(t,u) u*(1-u);
[T,u]=ode45(f,[0 10],0.01);
plot(T,u)
xlabel('time t')
ylabel('u(t)')

The output of progode1.m is shown in Figure 4.

Figure 4: Output of progode1.m
% This program solves the ode u’=u(1-u)
% numerically using ode45 for 20 initial values u(0).

f=@(t,u) u*(1-u);
for i=1:20
    [T,U]=ode45(f,[0 10],[(i-1)*.1]);
    plot(T,U);
end;

xlabel('time t')
ylabel('u(t)')

The output of progode2.m is shown in Figure 5.

Figure 5: Output of progode2.m
% progtra1.m
% iteration of the map
% \( F_\lambda(x,y) = (y, \lambda y (1-x)) \)
% starting at \( (x,y) = (0.5,0.6) \).

% plot of \( v(i) \) versus \( i \)
% convergence to a fixed point

lam=1.9;
clear v;
v(1)=0.5;
v(2)=0.6;
for i=3:200
    v(i)=lam*v(i-1)*(1-v(i-2));
end;
plot(v)
xlabel('i')
ylabel('v(i)')

The output of progtra1.m is shown in Figure 6.
Figure 6: Output of progtra1.m

% progtra2.m
% iteration of the map
% \( F_\lambda(x,y)=(y,\lambda y (1-x)) \)
% starting at \((x,y)=(0.5,0.6)\).
% % plot of the points \((v(i),v(i+1))\)
% in the phase plane
% % convergence to a fixed point
lam=1.9;
clear v;
v(1)=0.5;
v(2)=0.6;
for i=3:200
v(i)=lam*v(i-1)*(1-v(i-2));
end;
plot(v(1:199),v(2:200))
The output of progtra2.m is shown in Figure 7.
%progra3.m
% iteration of the map
% \( F_\lambda(x,y)=(y, \lambda y (1-x)) \)
% starting at \((x,y)=(0.5,0.6)\).
%plot of \( v(i) \) versus \( i \)
% no convergence to a fixed point

\texttt{lam}=2.1;
clear \texttt{v};
\texttt{v}(1)=0.5;%
\texttt{v}(2)=0.6;
for \texttt{i}=3:200
\texttt{v}(\texttt{i})=\texttt{lam}*(\texttt{v}(\texttt{i}-1)*(1-\texttt{v}(\texttt{i}-2)));
end;
plot(\texttt{v})
xlabel(‘i’)
ylabel(‘v(i)’)

The output of progra3.m is shown in Figure 8.
Figure 8: Output of progra3.m

**********************************************************
%progtra4.m
% iteration of the map
% F_\lambda(x,y)=(y,\lambda y (1-x))
% starting at (x,y)=(0.5,0.6).
% plot of the points (v(i),v(i+1))
% in the phase plane
% convergence to an invariant curve

lam=2.1;
clear v;
v(1)=0.5;
v(2)=0.6;
for i=3:1000
  v(i)=lam*v(i-1)*(1-v(i-2));
end;
plot(v(200:999),v(201:1000))
axis square
xlabel('v(i-1)')
ylabel('v(i)')
title('A trajectory (with lines) for the delayed logistic map with $\lambda = 2.1$')

The output of progtra4.m is shown in Figure 9.

Figure 9: Output of progtra4.m
The output of progtra5.m is shown in Figure 10.
A trajectory for the delayed logistic map with $\lambda = 2.1$

Figure 10: Output of progtra5.m

**********************************************************
%progtra6.m
% iteration of the map
% $F_\lambda(x,y) = (y, \lambda y(1-x))$
% starting at $(x,y) = (0.5, 0.6)$.
% plot of the points $(v(i), v(i+1))$
% in the phase plane
% convergence to a complicated limit set
lam=2.27;
clear v;
v(1)=0.5;
v(2)=0.6;
for i=3:1000
  v(i)=lam*v(i-1)*(1-v(i-2));
end;
The output of progtra6.m is shown in Figure 11.

Figure 11: Output of progtra6.m
The first output of proghys1.m is shown in Figure 12. The second output of proghys1.m is shown in Figure 13.

**Example for Hysteresis Phenomenon.** In the example, the term

\[
\lambda(t) = \sin t
\]

is considered as a slowly varying parameter. The variation of \(\lambda(t)\) is slow compared with the fast evolution of \(u(t)\) since 25 \(>>\) 1. At time \(t = 0\) the state \(u(0) = -1\) is at the stable fixed point \(u_1 = -1\) (for the equation \(u' = 25u(1 + u)(1 - u)\)). As \(t\) increases, the stable fixed point \(u_1\) disappears and \(u(t)\) moves rapidly to the stable fixed point \(u_3 \sim 1\). At time \(t = \pi\) the parameter value is \(\lambda(\pi) = 0\), again, i.e., \(\lambda(t)\) has returned to its initial value \(\lambda(0) = 0\). However, the state is \(u(\pi) \sim 1\), i.e., the state \(u(t)\) has not returned to its initial value \(u(0) = -1\).

In this very simple example, two different states of \(u\) correspond to the same value of the parameter \(\lambda\). This is also clearly shown in Figure 13 which shows \(u(t)\) versus \(\lambda(t)\).
Figure 12: First output of proghys1.m

Question: If we drive the concentration of $\text{CO}_2$ in the atmosphere back to pre-industrial levels, is there a guarantee that we will return to the pre-industrial climate?
4 Matlab Programs and Output, Continued

%funcmatP.m
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The function returns the (N+1)*(N+1) transition
% matrix P for the Gambler’s Ruin Problem
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function P=funcmatP(p,N)
q=1-p;
NN=N+1;
P=zeros(NN,NN);
P(1,1)=1;
P(NN,NN)=1;
for j=2:N
P(j+1,j)=p;
P(j-1,j)=q;
end
The output of the Matlab command

\[ P = \text{funcmatP}(4,5) \]

is shown below.

\[
\begin{array}{cccccc}
1.0000 & 0.6000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.6000 & 0 & 0 & 0 \\
0 & 0.4000 & 0 & 0.6000 & 0 & 0 \\
0 & 0 & 0.4000 & 0 & 0.6000 & 0 \\
0 & 0 & 0 & 0.4000 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.4000 & 1.0000 \\
\end{array}
\]
%funcmatA.m
function A=funcmatA(p,N,k,nmax)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% p and N are as in the Gambler's Ruin Problem
% k is the capital at time t=0
% nmax is the number of games
% The output matrix A of size (N+1)*(nmax+1)
% contains in its column of index t+1 the probabilities
% of having capital i+1 at time t.
% Precisely: A(i+1,t+1) is the probability of having capital
% i at time t.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
A=zeros(N+1,N+1);
A(k+1,1)=1;
P=funcmatP(p,N);
for j=1:nmax
A(:,j+1)=P*A(:,j);
end

The output of the Matlab command

\[
A = \text{funcmatA}(0.4, 5, 3, 7)
\]

is shown below.

\[
A = \\
\begin{pmatrix}
0 & 0 & 0 & 0.2160 & 0.2160 & 0.3715 & 0.3715 & 0.4711 \\
0 & 0 & 0.3600 & 0 & 0.2592 & 0 & 0.1659 & 0 \\
0 & 0.6000 & 0 & 0.4320 & 0 & 0.2765 & 0 & 0.1742 \\
1.0000 & 0 & 0.4800 & 0 & 0.2880 & 0 & 0.1797 & 0 \\
0 & 0.4000 & 0 & 0.1920 & 0 & 0.1152 & 0 & 0.0719 \\
0 & 0 & 0.1600 & 0.1600 & 0.2368 & 0.2368 & 0.2829 & 0.2829 \\
\end{pmatrix}
\]

Numerical computations show that the columns converge to

\[
\begin{pmatrix}
0.6398 \\
0 \\
0 \\
0 \\
0.3602 \\
\end{pmatrix}
\]

Consider the Gambler's Ruin Problem with \(p = 0.4, N = 5,\) and an initial capital \(k = 3.\) Then the probability of losing the initial capital of \(k = 3\) equals \(0.6398,\) the probability of winning 2 and ending the game with a capital 5 equals \(0.3602.\)
% gamesim.m

% random evolution of capital in Gambler's Ruin Problem

function G=gamesim(p,N,k,nmax)
G(1)=k;
for j=2:(nmax+1)
g=G(j-1);
if (g==0)||(g==N)
G(j)=g;
else
x=rand;
if (x<p)
G(j)=g+1;
else
G(j)=g-1;
end
end
end

The output of the repeated Matlab command

\[ G = \text{gamesim}(0.4,5,3,10) \]

is shown below.

G=gamesim(.4,5,3,10)

G =

\[
\begin{array}{ccccccccccc}
3 & 2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

G=gamesim(.4,5,3,10)

G =

\[
\begin{array}{ccccccccccc}
3 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

G=gamesim(.4,5,3,10)

G =

\[
\begin{array}{ccccccccccc}
3 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 0 & 0 \\
\end{array}
\]

G=gamesim(.4,5,3,10)
\[ G = \begin{bmatrix} 3 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ G = \text{gamesim}(0.4, 5, 3, 10) \]

\[ G = \begin{bmatrix} 3 & 2 & 3 & 2 & 1 & 2 & 3 & 2 & 1 & 2 & 1 \end{bmatrix} \]

\[ G = \text{gamesim}(0.4, 5, 3, 10) \]

\[ G = \begin{bmatrix} 3 & 4 & 3 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \]
%birth.m
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% numerical simulation of the simple stochastic
% birth process
% plot of the population size
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% 
x0=3;
maxstep=20;
T=zeros(1,maxstep+1);
for j=1:maxstep
y=rand;
T(j+1)=T(j)+log(1/y)/(x0-1+j);
end
plot(T,[x0:maxstep+x0],'.'); hold on
newt=[T(1:maxstep -1);T(2:maxstep)];
for j=1:maxstep -1
plot(newt(:,j),[x0+j-1 x0+j-1],'-'); hold on
end
y=[0:0.1:T(end)];
z=x0*exp(y);
plot(y,z,'-');
title('Simple birth process');
xlabel('time');
ylabel('size of population');

**********************************************************
The output of birth.m is shown in Figure 14.
5 General Remarks on Modeling

We first make some general remarks on two difficult problems, the weather/climate system and micro/macro models for fluids. In both cases, deterministic and probabilistic modeling plays a role. This motivates the subject of the course.

5.1 Weather and Climate Predictions

A main difference between weather prediction and climate prediction is the time scale: The time scale for the weather is about a day, and we can predict the weather for three or four days reasonably well. On the other hand, we would like to understand changes in climate occurring in 30 years, about $10^4$ days. There is also a significant difference in space scales: for the weather it may be 100 km, for the climate 1,000 km. We know the equations for weather prediction reasonably well. These are well-established fluid and thermodynamic equations of classical physics. On the other hand, no such fundamental equations are known for modeling climate changes. What are the relevant variables and what are the equations connecting them? (As we know, variables of interest are, for example, the average temperature and the $CO_2$ concentration in the atmosphere.) It is
well possible that no such equations will ever be found because they may simply not exist.

The partial differential equations used for the weather system are deterministic, but sensitive to small perturbations. The sensitivity limits the time scale for which one can predict the weather.

The weather and the climate are similar phenomena, but on different scales. It is a difficult problem to understand how the short time-scale weather system is related to the long time-scale climate system. Mathematics may help to identify relevant variables and to establish relationships between them. Using statistical arguments, one can hope to get rid of unimportant variations of the weather system, but the details of this process are far from clear and are the subject of ongoing research.

In the next section, we briefly discuss a similar, but somewhat easier situation: The relation between micro and macro models for fluids.

5.2 Small Scale and Large Scale Models for Fluids

A much studied and somewhat similar situation as for the weather/climate system occurs for fluid flows and thermodynamic relations. The situation is easier since we know good equations for the micro and the macro models. Macro models have been studied for centuries: thermodynamic relations \(^1\) and fluid equations like Euler’s equation (1752) and the Navier–Stokes equations (1822).

Roughly, micro models describe the bouncing around of a large number \((\sim 10^{23})\) of molecules. Think of a large number of bouncing billiard balls. These so-called kinetic models date back to Maxwell and Boltzmann. (Around 1860 Maxwell determined the velocity distribution of bouncing particles and thus introduced probability into physics. Boltzmann (1872) extended Maxwell’s work and addressed the interesting question how irreversibility arises in the macro world, given that kinetic models are time reversible.)

Boltzmann’s aim was to derive thermodynamics from mechanics, using statistical arguments. First serious attempts to derive the fluid equations from Boltzmann’s equation date back to Hilbert.

In principle, one should be able to derive all macroscopic fluid properties (like the exact temperature dependence of the viscosity coefficient) from molecular properties. It turns out that this is far from simple, and research in this area is still ongoing.

