1) Let $D$ denote the open unit disk and let $f \in \text{Aut}(D)$ have two distinct fixed points $z_1, z_2 \in D$:

$$f(z_1) = z_1, \quad f(z_2) = z_2, \quad z_1 \neq z_2.$$ 

Prove that $f$ is the identity, i.e, $f(z) \equiv z$.

**Hint:** First assume $z_1 = 0$.

**Solution:**

a) Assume first that $f(0) = 0, f(z_2) = z_2$ where $z_2 \neq 0$. We know from Schwarz Lemma that $f$ is a rotation, $f(z) = \alpha z$. Since

$$z_2 = f(z_2) = \alpha z_2, \quad z_2 \neq 0,$$

we obtain that $\alpha = 1$.

b) Recall that the Möbius transformation

$$\phi_c(z) = \frac{z - c}{1 - cz}, \quad z \in \mathbb{D},$$

is an automorphism of $\mathbb{D}$. Define

$$g = \phi_{z_1} \circ f \circ \phi_{-z_1}.$$ 

Obtain

$$g(0) = \phi_{z_1} \circ f(z_1) = \phi_{z_1}(z_1) = 0.$$ 

Also, if we set

$$z_3 = \phi_{z_1}(z_2)$$

then $z_3 \neq 0$ and $\phi_{-z_1}(z_3) = z_2$. Therefore,

$$g(z_3) = \phi_{z_1} \circ f(z_2) = \phi_{z_1}(z_2) = z_3.$$ 

We have shown that

$$g \in \text{Aut}(\mathbb{D}), \quad g(0) = 0, \quad g(z_3) = z_3 \neq 0.$$ 

Applying the result of a), we obtain $g = id$. This implies that $f = id$.

2) Let $f : \mathbb{H} \to \mathbb{D}$ be a biholomorphic map from the open upper half-plane $\mathbb{H}$ onto the open unit disk $\mathbb{D}$. Use your knowledge of $\text{Aut}(\mathbb{D})$ to prove that $f$ has the form

$$f(z) = e^{i\theta} \frac{z - P}{z - \overline{P}}$$

for some $P \in \mathbb{H}$ and some real $\theta$. 

Solution: Set $P = f^{-1}(0)$ and define
\[ g(z) = \frac{z - P}{z - \overline{P}}. \]
We know that $g : \mathbb{H} \to \mathbb{D}$ is biholomorphic. Define $h = g \circ f^{-1}$. Then $h \in Aut(\mathbb{D})$ and $h(0) = g(P) = 0$. Therefore,
\[ h = R_\alpha, \quad g \circ f^{-1} = R_\alpha, \quad f = R^{-1}_\alpha \circ g. \]

3) Consider the map
\[ z \to \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = -\frac{1}{2}(ie^{iz} + \frac{1}{ie^{iz}}) \]
on the half–strip
\[ S = \{ z = x + iy : -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad y > 0 \} . \]
By considering the auxiliary maps
\[
\begin{align*}
  z & \rightarrow ie^{iz} = w \\
  w & \rightarrow w + \frac{1}{w} = q \\
  q & \rightarrow -\frac{q}{2}
\end{align*}
\]
determine the image of $S$ under the map
\[ z \rightarrow \sin z \]
Also, determine the image of the boundary of $S$ under the map $z \rightarrow \sin z$.

Solution: a) If $z \in S$ then
\[ z = x + iy, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad y > 0 \]
The point
\[ e^{iz} = e^{ix}e^{-y} \]
lies in the open right half–disk. Multiplying by $i$, one obtains the open upper half–disk,
\[ \mathbb{D}_{\text{up}} = \mathbb{D} \cap \mathbb{H} . \]
The map
\[ z \rightarrow ie^{iz} \]
from $S$ to $\mathbb{D}_{\text{up}}$ is 1-1 and onto.

b) Consider the map
$w \rightarrow w + \frac{1}{w}$

for $w \in \mathbb{D}_{up}$. 

If $w \in \mathbb{D}_{up}$ then

$$w = re^{i\phi} = rc + irs, \quad 0 < r < 1, \quad 0 < \phi < \pi.$$ 

We have

$$w + \frac{1}{w} = (r + \frac{1}{r})c + i(r - \frac{1}{r})s$$

where

$$s = \sin \phi > 0 \quad \text{and} \quad r - \frac{1}{r} < 0.$$ 

It follows that

$$q = w + \frac{1}{w} \in -\mathbb{H},$$

where $-\mathbb{H}$ is the open lower half-plane. We claim that the map

$$w \rightarrow w + \frac{1}{w}$$

from $\mathbb{D}_{up}$ to $-\mathbb{H}$ is 1-1 and onto.

