MATH 584, Methods of Applied Mathematics II, Spring 2016
Prof. J. Lorenz, Instructor
Homework 2, assigned Feb. 10; due Feb. 24

1. Let

$$\Omega = \{(x,y) \ : \ 0 \le x \le L, \ 0 \le y \le 1\} \ .$$

Consider the 2D stationary incompressible Navier–Stokes equations in Ω with boundary conditions

$$u(x,0) = v(x,0) = u(x,1) = v(x,1) = 0 \quad \text{for} \quad 0 \le x \le L \ .$$

The boundary conditions correspond to walls at the top and the bottom of Ω. Assume that the flow is driven by the velocity gradient

$$p_x(x,y) \equiv \frac{p_L - p_0}{L}, \quad p_y(x,y) \equiv 0 \quad \text{in} \quad \Omega \ .$$

This correcponds to a pressure $p(0,y) = p_0$ at inflow and a pressure $p(L,y) = p_L$ at outflow, where we assume that $p_L < p_0$.

Determine a corresponding solution

$$\Big(u(x,y), v(x,y)\Big)$$

of the velocity field in Ω. (I do not know if the solution is unique, but there is a rather simple solution.)

2. Consider the wave equation in three space dimensions:

$$\phi_{tt} = c^2(\phi_{xx} + \phi_{yy} + \phi_{zz}) \ .$$

Proceed as in class and write the equation as a first–order symmetric hyperbolic system.

3. Consider the scalar PDE

$$u_t = a u_{xx} + 2b u_{xy} + c u_{yy}$$

where a, b, c are real constants.

Under what conditions on a, b, c can you prove an estimate

$$|e^{\hat{P}(ik)t}| \le C \ ,$$

where $C > 0$ is a constant independent of $t \ge 0$ and $k \in \mathbb{R}^2$?

Hint: The eigenvalues of the matix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

play a role.

4. We have shown in class how the complex function

$$F(z) = z + \frac{1}{z} = \phi(x, y) + i\psi(x, y), \quad |z| \geq 1 \ ,$$

gives us an irrotational, stationary solution

$$\Big(u(x, y), v(x, y), p(x, y)\Big)$$

of the incompressible Euler equations in the region

$$x^2 + y^2 \geq 1 \ .$$

Determine the points (x, y) where the speed is minimal and the points where the speed is maximal.