For both, the weather/climate system and the micro/macro fluid models, deterministic and probabilistic arguments are important. In this course, we want to study some simple deterministic ODEs and, to get acquainted with probabilistic arguments, we also want to study some simple models for random evolutions.

\(^1\) An example of a thermodynamic relation is the ideal gas law, \(pV = MRT\). Here \(p\) is the pressure, \(V\) the volume, \(M\) the mass, and \(T\) the absolute temperature of the ideal gas. The constant \(R\) depends on the gas. The ideal gas law is also called Boyle–Mariotte law and was found experimentally around 1660.
5.3 Rough Estimates on Numerical Weather Prediction

Recall the terminology

\[
\begin{align*}
Mega & : 10^6 \\
Giga & : 10^9 \\
Tera & : 10^{12} \\
Penta & : 10^{15}
\end{align*}
\]

A new laptop can perform about 20 Giga flops, i.e., \(20 \times 10^9\) floating point operations per second. Think of a floating point operation as the multiplication of two 16 digit numbers. The fastest machines now run at about one Penta flop.

In the following, we ignore storage problems and only count flops. The surface area of the earth is

\[A = 4\pi R^2 \sim 4 \times 3 \times 6.37^2 \times 10^6 \text{ km}^2 \sim 5 \times 10^8 \text{ km}^2.\]

If we put a mesh on the earth with 1 km meshsize, we need about

\[M_1 = 5 \times 10^8\]

meshpoints. (We ignore the trouble that the surface of the earth is not a square.) Discretizing height with 20 levels leads to

\[M_2 = 10^{10}\]

meshpoints. This gives us a rather good mesh discretizing the atmosphere.

Suppose at each meshpoint we store 10 scalar data, like 3 wind velocities, pressure, temperature, moisture, ... This leads to

\[D = 10^{11}\]

data at each given time. To perform a time step, we might need about 1000 operations per data point, leading to

\[N = 10^{14}\]

operations per time step.

If we have a penta flop machine we can perform 10 time steps per second. These very crude considerations show that numerical weather prediction is a reasonable numerical task. The equations governing the physics are well understood (heating by the sun, thermodynamics, fluid mechanics, ...)

In fact, weather prediction over a short time period of 2 to 3 days works rather well. However, the weather system is too sensitive to perturbations to allow much longer predictions. The main source of uncertainty seems to be the cloud cover. The conditions for clouds to form are quite delicate, and getting this wrong predicts the wrong weather.
6 Further Remarks

6.1 Differential Equations, Randomness, and Elementary Probability

A simple growth model dates back to Fibonacci. In the year around 1200, he considered the sequence $F_n$ defined by

$$F_1 = F_2 = 1, \quad F_{n+1} = F_n + F_{n-1}, \quad n = 2, 3, \ldots$$

The beginning of the sequence:

$$1, 1, 2, 3, 5, 8, \ldots$$

Using continuous time, simple growth models lead to exponential or logistic growth and are governed by the differential equations

$$\frac{du}{dt} = bu$$

and

$$\frac{du}{dt} = ru \left(1 - \frac{u}{K}\right).$$

Is there a relation to the Fibonacci sequence? While the solutions of the above differential equations are rather simple, corresponding discrete–time models are often much more complicated. We can see this by computer studies for the delayed logistic map,

$$u_{n+1} = \lambda u_n (1 - u_{n-1}), \quad n = 2, 3, \ldots$$

In this course we also want to study growth processes probabilistically. Matlab’s random number generator allows us to compute sample paths for stochastic models and to compare with deterministic evolution.

6.2 A Fibonacci Cartoon

We only count female rabbits; here $r$ is a small rabbit and $R$ is a grown–up. In each year all $r$ grow to become $R$ and each $R$ gives birth to one $r$. No rabbit ever dies. We start with one little rabbit, $r$.

<table>
<thead>
<tr>
<th>rabbits</th>
<th>number of rabbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>1</td>
</tr>
<tr>
<td>$R$</td>
<td>1</td>
</tr>
<tr>
<td>$Rr$</td>
<td>2</td>
</tr>
<tr>
<td>$RRr$</td>
<td>3</td>
</tr>
<tr>
<td>$RRRr$</td>
<td>5</td>
</tr>
<tr>
<td>$RRRRRr$</td>
<td>8</td>
</tr>
</tbody>
</table>

The number of rabbits is given by the Fibonacci sequence.
6.3 Path Integrals and Heat Engines

In the early eighteen hundreds, Sadi Carnot (1796 – 1832), a French engineer, asked the fundamental question: Is there a maximum work obtainable from heat flowing from one temperature to another? Carnot then abstracted the idea of a heat engine, which converts heat into mechanical work, and — amazingly — arrived at an idealized machine which has the best possible efficiency. Emile Clapeyron (1799–1864) formalized Carnot’s idea, introducing the Carnot cycle. As we will see, the mathematics involves path integrals in a \((p, V)\) space with pressure \(p\) and volume \(V\). Carnot’s ideas were later formalized as the 2nd law of thermodynamics, and led Kelvin to introduce absolute temperature.

6.4 Discussion

The subjects differential equations, randomness, probability, and heat engines are connected on a deeper level. Heat is random motion and motion is (often) described by differential equations. More precisely, the motion that we call heat involves so many particles \((\sim 10^{23})\) that, for practical reasons, we cannot avoid probabilistic descriptions and the notion of randomness. Though the beginnings of the subject date back more than a century, with work of Maxwell and Boltzmann, research in this area is still ongoing and many questions relating differential equations and randomness remain open. An acute problem is climate modeling. Here we believe to know the governing differential equations but, even using the fastest computers, there is no hope in solving them accurately numerically over a time period of 30 years, say. On the other hand, through decades of observations, statistical information for many quantities is available. How can one use this knowledge consistently in long term numerical simulations?

**Question:** What does the Fibonacci sequence \(F_n\) have to do with exponential growth? Can you find an approximation

\[ F_{n+1} \sim ae^{bn} \]

which is valid for large \(n\)?
7 Deterministic ODEs and Difference Equations

7.1 Exponential Growth

The simplest growth model is given by the ODE initial–value problem

\[ u'(t) = bu(t), \quad u(0) = a. \]

Here \( b \) is the birth rate per individual and \( a \) is the population size at time \( t = 0 \). The solution is \( u(t) = ae^{bt} \).

A standard derivation of the equation is as follows: If \( u(t) \) is the population size at time \( t \) and the birth rate per individual in the time interval \([t, t + \Delta t]\) is, approximately,

\[ b\Delta t \]

then

\[ u(t + \Delta t) \sim u(t) + bu(t)\Delta t. \]

Subtract \( u(t) \) from both sides, divide by \( \Delta t \), and let \( \Delta t \to 0 \) to obtain \( u' = bu \).

For this model the time axis is

\( T = [0, \infty) \).

The state space is \( A = \mathbb{R} \). We have a deterministic law of evolution.

One obtains the same equation with \( b = b_1 - d_1 \) where \( b_1 \) is the birth rate and \( d_1 \) is the death rate.

7.2 Logistic Growth

Exponential growth cannot continue for long. A somewhat more realistic model may be given by

\[ u'(t) = r\left(1 - \frac{u(t)}{K}\right)u(t). \]

Thus the birth rate \( b \) is replaced by

\[ b \sim r\left(1 - \frac{u(t)}{K}\right). \]

Here \( K > 0 \) is the carrying capacity of the system. If \( u(t) \ll K \) then \( u' \sim ru \), and we have approximately exponential growth. As \( u(t) \) approaches \( K \), the growth rate decays to zero.

The logistic equation can be solved using separation of variables. One obtains the general solution (depending on a free parameter \( \gamma \)):

\[ u(t) = \frac{K}{1 + \gamma e^{-rt}}, \]

and the initial condition \( u(0) = a \) yields
Thus

\[ u(t) = \frac{Ka}{a + (K - a)e^{-rt}}. \]

The following script solves the ODE

\[ u'(t) = r\left(1 - \frac{u(t)}{K}\right)u(t). \]

numerically for different initial conditions:

```matlab
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This script solves the ODE u'=ru(1-u/K).
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% definition of the RHS of the ODE y'=f(t,y)
% change r, K in here if needed
f=@(t,y) y*(1-y);

% loop for different initial conditions
for i=1:10
    % solving ODE numerically with ODE45
    % here f is the RHS of y'=f, it has to be function header
    % [0,10] is the interval where we want solution
    % [(i-1)*.2] is the initial value, i.e., y(0)
    [T,Y] = ode45(f,[0 10],[0]);
    % plotting all the curves in one plot
    % 'hold on' does keep the same plot for all iterations
    plot(T,Y,'-b'); hold on;
end;

The corresponding solutions are given in Figure 15.

7.3 Discrete–Time Growth Models

If one replaces the logistic evolution by the discrete time model

\[ \frac{1}{\Delta t}(u_{n+1} - u_n) = r\left(1 - \frac{u_{n-1}}{K}\right)u_n \]

one obtains evolution by the so-called delayed logistic map. The corresponding dynamics can be visualized by a point moving in the plane. The dynamics can already be quite complicated.

Let us show that the dynamics depends essentially only on one parameter. We write
Figure 15: Solutions $u(t)$ for initial conditions $u(0) = 0, 0.2, 0.4, ..., 1.8$

\[
\begin{align*}
    u_{n+1} &= u_n + r\Delta t(1 - u_{n-1}/K)u_n \\
    &= u_n \left(1 + r\Delta t - r\Delta tu_{n-1}/K\right) \\
    &= (1 + r\Delta t)u_n(1 - \beta u_{n-1})
\end{align*}
\]

with

\[
\beta = \frac{r\Delta t}{(1 + r\Delta t)K}.
\]

Set

\[
\lambda = 1 + r\Delta t.
\]

Then the discrete–time evolution is determined by

\[
u_{n+1} = \lambda u_n(1 - \beta u_{n-1}).
\]

With a constant $q$ (independent of $n$), to be determined, let

\[
u_n = q v_n
\]

for all $n$. Obtain
\[ qv_{n+1} = \lambdaqv_n(1 - \betaqv_{n-1}) . \]

Choosing \( q = 1/\beta \) we arrive at

\[ v_{n+1} = \lambda v_n(1 - v_{n-1}) . \quad (7.1) \]

The evolution is called evolution by the delayed logistic map with parameter \( \lambda \).

If we set

\[ F_\lambda(x, y) = (y, \lambda y(1 - x)) \]

then the evolution (7.1) becomes

\[ (v_{n-1}, v_n) \rightarrow F_\lambda(v_{n-1}, v_n) = (v_n, v_{n+1}) . \]

In other words, we iterate the map \( F_\lambda \).

Let us determine the fixed points of \( F_\lambda \): The equation

\[ F_\lambda(x, y) = (x, y) \]

leads to

\[ x = y \quad \text{and} \quad y = \lambda y(1 - x) . \]

We assume \( \lambda > 0 \). There is the trivial fixed point

\[ Q = (0, 0) \]

and the nontrivial fixed point

\[ P_\lambda = \left( 1 - \frac{1}{\lambda}, 1 - \frac{1}{\lambda} \right) . \]

We assume now that \( \lambda > 1 \).

To discuss the stability of \( P_\lambda \) we first compute the Jacobian of \( F_\lambda \):

\[ F_\lambda'(x, y) = \begin{pmatrix} 0 & 1 \\ -\lambda y & \lambda(1 - x) \end{pmatrix} . \]

Evaluation at \( (x, y) = P_\lambda \) yields

\[ F_\lambda'(P_\lambda) = \begin{pmatrix} 0 & 1 \\ 1 - \lambda & 1 \end{pmatrix} . \]

The eigenvalues \( \mu_{1,2} \) of this matrix are (for \( \lambda > \frac{5}{4} \)):

\[ \mu_{1,2} = \frac{1}{2} \pm i\sqrt{\lambda - \frac{5}{4}} . \]

We see that

\[ |\mu_{1,2}|^2 = \lambda - 1 . \]
It then follows that the eigenvalues $\mu_{1,2}$ lie inside the unit circle for $1 < \lambda < 2$, but outside the unit circle for $\lambda > 2$. As $\lambda$ crosses the value $\lambda_0 = 2$, the eigenvalues $\mu_{1,2}$ leave the unit circle, and the fixed point $P_\lambda$ becomes unstable.

One can show (and see numerically) that the map $F_\lambda(x,y)$ has a small invariant circle near the point $P_2 = \left( \frac{1}{2}, \frac{1}{2} \right)$ for $2 < \lambda < 2 + \varepsilon$. As $\lambda$ increases, this invariant circle disappears in a complicated fashion and the dynamics may become chaotic.

The bifurcation occurring at $\lambda_0 = 2$, leading from a stable fixed point to an invariant circle, is called a Neimark–Sacker bifurcation. The following script plots trajectories of the map $F_\lambda$ for different initial conditions:

