1. Consider the Cauchy problem

\[ u_t = Au_x, \quad u(x, 0) = f(x), \]

where \( A \in \mathbb{C}^{n \times n} \) is a constant matrix. Assume that all eigenvalues of \( A \) are real. Prove an estimate of the form

\[ \|u(\cdot, t)\| \leq Ke^{\alpha t}\|f\|_{H^q} \quad \text{for} \quad t \geq 0 \]

for all \( f \in M_0 \). Here \( K \) and \( q \) are independent of \( f \). How is the best \( q \) related to the Jordan form of \( A \)?

2. Let \( S \in \mathbb{C}^{n \times n} \) denote a nonsingular matrix with

\[ |S| \leq K_1 \quad \text{and} \quad |S^{-1}| \leq K_1. \]

For the Hermitian matrix \( H = S^*S \) prove an estimate of the form

\[ \frac{1}{K_2} I \leq H \leq K_2 I. \]

The meaning of this estimate is

\[ \frac{1}{K_2}|a|^2 \leq a^*Ha \leq K_2|a|^2 \quad \text{for all} \quad a \in \mathbb{C}^n. \]

3. Let

\[ u_t = \sum_{j=1}^{N} A_j D_j u + B u \]

denote a strongly hyperbolic system as defined in class. Prove that the Cauchy problem for the system is well-posed.

4. Consider a first order constant coefficient system

\[ u_t = \sum_{j=1}^{N} A_j D_j u \]

where \( A_j \in \mathbb{C}^{n \times n} \). Assume that there exists a vector \( k \in \mathbb{R}^N \) for which the matrix \( \sum_{j=1}^{N} k_j A_j \) has a non-real eigenvalue. Prove that the Cauchy problem for the system is not weakly well-posed. I.e., an estimate of the form

\[ \|u(\cdot, t)\| \leq Ke^{\alpha t}\|f\|_{H^q} \quad \text{for} \quad t \geq 0 \]

with \( K, \alpha, q \) independent of \( f \in M_0 \) does not hold.