1) Recall the definition of the function space

\[ X_{\text{per}} = \{ f : f : \mathbb{R} \to \mathbb{C}, f(x + 1) \equiv f(x), f \in C \} . \]

Let \( f \in X_{\text{per}} \) and assume that the Fourier coefficients satisfy the decay estimate

\[ |\hat{f}(k)| \leq \frac{A}{|k|^{r+1+\varepsilon}}, \quad k \in \mathbb{Z}, \quad k \neq 0 , \]

where \( r \geq 1 \) is an integer, \( \varepsilon > 0 \), and \( A \geq 0 \). Prove that \( f \in C^r \).

Hint: First let \( r = 1 \) and consider \((S_n f)'(x)\).

2) For \( f, g \in X_{\text{per}} \) define the convolution product by

\[ h(x) = (f * g)(x) = \int_0^1 f(x - y)g(y) \, dy, \quad x \in \mathbb{R} . \]

a) Prove that \( h \in X_{\text{per}} \).

b) How are the Fourier coefficients \( \hat{h}(k) \) related to \( \hat{f}(k) \) and \( \hat{g}(k) \)?

3) Let \( \mathbb{D} \) denote the open unit disk,

\[ \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} . \]

Determine the solutions

\[ u \in C^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}}) \]

of the following Dirichlet problems:

a)

\[ \Delta u = 0 \quad \text{in} \quad \mathbb{D}, \quad u(z) = \frac{1}{z} \quad \text{on} \quad \partial \mathbb{D} . \]

b)

\[ \Delta u = 0 \quad \text{in} \quad \mathbb{D}, \quad u(z) = \frac{1}{z\overline{z}} \quad \text{on} \quad \partial \mathbb{D} . \]

c)

\[ \Delta u = 0 \quad \text{in} \quad \mathbb{D}, \quad u(z) = \frac{1}{z^2} \quad \text{on} \quad \partial \mathbb{D} . \]

Hint: All solutions can be obtained by intelligent guessing.

4a) Check that the Poisson kernel

\[ P(r, \phi) = \frac{1 - r^2}{1 - 2r \cos \phi + r^2}, \quad 0 \leq r < 1, \quad \phi \in \mathbb{R} , \]
has the antiderivative

\[ Q(r, \phi) = 2 \arctan \left( \frac{1 + r}{1 - r} \tan(\phi/2) \right), \]

i.e., check that \( \partial Q/\partial \phi = P. \)

b) Consider the Dirichlet problem

\[ \Delta u = 0 \quad \text{in} \quad D, \quad u(z) = f(z) \quad \text{on} \quad \partial D \]

where \( f(z) \) is the discontinuous function

\[ f(e^{i\phi}) = \begin{cases} 
0 & \text{for} \quad 0 < \phi < \pi \\
1 & \text{for} \quad \pi < \phi < 2\pi 
\end{cases} \]

Proceed formally, as in class, to obtain a solution formula for \( u(r, \phi) \), ignoring that \( f \) is discontinuous.

c) For extra credit, use a computer to sketch the solution. For example, sketch lines of constant of \( u \) or try a 3D plot.