```matlab
% Script plots trajectories for the map
% v_n=lam*v_{n-1} (1-v_{n-2}).
%
% define lambda
lam=1;
% define size of array v
v=zeros(1,100);

% loop for different initial conditions
for i=1:10
    v(1)=(i-1)*.05;
    v(2)=(i-1)*.05;
end;

% obtaining v
for i=3:100
    v(i)=lam*v(i-1)*(1-v(i-2));
end;
% plotting all trajectories on the same plot
plot(v(2:100),v(1:99),'-');hold on;
end
title('Trajectories for \lambda=1');
xlabel('j');
ylabel('v_j');
```

The trajectories for $\lambda = 1$ are given in Figure 16. If we run the above script with $\lambda = 2$, then we observe invariant circles, like in Figure 17.

### 7.4 Stability of Fixed Points of ODEs; Hysteresis

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote a smooth map and denote the solution of

$$u' = f(u), \quad u(0) = u_0,$$
by $u(t, u_0)$. A point $u^* \in \mathbb{R}^n$ is a fixed point if $f(u^*) = 0$. Here is the technical definition of asymptotic stability of a fixed point:

**Definition:** A fixed point for the equation $u' = f(u)$ is called asymptotically stable if

a) For all $\varepsilon > 0$ there is $\delta > 0$ so that $|u^* - u_0| < \delta$ implies $|u^* - u(t, u_0)| < \varepsilon$ for $t \geq 0$;

b) there is $\delta > 0$ so that

$$\lim_{t \to \infty} |u^* - u(t, u_0)| = 0 \quad \text{if} \quad |u^* - u_0| < \delta .$$

In the scalar case, $n = 1$, it is not difficult to prove that a fixed point of the equation $u' = f(u)$ is asymptotically stable if

$$f'(u^*) < 0$$

and is unstable if $f'(u^*) > 0$.

**Example:** Consider the equation

$$u' = u(1 + u)(1 - u)$$

with two stable and one unstable fixed points.

**Example:** Consider the equation
Figure 17: Trajectories forming invariant circles

\[ u' = u(1 + u)(1 - u) + \lambda \]

where \( \lambda \) is a parameter. The number of fixed points depends on \( \lambda \). We can use this example to discuss the hysteresis phenomenon.

The following Matlab script describes the behavior of fixed points for variable \( \lambda \):

```matlab
function main;
    t=[0 15];
    u0=1;
    [t u]=ode45(@f,t,u0);
    figure(1);
    plot(t,u);
    figure(2);
    plot(sin(.6*t),u)
```

---

39
function dy=f(t,y)
dy=25*(y*(1+y)*(1-y)+sin(.6*t));

See Figures 18 and 19.

Figure 18: Hysteresis phenomenon: $u(t)$ versus $t$

One can also illustrate the hysteresis phenomenon using global variables in Matlab:

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This script plots the approximate fixed points
% for explicitly given lambda.
% 
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function main;
global lam i;
lam=[2:-.1:-2];
res=[];
for i=1:length(lam)
    [t u]=ode45(@(f, [0 10], [-1]);
    res=[res; u(end)];
Figure 19: Hysteresis phenomenon: $u(t)$ versus $\lambda$

end

plot(lam, res,'-*b');
xlabel('\lambda'); ylabel('u(t) for large times');
hold on

lam=[-2:.1:2];

res=[];
for i=1:length(lam)
    [t u]=ode45(@f, [0 10], [1]);
    res=[res; u(end)];
end

plot(lam, res,'-or')

function dy=f(t,y)
global lam i;
dy=(y.*(1-y).*(y+1)+lam(i)*50);

See Figure 20.
7.5 Projects

1. Solve the logistic differential equation numerically.

2. Study the delayed logistic map numerically. Observe the invariant circle and its disappearance.

3. Radiocarbon dating: Let’s first explain the physical basis for radiocarbon dating, which was discovered by W. Libley in 1949. Carbon atoms come in two forms, called isotopes: \(^{12}C\) and \(^{14}C\). The \(^{12}C\) atom has 6 protons and 6 neutrons in its nucleus; the \(^{14}C\) atom has 6 protons and 8 neutrons; there is no chemical difference between the atoms, but \(^{14}C\) is radioactive with a half–life of \(T = 5,568\) years. It is important to note that \(^{14}C\) is formed in the upper atmosphere through cosmic radiation, and the quotient

\[
q = \frac{\text{amount of } ^{14}C}{\text{amount of } ^{12}C}
\]

is constant in time in the atmosphere.

As long as a plant lives, its carbon obeys the quotient \(q\). When it dies, \(^{14}C\) decays through radioactive decay and is not reformed since the cosmic radiation does not reach the plant.

With a measuring instrument you measure

\[
M_1 = 6.68
\]
disintegrations per gram of carbon per minute from a living plant. A piece of old wood excavated at Mount Ararat gave a count of

\[ M_2 = 5.96 \]

disintegrations per gram of carbon per minute. How old is the wood? (See [Hale, Kocak, p. 21].)
8 The Gambler’s Ruin Problem

This is a very simple example for an evolution involving randomness.

8.1 The Game

We introduce randomness by throwing a coin. Let

\[ S = \{H, T\} \]

where \( H \) stands for heads and \( T \) stands for tails. Fix \( 0 < p < 1 \) and let \( q = 1 - p \). Assume that \( p \) is the probability for \( H \) to come up and \( q \) is the probability for \( T \) to come up.

(If \( p \neq \frac{1}{2} \) then the coin is not fair. Most games are not fair, indeed.)

We fix some integer \( N \in \mathbb{N} \) and let

\[ A = \{0, 1, \ldots, N\} \]

denote the state space. For example, we may choose \( N = 1,000 \). With \( X_t \in A \) we denote the capital of the gambler at time \( t \). The time axis is discrete,

\[ T = \{0, 1, 2, \ldots\} = \mathbb{N}_0 . \]

The capital \( X_t \in A \) is a random variable. We now describe how it evolves.

If at time \( t \) the capital is \( X_t \in \{1, 2, \ldots, N - 1\} \) then the gambler bets one dollar. If \( H \) shows up, then the gambler wins a dollar; if \( T \) shows up, the gambler loses a dollar. If \( X_t = 0 \) the gambler is already ruined and if \( X_t = N \) the gambler has already won. We then assume no further change of \( X_t \). Thus, we can describe the random evolution of \( X_t \) as follows:

If \( X_t = 0 \) then \( X_{t+1} = 0 \); if \( X_t = N \) then \( X_{t+1} = N \). If \( X_t = k \in \{1, 2, \ldots, N - 1\} \) then

\[ X_{t+1} = \begin{cases} 
    k + 1 & \text{with probability } p \\
    k - 1 & \text{with probability } q 
\end{cases} . \]

The gambler starts out with a capital

\[ X_0 = a \in A . \]

8.2 Some Questions

One can ask many questions. For example, let

\[ p = 0.49, \quad q = 0.51 , \]

and let
What is the probability for the gambler to reach $X_t = N = 1,000$ and to go home winning 200 Dollars? What is the probability to first get ruined, i.e., to reach $X_t = 0$ and to go home losing 800 Dollars? How long, on average, does the game last before $X_t = N$ or $X_t = 0$ is reached?

In order to simulate this numerically, we do not have to throw a coin, but can use Matlab’s random number generator. The call $Y = \text{rand}$ gives us a uniformly distributed random number $Y$ with $0 < Y < 1$. Then the event $0 < Y < p$ occurs with probability $p$, and the event $p \leq Y < 1$ occurs with probability $q = 1 - p$.

### 8.3 The Transition Matrix

The random evolution described above can be encoded in a so-called transition matrix

$$ P = (p_{ij})_{0 \leq i,j \leq N} \in \mathbb{R}^{(N+1) \times (N+1)} . $$

For $i, j \in A$ let

$$ p_{ij} = \text{prob}(X_{t+1} = i \mid X_t = j) . $$

Thus, $p_{ij}$ is the probability that $X_{t+1} = i$ under the assumption that $X_t = j$. (Note that in our game the probabilities $p_{ij}$ do not depend on $t$.)

For example, let $j = 1$, i.e., assume that $X_t = 1$. Then $X_{t+1} = 0$ with probability $q$ and $X_{t+1} = 2$ with probability $p$. The probability for

$$ X_{t+1} = i, \quad i \notin \{0, 2\} , $$

is zero. One then obtains that

$$ p_{01} = q, \quad p_{21} = p, \quad p_{i1} = 0 \quad \text{for} \quad i \notin \{0, 2\} . $$

The transition matrix is

$$ P = \begin{pmatrix}
1 & q & 0 & \cdots & 0 \\
0 & 0 & q & 0 \\
0 & p & 0 & q & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 0 & p & 1
\end{pmatrix} . $$

**Definition:** A real square matrix $P$ is called stochastic if $p_{ij} \geq 0$ for all $i, j$ and if all column sums equal one.

Clearly, the above transition matrix $P$ is stochastic.
8.4 Interpretation of Powers of $P$

Skip the subsection since the arguments in the next subsection are easier.

The matrix $P$ encodes the transition probabilities of the random variable $X_t$ in one time step. What happens after two or more time steps? How is this connected to matrix calculus?

We know that $$(P^2)_{ij} = \sum_k p_{ik} p_{kj}$$
and want to argue that $$(P^2)_{ij} = \text{prob}(X_{t+2} = i \mid X_t = j).$$

It is plausible that

$$\text{prob}(X_{t+2} = i \mid X_t = j) = \sum_k \text{prob}(X_{t+2} = i \text{ and } X_{t+1} = k \mid X_t = j).$$

Here

$$\text{prob}(X_{t+2} = i \text{ and } X_{t+1} = k \mid X_t = j) = \text{prob}(X_{t+1} = k \mid X_t = j) \cdot \text{prob}(X_{t+2} = i \mid X_{t+1} = k) = p_{ik} p_{kj}$$

The first equation may be the most doubtful one to accept and we have no way to prove its correctness since we deal with probabilities purely on an intuitive level. To make it plausible, let us assume that $X_t = j$ and consider the two events

$$E_1 = (X_{t+1} = k), \quad E_2 = (X_{t+2} = i).$$

Then the probability for $(E_1$ and $E_2$) to occur equals the (conditional) probability of the occurrence of $E_2$ under the assumption $E_1$ times the probability of $E_1$. In an abstract approach to probability, one makes this the definition of conditional probability.

One can generalize and obtain:

$$(P^n)_{ij} = \text{prob}(X_{t+n} = i \mid X_t = j).$$

8.5 Probability Density Vectors

Assume your capital at time $t = n$ is $k \in A = \{0, 1, \ldots, N\}$, thus

$$X_n = k.$$
Then we have

\[
\begin{align*}
\text{prob}(X_{n+1} = k + 1) &= p \quad \text{if} \quad 1 \leq k \leq N - 1 \\
\text{prob}(X_{n+1} = k - 1) &= q \quad \text{if} \quad 1 \leq k \leq N - 1 \\
\text{prob}(X_{n+1} = i) &= 0 \quad \text{if} \quad 1 \leq k \leq N - 1 \quad \text{and} \quad i \notin \{k - 1, k + 1\} \\
\text{prob}(X_{n+1} = 0) &= 1 \quad \text{if} \quad k = 0 \\
\text{prob}(X_{n+1} = i) &= 0 \quad \text{if} \quad k = 0 \quad \text{and} \quad i \neq 0 \\
\text{prob}(X_{n+1} = N) &= 1 \quad \text{if} \quad k = N \\
\text{prob}(X_{n+1} = i) &= 0 \quad \text{if} \quad k = N \quad \text{and} \quad i \neq N
\end{align*}
\]

We want to express these relations using the transition matrix \( P \). For each time \( n = 0, 1, \ldots \) define the probability density vector

\[
\pi^{(n)} \in \mathbb{R}^{N+1}
\]

by

\[
\pi_j^{(n)} = \text{prob}(X_n = j), \quad 0 \leq j \leq N.
\]

Let \( \mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^N \) denote the standard basis of \( \mathbb{R}^{N+1} \).

With these notations, the above relations can be expressed as follows: If \( X_n = k \), then

\[
\pi^{(n)} = \mathbf{e}^k \quad \text{and} \quad \pi^{(n+1)} = P\mathbf{e}^k.
\]

More generally, we claim:

**Lemma 8.1** If the probability density vector \( \pi^{(n)} \in \mathbb{R}^{N+1} \) is defined by

\[
\pi_j^{(n)} = \text{prob}(X_n = j), \quad 0 \leq j \leq N,
\]

and if the transition matrix \( P \in \mathbb{R}^{(N+1) \times (N+1)} \) is defined as above, then

\[
\pi^{(n+1)} = P\pi^{(n)}. \tag{8.1}
\]

To show this, consider first \( 2 \leq k \leq N - 2 \). We have

\[
\begin{align*}
\pi_k^{(n+1)} &= \text{prob}(X_{n+1} = k + 1) \\
&= p\text{prob}(X_n = k - 1) + q\text{prob}(X_n = k + 1) \\
&= p\pi_{k-1}^{(n)} + q\pi_{k+1}^{(n)} \\
&= (P\pi^{(n)})_k
\end{align*}
\]
The other cases for $k$, i.e., $k = 0, 1, N - 1, N$ are treated similarly.

Equation (8.1) is very important since it says that the probability density vectors

$$\pi^{(n)}, \quad n = 0, 1, \ldots$$

obey a simple deterministic law.

If your initial capital is $k \in A$, then

$$X_0 = k, \quad \pi^{(0)} = e^k$$

and

$$\pi^{(n)} = P^n e^k$$

is the $k$-th column of $P^n$. In this way, the transition matrix $P$ can be used to determine the evolution of the probability density vectors.

8.6 Realizations

Recall that

$$S = \{H, T\}, \quad \text{prob}(H) = p, \quad \text{prob}(T) = q = 1 - p .$$

Any specific choice like

$$HTTTH \ldots$$

of heads and tails together with an initial choice $X_0 = k$ leads to a realization of the random evolution,

$$X_0 = k, X_1, X_2, \ldots$$

We can use Matlab’s random number generator to obtain realizations of the game:

```matlab
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This script plots the result of 500 games.
% %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

p=.51; q=1-p;
N=50; expe=500;
res=zeros(1,expe);
stops=zeros(1,expe);
toomany=1000;
```
for ex=1:expe
    k=25;
    for i=2:toomany
        if (k==N)||(k==0)
            stops(ex)=i;
            i=toomany+1;
            inew=i;
            break;
        else
            x=rand;
            if x<=p
                k=k+1;
            else
                k=k-1;
            end;
        end
    end
    res(ex)=k;
end;

stops(find(stops==0))=toomany;
figure(1)
plot([1:expe],res,'.');
xlabel('Number of a game');
ylabel('Money at the end of gambling');
figure(2)
plot([1:expe],stops,'.')
xlabel('Number of the last step of the game');
ylabel('Number of the last step of the game');

See also Figures 21 and 22.

8.7 Probability of Ruin and Probability of Winning

As an example, assume that

\[ p = 0.49, \quad q = 0.51, \quad N = 100, \quad k = 50. \]  \hfill (8.2)

What is the probability that ruin occurs at some time in the future? (This means that \( X_n = 0 \) for some \( n \).)

What is the probability that winning occurs at some time in the future? (This means that \( X_n = N = 100 \) for some \( n \).)
Figure 21: Result of 500 games with $p = 0.51$ and $N = 50$

How long, on average, will the game last until either ruin or winning has occurred?

We can try this out by running many realizations, but for our simple game we can also develop an explicit formula.

**Project:** Carry out the details of the following: Fix $p, q = 1 - p$, and $N$. Let $\alpha_k$ denote the probability that ruin occurs at some time in the future if our initial capital is $X_0 = k$. We have

$$\alpha_0 = 1, \quad \alpha_N = 0.$$  \hspace{1cm} (8.3)

If $1 \leq k \leq N - 1$ and $X_0 = k$ then

$$X_1 = k + 1 \quad \text{with probability } p$$
$$X_1 = k - 1 \quad \text{with probability } q$$

If $X_1 = k + 1$ then the probability of ruin at some time in the future is $\alpha_{k+1}$.
If $X_1 = k - 1$ then the probability of ruin at some time in the future is $\alpha_{k-1}$.

We then convince ourselves that

$$\alpha_k = p\alpha_{k+1} + q\alpha_{k-1}.$$  

This is a homogeneous, second order difference equation,
Such an equation we can solve using the ansatz

\[ \alpha_k = \rho^k. \]

One obtains a quadratic for \( \rho \). If \( p \neq \frac{1}{2} \) one finds two distinct solutions for \( \rho \). Then, using the boundary conditions (8.3), one obtains

\[ \alpha_k = \left( \frac{q}{p} \right)^N - \left( \frac{q}{p} \right)^k \quad \text{if} \quad p \neq \frac{1}{2} \]

and

\[ \alpha_k = \frac{N - k}{N} \quad \text{if} \quad p = \frac{1}{2}. \]

**Remark:** We can also obtain the vector \( \alpha \) by solving a linear system \( D\alpha = e^{(0)} \) where the matrix \( D \) is obtained from \( P^T \) in a simple way.

Compute the \( \alpha \) vector for the data (8.2).

**Continuation of Project:** Fix \( p, q = 1 - p, \) and \( N \). Let \( \beta_k \) denote the probability that winning occurs at some time in the future if our initial capital is \( X_0 = k \). Derive boundary conditions and a difference equation for \( \beta_k \).
You should obtain that

\[ P^T \alpha = \alpha, \quad P^T \beta = \beta. \]

Remarkably, \( P^T \) has two eigenvectors to the eigenvalue 1. Can we see this by looking directly at the matrix \( P \)?

### 8.8 Project: Expected Time for Game

Carry out the details. Fix \( p, q = 1 - p \), and \( N \). Let \( \tau_k \) denote the expected time for ruin or winning to occur if \( X_0 = k \). We have

\[ \tau_0 = \tau_N = 0. \]

Convince yourself that the following difference equation is reasonable:

\[ \tau_k = p(1 + \tau_{k+1}) + q(1 + \tau_{k-1}), \quad 1 \leq k \leq N - 1. \]

Derive an explicit formula for \( \tau_k \).

**One Result:** For \( p = .4, N = 5 \), you should get

\[
\alpha = \begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5
\end{pmatrix} = \begin{pmatrix}
1 \\
.9242 \\
.8104 \\
.6398 \\
.3839 \\
0
\end{pmatrix}.
\]
9 Stochastic Model of a Simple Birth Process

9.1 The Forward Kolmogorov Equations

In this model, time is continuous, $0 \leq t < \infty$, and the state space is discrete,

$$A = \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

With

$$X_t \in \mathbb{N}_0$$

we denote the number of individuals in a population at time $t$. Clearly, $X_t$ must be an integer. In large populations the variable $X_t$ is often treated as a continuous variable, but in small populations it may be better to treat $X_t$ as an integer.

We let

$$p_j(t) := \text{prob}(X_t = j), \quad j \in \mathbb{N}_0.$$

Let us assume that at time $t = 0$ the population has $k$ individuals,

$$X_0 = k \in \mathbb{N}_0,$$

thus

$$p_j(0) = \text{prob}(X_0 = j) = \delta_{jk}, \quad j \in \mathbb{N}_0.$$

In our model, no deaths occur and the probability for an individual to give birth in a time interval of length $\Delta t$ is

$$b \Delta t$$

for small $\Delta t$. We choose $\Delta t$ so small that no birth or one birth will occur in the time interval from $t$ to $t + \Delta t$. We want to obtain an equation for

$$p_j(t + \Delta t) = \text{prob}(X_{t+\Delta t} = j).$$

There are two possibilities for $X_{t+\Delta t}$ to equal $j$: First, it is possible that at time $t$ there were $j - 1$ individuals and a birth occurred in the interval from $t$ to $t + \Delta t$. Second, at time $t$ there were $j$ individuals and no birth occurred in the interval from $t$ to $t + \Delta t$.

Let $b > 0$ denote the birth rate per individual. Then the probability of the first case is

$$p_{j-1}(t)b(j - 1)\Delta t.$$

The probability of the second case is

$$p_j(t)(1 - bj\Delta t).$$
(To arrive at the formula for the second case, note that the probability of a birth occurring in the interval from $t$ to $t + \Delta t$ is $bj\Delta t$ if one assume that at time $t$ there are $j$ individuals in the population.)

These assumptions yield

$$p_j(t + \Delta t) = p_{j-1}(t)b(j - 1)\Delta t + p_j(t)(1 - bj\Delta t) .$$

We can rewrite this as

$$\frac{1}{\Delta t} \left( p_j(t + \Delta t) - p_j(t) \right) = p_{j-1}(t)b(j - 1) - bp_j(t) .$$

For $\Delta t \to 0$ obtain

$$p_j'(t) = -bp_j(t) + b(j - 1)p_{j-1}(t), \quad j = 1, 2, \ldots$$

These are called the forward Kolmogorov equations of the model.

This is an infinite system of ODEs:

$$
\begin{align*}
p_1' &= -bp_1 \\
p_2' &= -2bp_2 + bp_1 \\
p_3' &= -3bp_3 + 2bp_2 \\
p_4' &= -4bp_4 + 3bp_3
\end{align*}
$$

etc.

### 9.2 Solution of the Forward Kolmogorov Equations

Recall the initial conditions

$$p_j(0) = \delta_{jk}$$

where $k = X_0$ is the size of the initial population.

Let us assume that $k \geq 1$. We obtain

$$p_j(t) \equiv 0, \quad j < k ,$$

and

$$p_k' = -kp_k, \quad p_k(0) = 1 ,$$

thus

$$p_k(t) = e^{-kt} .$$

We then can determine $p_{k+1}$ etc.

Interestingly, one can derive closed form expressions for the $p_j(t)$. First, with new unknown functions $\phi_j(t)$ let

$$p_j(t) = e^{-jt}\phi_j(t) .$$
(This transformation will eliminate the diagonal term in the system for the \( p_j \).

One obtains

\[ p'_j = -jp_j + e^{-jt} \phi'_j, \]

thus

\[ e^{-jt} \phi'_j = b(j-1)e^{-(j-1)t} \phi_{j-1}, \]

thus

\[ \phi'_j = (j-1)be^{jt} \phi_{j-1}. \]

Together with the initial conditions \( \phi_j(0) = \delta_{jk} \) we find that

\[ \phi_k \equiv 1 \]

and

\[ \phi'_j = (j-1)be^{jt} \phi_{j-1}, \quad \phi_j(0) = 0, \quad j \geq k + 1. \]

In principle, the \( \phi_j \) can be obtained by successive integrations.

The following turns out to work: With unknown coefficients \( a_j \) let

\[
\begin{align*}
\phi_j(t) &= a_j(e^{bt} - 1)^{j-k} \\
\phi_{j-1}(t) &= a_{j-1}(e^{bt} - 1)^{j-1-k}.
\end{align*}
\]

Then simple computations yield

\[ a_j(j-k) = a_{j-1}(j-1) \quad \text{for} \quad j \geq k + 1, \quad a_k = 1. \]

Thus, the coefficients \( a_j \) can be determined recursively.

It is not difficult to show that

\[ a_j = \binom{j-1}{k-1} = \frac{(j-1)!}{(k-1)!(j-k)!}, \quad j \geq k. \]

Thus we have obtained a closed form solution of the forward Kolmogorov equations with

\[ X_0 = k. \]

The formula is:

\[ p_j(t) = \binom{j-1}{k-1} e^{-jt}(e^{bt} - 1)^{j-k}, \quad j \geq k. \]
9.3 The Sum of the $p_j(t)$

In the following we assume

$$X_0 = k \geq 1.$$ 

Recall that

$$p_j(t) = \text{prob}(X_t = j), \quad j \in \mathbb{N}.$$ 

Since the random variable $X_t$ must take on exactly one of the values $j \in \mathbb{N}$ we expect that

$$\sum_{j=1}^{\infty} p_j(t) = 1$$

for all $t \geq 0$. If this would not hold, something must have gone wrong in our modeling process.

Define

$$p(t) = \sum_{j=1}^{\infty} p_j(t).$$

Then, since $p_j(0) = \delta_{jk}$, we have

$$p(0) = 1.$$ 

Also, if we look at the infinite ODE system satisfied by the $p_j(t)$ and proceed formally, we obtain

$$p'(t) = \sum_{j=1}^{\infty} p'_j(t)$$

$$= \sum_{j=1}^{\infty} (-jp_j(t)) + \sum_{j=1}^{\infty} (j - 1)p_{j-1}(t)$$

$$= 0.$$ 

Therefore, $p(t) = p(0) = 1$.

The argument is not quite correct since, in general, one is not allowed to differentiate a series term–by–term.

To sharpen our skills we want to give a rigorous argument using the explicit formula

$$p_j(t) = a_j e^{-jbt} (e^{bt} - 1)^{j-k}, \quad a_j = \binom{j-1}{k-1}.$$ 

Set

$$w = e^{-bt} \quad \text{for} \quad b > 0, \quad t > 0,$$
and
\[ z = 1 - w . \]

We then have
\[ 0 < z < 1 . \]

Also,
\[
p_j(t) = a_j w^j \left( \frac{1}{w} - 1 \right)^{j-k} = a_j w^k w^{j-k} \left( \frac{1}{w} - 1 \right)^{j-k} = a_j w^k (1 - w)^{j-k} = a_j (1 - z)^k z^{j-k}.
\]

We then have to show that
\[
\sum_{j \geq k} a_j (1 - z)^k z^{j-k} = 1 \quad \text{for} \quad 0 < z < 1 ,
\]
or, equivalently,
\[
\sum_{n=0}^{\infty} a_{k+n} z^n = (1 - z)^{-k}.
\]

Here
\[
a_{k+n} = \binom{k+n-1}{k-1} = \frac{(k+n-1)!}{(k-1)!n!} = \frac{1}{n!} k(k+1) \ldots (k+n-1).
\]

We set
\[ f(z) = (1 - z)^{-k} \]
and find that
\[ f^{(n)}(0) = k(k+1) \ldots (k+n-1) . \]

Therefore, the relation
\[
(1 - z)^{-k} = \sum_{n=0}^{\infty} \frac{1}{n!} k(k+1) \ldots (k+n-1) z^n \quad (9.1)
\]
