1. Consider an initial value problem for a parabolic system of the form

\[ u_t = \varepsilon u_{xx} + A(x,t)u_x + B(x,t)u + F(x,t) \quad \text{for} \quad x \in \mathbb{R}, \quad t \geq 0, \quad u(x,0) = f(x) \quad \text{for} \quad x \in \mathbb{R} \]

where \( \varepsilon > 0 \) and \( A(x,t) = A^*(x,t) \). (Thus, for \( \varepsilon = 0 \), the system is symmetric hyperbolic.)

Assume that all functions are \( C^\infty \) and 1–periodic in \( x \).

We know that for \( \varepsilon > 0 \) there exists a unique smooth solution \( u_\varepsilon(x,t) \) for \( 0 \leq t < \infty \).

a) Prove an energy estimate for \( u_\varepsilon(x,t) \) with a bound independent of \( \varepsilon > 0 \).

b) Prove bounds for all derivatives of \( u_\varepsilon(x,t) \) with constants independent of \( \varepsilon > 0 \).

c) Consider the limit \( \varepsilon \to 0^+ \). Prove that the solution \( u_\varepsilon(x,t) \) and its derivatives converge to a solution of the symmetric hyperbolic system obtained for \( \varepsilon = 0 \).

d) Prove that the solution of the initial value problem for the symmetric hyperbolic system is unique.

2. Consider an initial value problem for a strongly parabolic system of the form

\[ u_t = A(x,t)u_{xx} \quad \text{for} \quad x \in \mathbb{R}, \quad t \geq 0, \quad u(x,0) = f(x) \quad \text{for} \quad x \in \mathbb{R} \]

where

\( A(x,t) + A^*(x,t) \geq 2\delta I, \quad \delta > 0 \).

Assume that all functions are \( C^\infty \) and 1–periodic in \( x \).

Fix any \( T > 0 \). By considering

\[ \frac{d}{dt} \left( t^2 u_{xx}(\cdot,t), u_{xx}(\cdot,t) \right) \]

and using the estimates proved in class for

\[ \frac{d}{dt} \left( u(\cdot,t), u(\cdot,t) \right) \]

and

\[ \frac{d}{dt} \left( tu_x(\cdot,t), u_x(\cdot,t) \right) \]

prove a solution estimate of the form

\[ \max_{0 \leq t \leq T} t^2 \| D^2 u(\cdot,t) \|^2 \leq C_T \| f \|^2 \quad \text{where} \quad D = \partial / \partial x. \]
Here $C_T$ is a constant independent of $f$. Note that the bound does not use derivatives of $f$.

3. Under the same assumptions as in Problem 2, prove a solution estimate of the form

$$\max_{0 \leq t \leq T} t^p \|D^p u(\cdot, t)\|^2 \leq C_T \|f\|^2$$

where $D = \partial / \partial x$

for all $p = 3, 4, \ldots$ Here $C_T$ is a constant which will depend on $p$, but will be independent of $f$.

4. Let $A \in \mathbb{C}^{n \times n}$ denote a matrix satisfying

$$A + A^* \geq 2\delta I > 0.$$ 

Prove that $A$ is nonsingular and

$$|A^{-1}| \leq \frac{1}{\delta}$$

where $|\cdot|$ is the matrix norm corresponding to the Euclidean vector norm.