Let $q \in -\mathbb{H}$ be given. The equation

$$w + \frac{1}{w} = q$$

is equivalent to

$$w^2 - qw + 1 = 0$$

with solutions

$$w_{1,2} = \frac{q}{2} \pm \sqrt{\frac{q^2}{4} - 1}.$$ 

It follows that

$$w_1 = w_2$$

if and only if $q = \pm 2i$. Since $2i \notin -\mathbb{H}$ and $-2i \notin -\mathbb{H}$ one obtains that $w_1 \neq w_2$ if $q \in -\mathbb{H}$.

We have

$$w_1 w_2 = 1, \quad w_1 + w_2 = q$$

and may assume that

$$|w_1| \leq 1 \leq |w_2|.$$
Suppose that $|w_1| = 1$. Then we have

$$w_1 = e^{i\phi}, \quad w_2 = e^{-i\phi},$$

thus

$$q = w_1 + w_2 \in \mathbb{R}.$$ 

This contradicts the assumption that $q \in -\mathbb{H}$. It follows that

$$|w_1| < 1 < |w_2|$$

and we write

$$w_1 = re^{i\phi} = r(c + is) \quad \text{with} \quad 0 < r < 1.$$ 

From

$$q = w_1 + \frac{1}{w_1} = (r + \frac{1}{r})c + i(r - \frac{1}{r})s$$

and

$$r - \frac{1}{r} < 0 \quad \text{and} \quad \text{Im} \ q < 0$$

it follows that

$$s = \sin \phi > 0.$$ 

Therefore,

$$0 < \phi < \pi.$$ 

We have shown that

$$w_1 = re^{i\phi} \in \mathbb{D}_{up}.$$ 

c) Clearly, the map

$$q \rightarrow -\frac{q}{2}$$

from $-\mathbb{H}$ to $\mathbb{H}$ is 1-1 and onto.

Together, the map

$$z \rightarrow \sin z$$

from $S$ to $\mathbb{H}$ is 1-1 and onto.

**Mapping of the boundary:** The boundary of $S$ consists of three parts:
We apply the map $z \to \sin z$ in three steps, as above.

**A:**

\[
\begin{align*}
z &= -\frac{\pi}{2} + iy, \quad 0 \leq y < \infty \\
w &= ie^{iz} = e^{-y} \in (0, 1] \\
q &= w + \frac{1}{w} \in [2, \infty) \\
\sin z &= -\frac{q}{2} \in (-\infty, -1]
\end{align*}
\]

**B:**

\[
\begin{align*}
z &= x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\
w &= ie^{ix} \quad \text{in upper semi–circle} \\
q &= w + \frac{1}{w} \in [-2, 2] \\
\sin z &= -\frac{q}{2} \in [-1, 1]
\end{align*}
\]

**C:**

\[
\begin{align*}
z &= \frac{\pi}{2} + iy, \quad 0 \leq y < \infty \\
w &= ie^{iz} = -e^{-y} \in [-1, 0) \\
q &= w + \frac{1}{w} \in (-\infty, -2] \\
\sin z &= -\frac{q}{2} \in [1, \infty)
\end{align*}
\]

One obtains that the boundary of $S$ gets mapped to the real line, which is the boundary of $\mathbb{H}$.

4) Let

\[Q_1 = \{x = x + iy : x > 0, y > 0\} \quad \text{and} \quad Q_2 = \{x = x + iy : x < 0, y > 0\}\]

denote the open 1st and 2nd quadrants. Let

\[F(z) = \frac{i - z}{i + z}\]

denote the Cayley transform. Determine the sets $F(Q_1)$ and $F(Q_2)$ and sketch them in the complex plane.
**Solution:** First let \( z = x + iy \) with \( y > 0 \). We then have

\[ |i - z| < |i + z| , \]

thus \( F(z) \in \mathbb{D} \). Next, let \( z = x + iy \in Q_1 \), thus \( y > 0 \) and \( x > 0 \).

Setting

\[ \gamma = (i + z)^{-2} > 0 , \]

we have

\[
F(z) = \frac{i - z}{i + z} = \frac{i - z}{i + z} \cdot \frac{-i + z}{-i + z} = \gamma(i - z)(-i + \overline{z}) = \gamma(1 - |z|^2 + i(z + \overline{z})) = \gamma(1 - |z|^2 + 2ix)
\]

Since \( x > 0 \) it follows that \( F(z) \) lies in the open upper half-plane \( \mathbb{H} \), \( F(z) \in \mathbb{D} \cap \mathbb{H} \). Also, if \( z \in Q_2 \) then \( x < 0 \) and \( F(z) \in \mathbb{D} \cap (-\mathbb{H}) \). Since \( F \) is a bijection, one obtains that

\[
F(Q_1) = \mathbb{D} \cap \mathbb{H} \quad \text{and} \quad F(Q_2) = \mathbb{D} \cap (-\mathbb{H}) .
\]