is the Taylor series expansion of the function \( f(z) = (1 - z)^{-k} \) about \( z = 0 \). In a course on complex variables we will learn that (9.1) holds for all complex \( z \) with \( |z| < 1 \).
9.4 The Expected Value of $X_t$

Recall our assumption

$$X_0 = k \in \mathbb{N}$$

which implies $p_j(t) = 0$ for $j < k$. Since

$$\text{prob}(X_t = j) = p_j(t)$$

the expected value, or mean, of the random variable $X_t$ is, by definition,

$$E(X_t) = \sum_{j \geq k} j p_j(t).$$

(This might need some motivation.)

We claim that

$$E(X_t) = ke^{bt}. \quad (9.2)$$

Thus, the expected value of $X_t$ agrees with the value of the deterministic variable $u(t)$ solving

$$u' = bu, \quad u(0) = k,$$

which is reasonable.

**Proof of (9.2):** Using the same notations as above, we have

$$\sum_{j \geq k} jp_j(t) = \sum_{j \geq k} ja_j(1 - z)^k z^{j-k}$$

and

$$ke^{bt} = k(1 - z)^{-1}.$$ 

We must show that

$$\sum_{j \geq k} ja_j z^{j-k} = k(1 - z)^{-k-1},$$

or,

$$\sum_{n=0}^{\infty} (k + n)a_{n+k} z^n = k(1 - z)^{-k-1}.$$

It is not difficult to see that this follows by differentiating the power series expansion (9.1) term-by-term. In a course on complex variables we will learn that this is justified.
9.5 The Variance of $X_t$

Let us start with some general remarks on the notion of variance of some random variable $X_t$. The $t$–dependence is unimportant here.

Consider two random variables, $X_t$ and $Y_t$, and assume that $X_t$ takes on the values 9, 10, 11, each with probability $\frac{1}{3}$. The mean of $X_t$ is

$$E(X_t) = \frac{1}{3}(9 + 10 + 11) = 10 .$$

The second random variable $Y_t$ takes on the values 0, 10, 20, each with probability $\frac{1}{3}$. The mean of $Y_t$ is

$$E(Y_t) = \frac{1}{3}(0 + 10 + 20) = 10 .$$

Both random variables have the same mean, but $X_t$ varies much less about the mean than $Y_t$. The variance of a random variable is a quantitative measure for the variation about the mean.

Let $\alpha_j$ denote a sequence of real numbers and let $X_t$ denote a random variable taking values in $\{\alpha_1, \alpha_2, \ldots\}$. Let

$$p_j(t) = \text{prob}(X_t = \alpha_j), \quad \sum_j p_j(t) = 1 .$$

Then, by definition,

$$\mu_t = E(X_t) = \sum_j \alpha_j p_j(t)$$

is the mean of $X_t$. Note that, for fixed $t$, $\mu_t$ is a number, not a random variable. The random variable

$$Y_t = X_t - \mu_t$$

takes on the values $\alpha_j - \mu_t$ with probability $p_j(t)$. Therefore, the mean of $Y_t$ is

$$E(Y_t) = E(X_t - \mu_t) = \sum_j (\alpha_j - \mu_t) p_j(t) = \sum_j \alpha_j p_j(t) - \mu_t \sum_j p_j(t) = 0 .$$

Thus, the random variable $X_t - \mu_t$ has mean zero. The number

$$\sigma_t^2 = E((X_t - \mu_t)^2)$$

is called the variance of $X_t$.

Note that
\[ Z_t = (X_t - \mu_t)^2 = X_t^2 - 2\mu_t X_t + \mu_t^2 \]
takes the value
\[ \alpha_j^2 - 2\mu_t \alpha_j + \mu_t^2 \]
with probability \( p_j(t) \). Therefore,
\[
\sigma_t^2 = \sum (\alpha_j^2 - 2\mu_t \alpha_j + \mu_t^2) p_j(t) \\
= \sum \alpha_j^2 p_j(t) - 2\mu_t \sum \alpha_j p_j(t) + \mu_t^2 \\
= \sum \alpha_j^2 p_j(t) - \mu_t^2 \\
= E(X_t^2) - \mu_t^2.
\]

**Application:** We want to apply the general concept to the random variable \( X_t \) modeling the simple birth process. Recall that \( X_0 = k \) and \( X_t \) takes on the value \( \alpha_j = j \) with probability
\[
p_j(t) = a_j e^{-jbt}(e^{bt} - 1)^{j-k}, \quad a_j = \binom{j-1}{k-1} = \frac{(j-1)!}{(k-1)!(j-k)!}, \quad j \geq k.
\]
We claim that
\[
\sigma_t^2 = E(X_t^2) - \mu_t^2 = k e^{bt}(e^{bt} - 1)
\] (9.3)
where
\[ \mu_t = ke^{bt}. \]
Using the notations introduced above, we must show
\[
E(X_t^2) = k^2 e^{2bt} + ke^{2bt} - ke^{bt}
\]
or,
\[
\sum_{j \geq k} j^2 p_j(t) = k(k+1)w^{-2} - kw^{-1}
\]
with
\[ p_j(t) = a_j w^k (1 - w)^{j-k}. \]
In other words, we must show
\[
\sum_{j \geq k} j^2 a_j (1 - w)^{j-k} = k(k+1)w^{-k-2} - kw^{-k-1}
\]
or,
\[
\sum_{j \geq k} j^2 a_j z^{j-k} = k(k+1)(1-z)^{-k-2} - k(1-z)^{-k-1}.
\] (9.4)

Recall that we have obtained the Taylor series for

\[ f(z) = (1 - z)^{-k}. \]

The right side of (9.4) is

\[ rhs = f''(z) - f'(z). \]

Here

\[
\begin{align*}
  f(z) & = (1 - z)^{-k} \\
         & = \sum_{n=0}^{\infty} \frac{1}{n!} k(k+1) \ldots (k+n-1) z^n \\
  f'(z) & = k(1 - z)^{-k-1} \\
         & = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} k(k+1) \ldots (k+n-1) z^{n-1} \\
         & = \sum_{n=0}^{\infty} \frac{1}{n!} k(k+1) \ldots (k+n) z^n \\
  f''(z) & = k(k+1)(1 - z)^{-k-2} \\
         & = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} k(k+1) \ldots (k+n) z^{n-1} \\
         & = \sum_{n=0}^{\infty} \frac{1}{n!} k(k+1) \ldots (k+n+1) z^n
\end{align*}
\]

Therefore,

\[
\begin{align*}
  rhs & = f''(z) - f'(z) \\
       & = \sum_{n=0}^{\infty} \frac{1}{n!} \left(k(k+1) \ldots (k+n+1) - k(k+1) \ldots (k+n)\right) z^n \\
       & = \sum_{n=0}^{\infty} \frac{1}{n!} \left(k(k+1) \ldots (k+n)\right) (k+n) z^n \\
       & = \sum_{n=0}^{\infty} \frac{1}{n!} \left(k(k+1) \ldots (k+n-1)\right) (k+n)^2 z^n \\
       & = \sum_{n=0}^{\infty} a_{k+n} (k+n)^2 z^n
\end{align*}
\]

The left side in (9.4) is
lhs = \sum_{j \geq k} j^2 a_j z^{j-k} = \sum_{n=0}^{\infty} a_{k+n}(k+n)^2 z^n .

This proves the formula (9.4). We have shown:

**Theorem 9.1** For the random variable \( X_t \) defined by the birth process we have

\[
\begin{align*}
\mu_t &= E(X_t) = ke^{bt}, \\
\sigma_t^2 &= E((X_t - \mu_t)^2) = ke^{bt}(e^{bt} - 1).
\end{align*}
\]

Here \( X_0 = k \) is the size of the population at time \( t = 0 \).

### 9.6 Interevent Times and Stochastic Realizations

Recall that \( X_t \) denotes the size of the population at time \( t \) where \( X_0 = k \) is a given integer. The random variable \( X_t \) takes values in \( \mathbb{N} \), and we have computed the probabilities

\[
\text{prob}(X_t = j) = p_j(t), \quad j \geq k, \quad t \geq 0,
\]

using the forward Kolmogorov equations.

Any realization of \( X_t \) should have a graph like this:

***************

put figure

***************

For any realization of \( X_t \) there is a sequence of times

\[ T_0 = 0 < T_1 < T_2 < \ldots \]

so that

\[ X_t = k + j \quad \text{for} \quad T_j \leq t < T_{j+1}, \quad j = 0, 1, \ldots \]

A jump of \( X_t \) from \( k + j - 1 \) to \( k + j \) takes place at time \( T_j, j = 1, 2, \ldots \)

If

\[ T_{j+1} = T_j + \tau_j \]

then the positive times \( \tau_j \) are called the interevent times. These numbers are random variables, taking values in \( 0 \leq \tau < \infty \). We claim:

**Theorem 9.2** For \( \tau \geq 0 \) and \( j = 0, 1, \ldots \) we have

\[
\text{prob}(\tau_j > \tau) = e^{-b(k+j)\tau} .
\]
Proof: Define

\[ G_j(\tau) = \text{prob}(\tau_j > \tau). \]

(For our argument we assume \( G_j \) to be a smooth function.)

It is plausible that

\[ G_j(0) = 1. \]

A general law of probability is the following: If \( E_1 \) and \( E_2 \) are two events, then

\[ \text{prob}(E_1 \text{ and } E_2) = \text{prob}(E_1) \text{prob}(E_2|E_1). \] (9.5)

Here

\[ \text{prob}(E_2|E_1) \]

is the probability of the occurrence of \( E_2 \) under the assumption of the occurrence of \( E_1 \), also called conditional probability. (In an abstract approach to probability theory, one uses (9.5) to define the conditional probability \( \text{prob}(E_2|E_1) \).)

In the following, let \( \Delta \tau > 0 \) and let us apply the general equation (9.5) where \( E_1 \) is the event \( \tau_j > \tau \) and \( E_2 \) is the event that no jump occurs in the time interval from \( T_j + \tau \) to \( T_j + \tau + \Delta \tau \). Then we have

\[
G_j(\tau + \Delta \tau) = \text{prob}(\tau_j > \tau + \Delta \tau) \\
= \text{prob}(E_1 \text{ and } E_2) \\
= \text{prob}(E_1) \text{prob}(E_2|E_1) \\
= \text{prob}(\tau_j > \tau) \text{prob}(E_2|E_1) \\
= G_j(\tau) \text{prob}(E_2|E_1).
\]

If \( E_1 \) occurs then

\[ X_{T_j+\tau} = k + j \]

and, under our modeling assumption,

\[ \text{prob}(E_2|E_1) \sim 1 - b(k + j)\Delta \tau \]

for \( \Delta \tau > 0 \) small. One obtains

\[ G_j(\tau + \Delta \tau) - G_j(\tau) \sim -b(k + j)\Delta \tau G_j(\tau). \]

Divide by \( \Delta \tau \) and take the limit \( \tau \to 0 \) to find that

\[ G'_j(\tau) = -b(k + j)G_j(\tau). \]
Together with the initial condition \( G_j(0) = 1 \) we obtain the formula 
\[ G_j(\tau) = e^{-b(k+j)\tau} \]

**Numerical Realization:** We want to show how we can obtain random 
\( \tau_j \)-values which satisfy

\[ \text{prob}(\tau_j > t) = e^{-b(k+j)t} \quad \text{for} \quad 0 \leq t < \infty. \]  
\[ (9.6) \]

First, we make a small and unimportant transformation so that we can use standard terminology: The function 
\[ F_j(t) = \text{prob}(\tau_j \leq t) = 1 - e^{-b(k+j)t}, \quad t \geq 0, \]

is called the cumulative distribution function of the random variable \( \tau_j \). Here \( \tau_j \) takes random values in the interval \( 0 \leq t < \infty \).

Matlab’s random number generator gives us a random variable \( Y \) which is uniformly distributed in the interval from 0 to 1, thus

\[ \text{prob}(Y \leq Y_0) = Y_0 \quad \text{for} \quad 0 < Y_0 < 1. \]

The following result is almost trivial:

**Theorem 9.3** Let \( Y \) be a uniformly distributed random variable taking values in \((0, 1)\). Let \( F : [0, \infty) \to (0, 1) \) be a strictly increasing continuous function with

\[ F(0) = 0, \quad \lim_{t \to \infty} F(t) = 1. \]

Then the random variable

\[ T = F^{-1}(U) \]

takes values in \( 0 \leq t < \infty \) and has the cumulative distribution function \( F(t) \).

**Proof:** We must show that

\[ \text{prob}(T \leq t) = F(t) \quad \text{for} \quad 0 \leq t < \infty. \]

We have

\[ \text{prob}(T \leq t) = \text{prob}(F^{-1}(U) \leq t) \\
= \text{prob}(U \leq F(t)) \\
= F(t) \]

\[ \diamond \]

We now undo our trivial transformation. If \( Y \) is uniformly distributed in \((0, 1)\) then \( 1 - Y \) is also uniformly distributed in \((0, 1)\). Therefore, to get numerical values for the \( \tau_j \) we proceed as follows:

1) Choose \( 0 < Y < 1 \) at random, uniformly distributed.
2) Solve

\[ Y = e^{-b(k+j)\tau_j}, \]

i.e.,

\[ \tau_j = \frac{\ln(1/Y)}{b(k+j)}. \]

Then (9.6) holds.

The following Matlab script is used to obtain \( \tau_j, j = 1, 2, \ldots, 40 \) for \( X_0 = 1 \):

```matlab
x0=3;
maxstep=40;
tau=zeros(1,maxstep+1);
for i=1:maxstep
    y=rand;
    tau(i+1)=tau(i)+log(1/y)/(x0-1+i);
end
plot(tau,[x0:maxstep+x0],'.');hold on
newt=[tau(1:maxstep-1); tau(2:maxstep)];
for i=1:maxstep-1
    plot(newt(:,i),[x0+i-1 x0+i-1],'-');hold on;
end
y=[0:.1:tau(end)];
z=x0*exp(y);
plot(y,z,'-'); hold on;
title('Simple birth process');
xlabel('Time');
ylabel('Size of population');
```

The plot is given in Figure 23.

Using the code above we can check formula

\[ p_j(t) = \left( \frac{j - 1}{k - 1} \right) e^{-jbt} (e^{bt} - 1)^{j-k}, \quad j \geq k, \]
by numerical simulations.

The following script plots values of $p_j(3)$ obtained numerically and analyti-
cally for $X_0 = 1, 2, 3, 4$.

```matlab
% This script plots probabilities $p_j(t)$ for $t=3$
% calculated numerically via 90000 experiments
% together with $p_j(t)$ obtained analytically.
%
% defining number of experiments
exper=90000;

% loop for different initial conditions
for x0=1:4
    maxstep=150;

    res=zeros(1,exper);

    for l=1:exper
```

Figure 23: Simple birth process
tau=0;
for i=1:maxstep
    y=rand;
    tau=tau+log(1/y)/(x0-1+i);
    if tau>=3
        res(l)=i;
        break;
    end;
end;

notinform=length(find(res==0));
probab=zeros(1,max(res));
analytprobab=zeros(max(res),1);

for l=1:max(res)
    probab(l)=length(find(res==l));
    if l>=x0
        % This part could be done more efficiently
        analytprobab(l)=factorial(l)/factorial(x0-1)/...
            factorial(l+1-x0)*exp(-3*l)*(exp(3)-1)^(l-x0);
    end;
end;

plot([1:max(res)],analytprobab,'-');hold on;
plot([1:max(res)],probab/(exper-notinform),'.');
end;
title('p_j(3) for X_0 from 1 to 4');
xlabel('j');
ylabel('p_j(3)');

The corresponding plot is given in Figure 24.
10 Introduction to Kinetic Theory

In kinetic theory, one tries to connect micro models for fluids or gases with macro models, as described by thermodynamics and continuum mechanics.

Some dates:
Robert Boyle (1627–1691),
Daniel Bernoulli (1700–1782),
Sadi Carnot (1796–1832),
Emile Clapeyron (1799–1864),
James Prescott Joule (1818–1889),
James Clark Maxwell (1831–1879).

10.1 Daniel Bernoulli: Pressure, Volume, and Particle Velocity

We start with a result due to Daniel Bernoulli, 1738.

Suppose you have a box \( B = [0, L]^3 \) with volume \( V = L^3 \) which contains \( N \) particles, each of mass \( m \). Let \( p \) denote the pressure in the gas and let \( T \) denote its temperature.

Suppose we can change the volume of the box by moving a wall. This may change the temperature, \( T \), but we wait until the temperature is again the temperature of the surroundings. Let \( p_1, V_1 \) denote pressure and volume after the wall has been moved.
One observes that (for an ideal gas)

\[ pV = p_1V_1 . \]

This law, \( pV = \text{const} \) at constant temperature, is called Boyle’s law or, by the French, Mariotte’s law. (In German: Boyle–Mariotte’s law.)

In trying to understand the law, D. Bernoulli obtained a first result of kinetic theory.

Suppose all the particles move at the same speed, \( v \). Also, for simplicity, assume that at any given time, \( \frac{N}{6} \) of the particles move in the positive \( x \) direction, i.e., have velocity

\[ \mathbf{v} = (v, 0, 0) . \] (10.1)

(This simplifying assumption is a strike of genius of Bernoulli which makes the problem treatable; remarkably, the outcome is essentially correct.)

Let \( W \) denote the wall parallel to the \( yz \)-plane at \( x = L \). Consider a particle with velocity \( \mathbf{v} = (v, 0, 0) \) before it hits the wall \( W \). After it hits \( W \), its velocity is \( -v \).

Newton’s law,

\[ m \frac{d}{dt} \mathbf{v}(t) = \text{force} , \]

cannot be directly used if we have an instantaneous change of velocity from \( (v, 0, 0) \) to \( (-v, 0, 0) \) because the required force would be infinite. However, we can argue as follows: Let

\[ \mathbf{v}(t) = (v, 0, 0) \]

and

\[ \mathbf{v}(t + \Delta t) = (-v, 0, 0) . \]

We then have

\[ m\left( \mathbf{v}(t + \Delta t) - \mathbf{v}(t) \right) \sim \text{force} \cdot \Delta t \]

where the force is the force of the wall acting on the particle. Conversely, if \( \mathbf{F} = (f, 0, 0) \) is the force of the particle applied to \( W \), then

\[ -m\left( \mathbf{v}(t + \Delta t) - \mathbf{v}(t) \right) \sim \mathbf{F} \cdot \Delta t . \]

This yields:

\[ 2mv \sim f \Delta t . \]

In a time interval of length \( \Delta t \), the number of particles that will hit the wall \( W \) is

\[ N_1 = \frac{N}{6} \cdot \frac{v \Delta t}{L} . \]
Thus, if $F$ is the magnitude of the total force on the wall $W$, then

$$\frac{N}{6} \cdot \frac{v \Delta t}{L} \cdot 2mv \sim F \Delta t.$$ 

We can divide by $\Delta t$, obtaining

$$F = \frac{2N}{3} \cdot \frac{1}{2}mv^2 \cdot \frac{1}{L}.$$ 

The pressure on the wall $W$ is

$$p = \frac{F}{L^2} = \frac{2N}{3} \cdot \frac{1}{2}mv^2 \cdot \frac{1}{L^3}.$$ 

In other words,

$$pV = \frac{2N}{3} \cdot \frac{1}{2}mv^2.$$ 

Here $\frac{1}{2}mv^2$ is the kinetic energy of a particle. This is D. Bernoulli’s result. It indicates, remarkably, that constant temperature corresponds to constant kinetic energy of the particles.

As it stands, the result only needs one modification: One has to replace $v^2$ by its mean. A correct result is:

$$pV = \frac{2N}{3} \cdot \frac{m}{2} \langle v^2 \rangle$$

where $\langle v^2 \rangle$ is the mean of $v^2$ over all particle. (In fact, how to take the mean, is not completely clear.)

It was also discovered later that there is a universal constant $k$ so that

$$\frac{3}{2} kT = \frac{m}{2} \langle v^2 \rangle$$

where $T$ is the absolute temperature of the gas. The constant $k$ is Boltzmann’s constant,

$$k = 1.3804 \cdot 10^{-23} \text{ Joule/K}.$$ 

Here $K$ is one degree Kelvin and

$$1 \text{ Joule} = \text{Newton meter} = 1 \text{ Watt sec}.$$ 

(One Newton is about the weight of an apple.)

### 10.2 Maxwell’s Velocity Distribution

The Maxwell distribution was determined by Maxwell in 1860.

**Auxiliary Results:**
\[
\int_0^\infty e^{-x^2} \, dx = \frac{1}{2} \sqrt{\pi} \\
\int_0^\infty x^2 e^{-x^2} \, dx = \frac{1}{4} \sqrt{\pi} \\
\int_0^\infty x^4 e^{-x^2} \, dx = \frac{3}{2^3} \sqrt{\pi} \\
\int_0^\infty x^{2n} e^{-x^2} \, dx = \frac{1 \cdot 3 \cdots (2n - 1)}{2^{n+1}} \sqrt{\pi}
\]

Proof: Let \( J = \int_{-\infty}^\infty e^{x^2} \, dx \). Then, using polar coordinates,

\[
J^2 = \int_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dxdy \\
= \int_0^{2\pi} \int_0^\infty re^{-r^2} \, drd\phi \\
= \pi \int_0^\infty 2re^{-r^2} \, dr \\
= \pi \int_0^\infty e^{-\xi} \, d\xi \\
= \pi
\]

This proves the first formula.

To prove the 2nd formula, we use integration by parts:

\[
\int_0^\infty x^2 e^{-x^2} \, dx = \int_0^\infty \frac{x}{2} (2xe^{-x^2}) \, dx \\
= \int_0^\infty \frac{x}{2} (-e^{-x^2})' \, dx \\
= \int_0^\infty \frac{1}{2} e^{-x^2} \, dx \\
= \frac{1}{4} \sqrt{\pi}
\]

The 3rd formula follows similarly,

\[
\int_0^\infty x^4 e^{-x^2} \, dx = \int_0^\infty \frac{x^3}{2} (2xe^{-x^2}) \, dx \\
= \int_0^\infty \frac{x^3}{2} (-e^{-x^2})' \, dx \\
= \int_0^\infty \frac{3}{2} e^{-x^2} \, dx \\
= \frac{3}{2^3} \sqrt{\pi}
\]
The general case follows by induction in $n$.

In the following, $B$ denotes a box in $\mathbb{R}^3$. (One could allow for more general sets, namely Lebesgue measurable sets.) With $v \in \mathbb{R}^3$ we denote the velocity of a randomly chosen particle in a gas of temperature $T$. (We assume that we have chosen units for space and time so that $v \in \mathbb{R}^3$ is dimensionless.)

Maxwell’s velocity distribution can be derived from the following assumptions:

1. There is a smooth function $f : \mathbb{R}^3 \to [0, \infty)$ so that for every box $B$:

\[
\text{prob}(v \in B) = \int_B f(v) \, dv.
\]

(The function $f(v)$ is called the probability density of the velocity distribution.)

We have the normalization

\[
\int_{\mathbb{R}^3} f(v) \, dv = 1.
\]

2. Let $v = (v_1, v_2, v_3)$. For $j = 1, 2, 3$ there is a smooth function $g_j : \mathbb{R} \to [0, \infty)$ so that, for every interval $I \subset \mathbb{R}$:

\[
\text{prob}(v_j \in I) = \int_I g_j(z) \, dz.
\]

3. For reasons of symmetry:

\[g_1 = g_2 = g_3 =: g.\]

4. (Independence) If $B = I_1 \times I_2 \times I_3$ then

\[
\text{prob}(v \in B) = \text{prob}(v_1 \in I_1) \text{prob}(v_2 \in I_2) \text{prob}(v_3 \in I_3).
\]

5. There is a smooth function $\Phi : [0, \infty) \to [0, \infty)$ with

\[f(v) = \Phi(|v|^2) \quad \text{where} \quad |v|^2 = v_1^2 + v_2^2 + v_3^2.
\]

Let us now draw some conclusions from these assumptions. Fix $w \in \mathbb{R}^3$ and let $h > 0$. Let

\[B_h = [w_1, w_1 + h] \times [w_2, w_2 + h] \times [w_3, w_3 + h].\]

Then we have

\[
\text{prob}(v \in B_h) = \int_{B_h} \Phi(|v|^2) \, dv
\]

\[= \int_{w_1}^{w_1 + h} g(z)dz \int_{w_2}^{w_2 + h} g(z)dz \int_{w_3}^{w_3 + h} g(z)dz.
\]

Divide by $h^3$ and let $h \to 0$ to obtain

\[\Phi(|w|^2) = g(w_1)g(w_2)g(w_3), \quad w \in \mathbb{R}^3.
\]
Let \( w_1 = z, w_2 = w_3 = 0 \). Obtain:

\[
\Phi(z^2) = g(z)g^2(0).
\]

This shows that

\[
g(z) = \frac{\Phi(z^2)}{g^2(0)}.
\]

(10.2)

Let \( y = w_1^2, z = w_2^2, w_3 = 0 \). Obtain:

\[
\Phi(y + z) = g(w_1)g(w_2)g(0) = \frac{\Phi(y)}{g^2(0)} \cdot \frac{\Phi(z)}{g^2(0)} \cdot g(0).
\]

In other words:

\[
\Phi(y + z) = \frac{\Phi(y)\Phi(z)}{g^3(0)} \quad \text{for} \quad y, z \in \mathbb{R}.
\]

Differentiate in \( y \):

\[
\frac{d}{dy} \Phi(y + z) = \frac{\Phi'(y)\Phi(z)}{g^3(0)} \quad \text{for} \quad y, z \in \mathbb{R}.
\]

Put \( y = 0 \):

\[
\Phi'(z) = -K\Phi(z), \quad K = -\frac{\Phi'(0)}{g^3(0)}.
\]

Therefore, \( \Phi(z) \) is an exponential:

\[
\Phi(z) = C^3 e^{-Kz}.
\]

The normalization condition determines \( C \) in terms of \( K > 0 \):

\[
1 = \int_{\mathbb{R}^3} \Phi(|v|^2) \, dv
= C^3 \int e^{-K|v|^2} \, dv
= C^3 \left( \int_{-\infty}^{\infty} e^{-Kz^2} \, dz \right)^3
= (2C)^3 \left( \int_{0}^{\infty} e^{-Kz^2} \, dz \right)^3.
\]

Since

\[
\int_{0}^{\infty} e^{-Kz^2} \, dz = \frac{1}{2} \sqrt{\pi/K}
\]

one finds that

\[
C = \sqrt{K/\pi}.
\]
Maxwell’s velocity distribution depends only on one parameter, $K > 0$. We have

$$f(v) = \Phi(|v|^2) = \left(\frac{K}{\pi}\right)^{3/2} e^{-K|v|^2}.$$  

It is common to introduce another function,

$$F(z) = \frac{4}{\sqrt{\pi}} K^{3/2} z^2 e^{-Kz^2}, \quad 0 \leq z < \infty.$$  

This function arises as follows. We have for $v_0 > 0$:

$$\text{prob}(|v| \leq v_0) = \int_{|v| \leq v_0} f(v) \, dv = \left(\frac{K}{\pi}\right)^{3/2} \int_{|v| \leq v_0} e^{-K|v|^2} \, dv = \left(\frac{K}{\pi}\right)^{3/2} 4\pi \int_0^{v_0} z^2 e^{-Kz^2} \, dz = \frac{4}{\sqrt{\pi}} K^{3/2} \int_0^{v_0} z^2 e^{-Kz^2} \, dz = \int_0^{v_0} F(z) \, dz.$$

**Theorem 10.1** Let

$$F(z) = \frac{4}{\sqrt{\pi}} K^{3/2} z^2 e^{-Kz^2}, \quad 0 \leq z < \infty.$$  

Under Maxwell’s assumptions, we have a velocity distribution satisfying

$$\text{prob}(v_0 \leq |v| \leq v_1) = \int_{v_0}^{v_1} F(z) \, dz.$$  

Let us use Maxwell’s distribution to compute the mean kinetic energy of a particle of mass $m$:

$$\frac{m}{2} \langle |v|^2 \rangle = \frac{m}{2} \int |v|^2 f(v) \, dv = \frac{m}{2} \left(\frac{K}{\pi}\right)^{3/2} 4\pi \int_0^\infty z^4 e^{-Kz^2} \, dz = \frac{2m\pi(K/\pi)^{3/2}K^{-5/2}}{8\sqrt{\pi}} = \frac{3}{4} \cdot \frac{m}{K}.$$  

Using Boltzmann’s constant $k$ and the absolute temperature $T$ we also have

$$\frac{m}{2} \langle |v|^2 \rangle = \frac{3kT}{2}.  \quad 74$$
This yields

\[ K = \frac{m}{2kT} \]

One obtains:

\[ F(z) = \sqrt{\frac{2}{\pi}} \left( \frac{m}{kT} \right)^{3/2} z^{3/2} e^{-mz^2/2kT}, \quad 0 \leq z < \infty. \] (10.3)

**Theorem 10.2** In a gas of temperature \( T \) where the particles have mass \( m \), the velocity distribution satisfies

\[ \text{prob}(v_0 \leq |v| \leq v_1) = \int_{v_0}^{v_1} F(z) \, dz. \]

Here \( F(z) \) is given in equation (10.3).

### 10.3 Optimal Heat Engines: Carnot’s Cycle

In the following, \( T_1 \) and \( T_2 \) denote absolute temperatures on the Kelvin scale.

Energy comes in different forms. Most familiar is heat energy and mechanical energy. The unit of heat energy is 1\text{cal}, which is the heat energy which increases the temperature of 1\text{g} of water by 1\text{degree Celsius}. The unit of mechanical energy is

\[ 1\text{Joule} = 1\text{Newton meter} = \frac{1\text{kg} \cdot \text{m}^2}{\text{sec}^2}. \]

Here 1\text{Newton} is the unit of force, approximately the weight of an apple. Thus, we can think of 1\text{Joule} as the mechanical energy to lift an apple by one meter.

It is easy to turn mechanical energy into heat through friction. By careful experiments, Joule found that 1\text{cal} corresponds to 4.185\text{Joule}.

Can we turn heat energy into mechanical energy? A machine which does this is called a heat engine. Sadi Carnot described a heat engine which absorbs heat energy \( Q \) at a temperature \( T_1 \), turns a part of it into mechanical work \( W \), and releases the energy \( Q - W \) as heat energy at the lower temperature \( T_2 < T_1 \).

The efficiency of a heat engine is defined as

\[ \eta = \frac{W}{Q}, \]

where \( W \) is the mechanical (useful) energy which is produced and \( Q \) is the heat energy taken in at the high temperature \( T_1 \). Carnot gave an intuitive argument that his idealized machine has optimal efficiency. This argument can be formalized as the second law of thermodynamics.

The argument is as follows: Carnot’s engine can be run forward, as a heat engine turning heat into mechanical work, or backward as a refrigerator. Would there be a heat engine with better efficiency than Carnot’s heat engine, we could combine it with Carnot’s refrigerator. The outcome would be a machine which uses heat from a source at low temperature \( T_2 \), turns part of it into mechanical
work and releases another part at high temperature $T_1$. That this is impossible is intuitively obvious.

**More detailed description:** Suppose you have an amount of heat energy $Q$ at temperature $T_1$, and your surroundings have temperature $T_2 < T_1$. How much of the heat energy $Q$ can you turn into mechanical work? It turns out that the maximal amount is

$$W = \frac{T_1 - T_2}{T_1} Q .$$  

(10.4)

Sadi Carnot was the first to have the insight that only a certain part of heat energy can be turned into mechanical work. In addition, he described an idealized machine which has the optimal efficiency satisfying (10.4). Most remarkably, this happened before the principle of energy conservation was understood and absolute temperature was introduced. Kelvin used (10.4) to define his temperature scale. (In this way, the thermometer does not depend on any substance.) The ideas of Carnot lead to a formulation of the 2nd law of thermodynamics.

In the language of kinetic theory, heat is the random motion of a large number of particles. A major challenge is to understand (10.4) and the 2nd law of thermodynamics from the point of view of statistical mechanics. Work on this subject was initiate by Boltzmann.

**The Carnot Cycle** The cycle is drawn in a $(V,p)$ diagram, a visualization due to Emile Clapeyron.

We want to work out the details of a Carnot cycle for an ideal gas. We have

$$pV = RT$$
$$U = C_V T$$

where $p, V, T, U$ are the pressure, the volume, the absolute temperature, and the internal energy of the gas. $R$ and $C_V$ are positive constants of dimension energy/degree which depend on the gas, but not on $p, V, T, U$.

Before describing the Carnot cycle, we want to derive the equation

$$p V^\gamma = \text{const}, \quad \gamma = 1 + \frac{R}{C_V},$$

for an adiabatic expansion. Consider a path

$$(p(t), V(t)), \quad 0 \leq t \leq 1,$$

which describes an adiabatic expansion. We have

$$R U' = C_V R T' = C_V (p' V + p V').$$

Also, at every instant,

$$dU = -pdV,$$

since the work $pdV$ done by the gas has during the expansion has to reduce the internal energy $U$; this is because the expansion is assumed to be adiabatic, i.e., without energy exchange with the surroundings.
Obtain:

\[-RpV' = RU' = CV(p'V + pV'),\]

thus

\[\frac{p'}{p} + (1 + \frac{R}{CV}) \frac{V'}{V} = 0,\]

thus

\[(\ln p)' + \gamma(\ln V)' = 0.\]

Integration yields that

\[pV\gamma = \text{const}\]
during an adiabatic expansion. From \(pV\gamma = \text{const}\) and \(p = RT/V\) obtain

\[TV\gamma^{-1} = \text{const}\] (10.5)
during an adiabatic process.

**The Carnot Cycle.** The cycle consists of four processes. In each process the state of the machine, specified by \((V, p)\), changes slowly. The illustration of the cycle in a \((V, p)\) diagram is due to Emile Clapeyron (1799–1864).

**Process 1:** isothermal expansion from \((V_1, p_1)\) to \((V_2, p_2)\) at temperature \(T_1\).

The work done by the machine is (use that \(p = RT_1/V\)):

\[W_1 = \int_0^1 p(t)V'(t) \, dt = RT_1 \int_0^1 v'/V \, dt = RT_1 \ln(V_2/V_1)\]

This work equals the heat energy absorbed by the machine at temperature \(T_1\):

\[W_1 = Q_1 = RT_1 \ln(V_2/V_1)\]

Since the temperature \(T_1\) does not change, the internal energy \(U\) does not change during the process.

**Process 2:** adiabatic expansion from \((V_2, p_2)\) to \((V_3, p_3)\); no heat exchange. The gas cools to a temperature \(T_2 < T_1\). The expansion still performs mechanical work. The energy for the work is provided by the internal energy, which drops by

\[U_2 = CV(T_1 - T_2)\]
Also, because of (10.5):
\[ T_1 V_2^{\gamma - 1} = T_2 V_3^{\gamma - 1}. \]  
(10.6)

**Process 3:** isothermal compression from \((V_3, p_3)\) to \((V_4, p_4)\) at temperature \(T_2\).

Similarly as in Process 1, we can compute the mechanical work that needs to be provided. It is
\[ W_3 = RT_2 \ln(V_3/V_4). \]
Since the compression occurs at constant temperature \(T_2\), this energy equals the heat energy released by the machine to the surroundings.

**Process 4:** adiabatic compression from \((V_4, p_4)\) to \((V_1, p_1)\); no heat exchange. The gas heats from temperature \(T_2\) to \(T_1\). The internal energy rises by
\[ U_2 = C_V(T_1 - T_2). \]

Also, because of (10.5):
\[ T_2 V_4^{\gamma - 1} = T_1 V_1^{\gamma - 1}. \]  
(10.7)

This ends the Carnot cycle.

Because of (10.6) and (10.7) one obtains
\[ \frac{V_2}{V_1} = \frac{V_3}{V_4}. \]

The efficiency \(\eta\) of the machine is defined as the ratio of the total mechanical work done by the machine divided by the heat energy taken from the hot source. We have
\[ \eta \quad \begin{align*} = & \quad \frac{RT_1 \ln(V_2/V_1) - RT_2 \ln(V_3/V_4)}{RT_1 \ln(V_2/V_1)} \\ = & \quad \frac{T_1 - T_2}{T_1} \end{align*} \]

The point of Carnot’s machine is that it goes only through states of thermodynamic equilibrium; therefore, it can be run forward and backwards through the cycle. In the way we have described it (forward), the mechanical work performed by the machine is
\[ W = W_1 - W_3. \]

The machine absorbs heat energy \(Q_1 = W_1\) (at temperature \(T_1\)) and releases heat energy \(Q_3 = W_3\) at temperature \(T_2\). Conservation of energy holds:
\[ W_1 - W_3 = Q_1 - Q_3. \]
Would there be a more efficient machine, one could combine it with Carnot’s machine run backwards (as a refrigerator) and deliver heat from the cold source (at \( T_2 \)) to the hot source (at \( T_1 \)) without doing any work. This is not reasonable. In fact, that this is impossible is called the 2nd law of thermodynamics. This argument implies that Carnot’s machine is the most efficient.

**Illustration.** To illustrate the formula

\[
\eta = \frac{T_1 - T_2}{T_1}
\]

for the efficiency of Carnot’s idealized machine, assume that

\[ T_1 = 373K, \quad T_2 = 273K, \]

i.e., \( T_1 \) corresponds to the boiling point and \( T_2 \) to the freezing point of water. The efficiency is

\[
\eta = \frac{100}{373} = 0.2681.
\]

It is clear that one can raise the efficiency by increasing \( T_1 \).

It is not possible to build practical machines that reach the optimal efficiency. In fact, to reach 30\% of optimal efficiency, is already nontrivial from an engineering point of view.
11 Projects On Deterministic Differential and Difference Equations

11.1 Fibonacci Sequence
The ansatz $F_n = r^n$ leads to

$$r^2 - r - 1 = 0, \quad r_1 = \frac{1}{2}(1 + \sqrt{5}), \quad r_2 = \frac{1}{2}(1 - \sqrt{5}).$$

Since $r_1 r_2 = -1$ it is easy to see that

$$-1 < r_2 < 0 < 1 < r_1.$$

The general form

$$F_{n+1} = ar_1^n + br_2^n, \quad n = 0, 1, \ldots$$

and the initial condition

$$F_1 = F_2 = 1$$

leads to

$$F_{n+1} = \frac{1}{2\sqrt{5}} \left( (\sqrt{5} + 1) \left( \frac{\sqrt{5} + 1}{2} \right)^n + (\sqrt{5} - 1)(-1)^n \left( \frac{\sqrt{5} - 1}{2} \right)^n \right).$$

Since $|r_2| < 1$ one finds that, for large $n$:

$$F_{n+1} \sim \frac{\sqrt{5} + 1}{2\sqrt{5}} r_1^n = \frac{\sqrt{5} + 1}{2\sqrt{5}} e^{\alpha n}$$

with

$$\alpha = \ln r_1.$$

11.2 Radiocarbon Dating
This is an application of exponential decay, $u' = -bu$.

Let’s first explain the physical basis for radiocarbon dating, which was discovered by W. Libley in 1949. Carbon atoms come in two forms, called isotopes: $^{12}C$ and $^{14}C$. The $^{12}C$ atom has 6 protons and 6 neutrons in its nucleus; the $^{14}C$ atom has 6 protons and 8 neutrons; there is no chemical difference between the atoms, but $^{14}C$ is radioactive with a half-life of $T = 5,568$ years. It is important to note that $^{14}C$ is formed in the upper atmosphere through cosmic radiation, and the quotient

$$q = \text{amount of } ^{14}C / \text{amount of } ^{12}C$$

is constant in time in the atmosphere.
As long as a plant lives, its carbon obeys the quotient $q$. When it dies, $^{14}\text{C}$ decays through radioactive decay and is not reformed since the cosmic radiation does not reach the plant.

With a measuring instrument you measure

$$M_1 = 6.68$$

disintegrations per gram of carbon per minute from a living plant. A piece of old wood excavated at Mount Ararat gave a count of

$$M_2 = 5.96$$
disintegrations per gram of carbon per minute. How old is the wood? (See [Hale, Kocak, p. 21].)

### 11.3 Logistic Growth

Consider the logistic differential equation,

$$u' = r\left(1 - \frac{u}{K}\right)u$$

with $r > 0, K > 0$. Determine its fixed points and their stability. Plot some solutions.

### 11.4 The Delayed Logistic Map

A discrete version of the logistic differential equation is given by the delayed logistic map,

$$v_{n+1} = \lambda v_n(1 - v_{n-1}), \quad \lambda > 0.$$  

Determine the fixed points and their stability. Plot some trajectories and observe the Neimark–Sacker bifurcation at $\lambda = 2$. Observe the invariant circle and its breaking as $\lambda$ increases. (The details of the breaking process are complicated and not completely understood.)

### 11.5 Hysteresis

1) Consider the equation

$$\frac{du}{dt} = u(1 + u)(1 - u) + \lambda, \quad u(0) = u_0,$$

where $\lambda$ is a real parameter.

a) For which values of $\lambda$ does the equation have one, two, or three fixed points? Discuss the stability of the fixed points.

b) Sketch the fixed points as functions of $\lambda$.

2) Consider the equation

$$\frac{du}{dt} = \frac{1}{\varepsilon} \left( u(1 + u)(1 - u) + \lambda(t) \right), \quad u(0) = u_0,$$
where $\varepsilon > 0$ and where $\lambda(t)$ is a slowly varying function that you may choose.

By solving the equation numerically, try to demonstrate the hysteresis phenomenon.

3) Consider the differential system

\[
\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - (u_1^2 + u_2^2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
\]

Experiment with this system numerically for $-1 \leq \lambda \leq 1$ and starting points near the origin. For which values of $\lambda$ do you find the origin to be stable? What happens with the dynamics if the origin is unstable?
12 Projects on Gambler’s Ruin

12.1 Numerical Simulations

Implement the game and take the values
\[ p = 0.49, \quad q = 0.51, \quad N = 100, \quad k = 50, \]
for example. By running simulations, try to get approximate answers for the following:

a) What is the probability that ruin occurs at some time in the future? (This means that \(X_n = 0\) for some \(n\).)

b) What is the probability that winning occurs at some time in the future? (This means that \(X_n = N = 100\) for some \(n\).)

c) How long, on average, will the game last until either ruin or winning has occurred?

12.2 Derivation of Formulas

Carry out the details of the following: Fix \(p, q = 1 - p\), and \(N\). Let \(\alpha_k\) denote the probability that ruin occurs at some time in the future if our initial capital is \(X_0 = k\). We have
\[ \alpha_0 = 1, \quad \alpha_N = 0. \]

If \(1 \leq k \leq N - 1\) and \(X_0 = k\) then
\[ X_1 = k + 1 \quad \text{with probability } p \]
\[ X_1 = k - 1 \quad \text{with probability } q \]

If \(X_1 = k + 1\) then the probability of ruin at some time in the future is \(\alpha_{k+1}\).
If \(X_1 = k - 1\) then the probability of ruin at some time in the future is \(\alpha_{k-1}\).

We then convince ourselves that
\[ \alpha_k = p\alpha_{k+1} + q\alpha_{k-1}. \]

This is a homogeneous, second order difference equation,
\[ p\alpha_{k+1} - \alpha_k + q\alpha_{k-1} = 0, \quad 1 \leq k \leq N - 1. \]

We can solve such an equation using the ansatz
\[ \alpha_k = \rho^k. \]

One obtains a quadratic for \(\rho\). If \(p \neq \frac{1}{2}\) one finds two distinct solutions for \(\rho\).

Then, using the boundary conditions (12.2), one obtains
\[ \alpha_k = \frac{(q/p)^N - (q/p)^k}{(q/p)^N - 1} \quad \text{if } \ p \neq \frac{1}{2} \]

and
\[ \alpha_k = \frac{N - k}{N} \quad \text{if} \quad p = \frac{1}{2}. \]

Remark: We can also obtain the vector \( \alpha \) by solving a linear system \( D\alpha = e^{(0)} \) where the matrix \( D \) is obtained from \( P^T \) in a simple way.

Compute the \( \alpha \) vector for the data (12.1) and compare with the result obtained by numerical simulations.

12.3 Expected Time for Game

Carry out the details.

Fix \( p, q = 1 - p, \) and \( N \). Let \( \tau_k \) denote the expected time for ruin or winning to occur if \( X_0 = k \). We have

\[ \tau_0 = \tau_N = 0. \]

Convince yourself that the following difference equation is reasonable:

\[ \tau_k = p(1 + \tau_{k+1}) + q(1 + \tau_{k-1}), \quad 1 \leq k \leq N - 1. \]

Derive an explicit formula for \( \tau_k \).

Evaluate your formula for the data (12.1) and compare with the result obtained by numerical simulations.
13 Projects on the Simple Birth Process

1. Run some stochastic realizations of the birth process with

\[ b = 1, \quad X_0 = 1, \]

where you compute the interevent times \( \tau_0, \tau_1, \ldots \) by

\[ \tau_j = \frac{\ln(1/Y)}{1 + j}, \quad 0 < Y = rand < 1. \]

Plot the corresponding piecewise constant random variable \( X_t \).

2. In the same plot show the expected value

\[ \mu_t = E(X_t) = e^t \]

and

\[ \mu_t \pm \sigma_t^2 \]

where

\[ \sigma_t^2 = e^t(e^t - 1) \]

is the variance of \( X_t \).

3. We have computed (for \( b = 1, X_0 = k = 1 \)):

\[ p_j(t) = \text{prob}(X_t = j) = e^{-jt}(e^t - 1)^{j-1}, \quad j = 1, 2, \ldots \]

Fix \( t = 3 \), for example, and plot \( p_j(t) \) as a function of \( j \). Try to confirm the result by stochastic realizations and many evaluations of \( X_3 \).
14 Comments on Projects

14.1 Carbon Dating

For $u' = -bu$ obtain

$$u(t) = u_0 e^{-bt}.$$  

The half-life $T = 5568$ years determines the decay rate $b$:

$$u(T) = \frac{1}{2} u_0 = u_0 e^{-bT},$$  

thus

$$e^{-bT} = \frac{1}{2}, \quad b = \frac{\ln 2}{T}.$$  

If $t$ is the age of the wood, obtain

$$6.68 e^{-bt} = 5.96,$$  

thus

$$e^{bt} = 6.68/5.96,$$  

thus

$$t = \frac{1}{b} \ln(6.68/5.96) = T \cdot \frac{\ln(6.68) - \ln(5.96)}{\ln(2)}.$$  

14.2 Gambler’s Ruin

The quadratic equation

$$pp^2 - p + q = 0$$

has the solutions

$$\rho_{1,2} = \frac{1}{2p} \pm \sqrt{\frac{1}{4p^2} - \frac{q}{p}}$$

$$= \frac{1}{2p} \pm \sqrt{\frac{1 - 4pq}{4p^2}}.$$  

Here

$$1 = (p + q)^2 = p^2 + 2pq + q^2,$$  

thus

$$1 - 4pq = (p - q)^2.$$  

Therefore,
\[ \rho_{1,2} = \frac{1}{2p} \pm \frac{|p - q|}{2p} . \]

**Case 1:** \( p \neq \frac{1}{2} \). In this case

\[ \rho_1 = \frac{1}{2p} (p + q + p - q) = 1 \]

and

\[ \rho_2 = \frac{1}{2p} (p + q - p + q) = q/p . \]

The general solution of the difference equation is

\[ \alpha_k = c_1 + c_2 (q/p)^k . \]

Matching the boundary conditions yields the formula for \( \alpha_k \).

**Case 2:** \( p = \frac{1}{2} \). In this case

\[ \rho_{1,2} = 1 \]

is a double root of the quadratic. The general solution of the difference equation is

\[ \alpha_k = c_1 + c_2 k . \]

Matching the boundary conditions yields

\[ \alpha_k = \frac{N - k}{N} . \]

**Example 1:** Let

\[ p = q = \frac{1}{2}, \quad N = 100, \quad k = 50. \]

The probability of ruin at some time in the future is

\[ \alpha_{50} = \frac{1}{2} . \]

**Example 2:** Let

\[ p = .49, \quad q = .51, \quad N = 100, \quad k = 50. \]

Let \( Q = .51/.49 \). The probability of ruin at some time in the future is

\[ \alpha_{50} = \frac{Q^{100} - Q^{50}}{Q^{100} - 1} \sim .880825???. \]

**14.3 Gambler’s Winning**

It is interesting to note that \( \alpha_k + \beta_k = 1 \). Thus, the probability for the game to last forever is 0, though this is clearly possible if \( N \geq 4 \).
14.4 Expected Time for Game

We assume here that $p \neq \frac{1}{2}$.

The difference equations for $\tau_k$ are

$$p\tau_{k+1} - \tau_k + q\tau_{k-1} = -1$$

with boundary conditions

$$\tau_0 = \tau_N = 0.$$

To find a special solution of the inhomogeneous difference equations, let

$$\tau_k = ck$$

where $c$ needs to be determined. One obtains

$$pc(k + 1) - ck + qc(k - 1) = -1,$$

thus

$$c = \frac{1}{q - p}.$$

Then the general solution of the difference equations reads

$$\tau_k = c_1 + c_2 \left(\frac{q}{p}\right)^k + \frac{k}{q - p}.$$

Using the boundary conditions, one finds that

$$\tau_k = \frac{k}{q - p} - \frac{N}{q - p} \cdot \left(\frac{q}{p}\right)^k - 1.$$

For example, if

$$N = 1,000, \quad k = 500, \quad p = 0.49, \quad q = 0.51,$$

the formula yields

$$\tau_{500} = 24,999.99989\ldots$$

It is quite difficult to obtain this accuracy by simulations.