Functions of a Complex Variable II
Math 562, Spring 2018
Jens Lorenz
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Department of Mathematics and Statistics,
UNM, Albuquerque, NM 87131

Contents

1 Notations 5

2 The Schwarz Lemma and $Aut(\mathbb{D})$ 5
  2.1 Schwarz Lemma ............................................. 5
  2.2 Biholomorphic Maps and Automorphisms ................. 6
  2.3 The Automorphism Group of $\mathbb{D}$ ..................... 7
  2.4 The Schwarz–Pick Lemma ................................ 9
  2.5 Remarks .................................................... 10

3 Linear Fractional Transformations and the Riemann Sphere 12
  3.1 The Riemann Sphere ....................................... 12
  3.2 Linear Fractional Transformations ........................ 12
  3.3 The Automorphisms of the Riemann Sphere ............... 14
  3.4 A Remarkable Geometric Property of Linear Fractional Transformations ............................................. 16
    3.4.1 Analytical Description of Circles .................... 16
    3.4.2 Circles under Reciprocation .......................... 17
    3.4.3 Straight Lines under Reciprocation .................. 18
  3.5 Example: The Cayley Transform in the Complex Plane .... 19
  3.6 The Cayley Transform of a Matrix ....................... 20

4 The Riemann Mapping Theorem 22
  4.1 Statement and Overview of Proof .......................... 22
  4.2 The Theorem of Arzela–Ascoli ............................ 24
  4.3 Montel’s Theorem ......................................... 26
  4.4 Auxiliary Results on Logarithms and Square Roots ....... 29
  4.5 Construction of a Map in $\mathcal{F}$ ...................... 30
  4.6 A Bound for $|f'(0)|$ .................................... 32
  4.7 Application of the Theorems of Montel and Hurwitz .... 33
  4.8 Proof That $f$ is Onto .................................... 34
4.9 Examples of Biholomorphic Mappings .................................. 36

5 An Introduction to Schwarz–Christoffel Formulas .................. 37
5.1 General Power Functions ........................................... 37
5.2 An Example of a Schwarz–Christoffel Map ....................... 38

6 Meromorphic Functions with Prescribed Poles ................. 42
6.1 Meromorphic Functions on \( \mathbb{C} \) .................................. 42
6.2 The Mittag–Leffler Theorem ..................................... 42
6.3 An Example .......................................................... 45

7 Infinite Products ......................................................... 47
7.1 Infinite Products of Complex Numbers ......................... 47
  7.1.1 Two Definitions of Convergence ............................. 47
  7.1.2 Examples ....................................................... 48
7.2 Infinite Products of Numbers: Convergence Theory .......... 51
7.3 Infinite Products of Functions .................................. 56
7.4 Example: The Product Formula for the Sine Function ...... 59
7.5 Euler’s Constant .................................................... 60
7.6 The Gauss Formula for \( \Gamma(z) \) and Weierstrass’ Product Formula
  for \( 1/\Gamma(z) \) .................................................. 61
  7.6.1 A Formula for \( (\Gamma'/\Gamma)'/\Gamma \) ......................... 65
7.7 Entire Functions with Prescribed Zeros ......................... 65
  7.7.1 Construction of \( f(z) \); Motivation ......................... 66
  7.7.2 Weierstrass’ Canonical Factors ........................... 67

8 The Bernoulli Numbers and Applications ....................... 72
8.1 The Bernoulli Numbers ........................................... 72
8.2 The Taylor Series for \( z \cot \pi z \) in Terms of Bernoulli Numbers ................................. 74
8.3 The Mittag–Leffler Expansion of \( \pi z \cot(\pi z) \) .............. 75
8.4 The Values of \( \zeta(2m) \) ............................................. 76
8.5 Sums of Powers and Bernoulli Numbers ....................... 76

9 The Riemann Zeta–Function ........................................... 79
9.1 Definition for \( \Re z > 1 \) ............................................. 79
9.2 Simple Bounds for \( s > 1 \) .......................................... 79
9.3 Meromorphic Continuation of \( \zeta(z) \) to \( \Re z > 0 \) .......... 80
9.4 Analytic Continuation of \( \zeta(z) \) to \( \mathbb{C} \setminus \{1\} \) ........ 82
9.5 Euler’s Product Formula for \( \zeta(z) \) ............................. 85
9.6 The Sum \( \sum \frac{1}{p} \) and the Prime Number Theorem .......... 87
9.7 A Functional Equation ......................................... 89
9.8 Growth Estimates for the \( \zeta \)–Function .......................... 90

10 Analytic Continuation .................................................. 94
10.1 Analytic Continuation Using the Cauchy Riemann Equations .... 94
10.2 Exponential Decay of Fourier Coefficients and the Strip of Analyticity ..................................... 94
10.3 The Schwarz Reflection Principle ................................ 94
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.4</td>
<td>Examples for Analytic Continuation</td>
<td>94</td>
</tr>
<tr>
<td>10.5</td>
<td>Riemann Surfaces: Intuitive Approach</td>
<td>94</td>
</tr>
<tr>
<td>10.6</td>
<td>Riemann Surfaces: Germs, Sheafs, and Fibers</td>
<td>94</td>
</tr>
<tr>
<td>11</td>
<td>Fourier Series</td>
<td>95</td>
</tr>
<tr>
<td>11.1</td>
<td>Convergence Results: Overview</td>
<td>95</td>
</tr>
<tr>
<td>11.2</td>
<td>The Dirichlet Kernel</td>
<td>97</td>
</tr>
<tr>
<td>11.3</td>
<td>The Fejér Kernel</td>
<td>98</td>
</tr>
<tr>
<td>11.4</td>
<td>Convergence of $\sigma_n f$ in Maximum Norm</td>
<td>100</td>
</tr>
<tr>
<td>11.5</td>
<td>Weierstrass’ Approximation Theorem for Trigonometric Polynomials</td>
<td>101</td>
</tr>
<tr>
<td>11.6</td>
<td>Convergence of $S_n f$ in $L_2$</td>
<td>101</td>
</tr>
<tr>
<td>11.7</td>
<td>The Lebesgue Constant of the Dirichlet Kernel</td>
<td>103</td>
</tr>
<tr>
<td>11.7.1</td>
<td>The Lebesgue Constant as Norm of a Functional</td>
<td>105</td>
</tr>
<tr>
<td>11.8</td>
<td>Divergence of the Fourier Series of a Continuous Function at a Point</td>
<td>105</td>
</tr>
<tr>
<td>11.9</td>
<td>Isomorphy of $L_2(0,1)$ and $l_2$</td>
<td>107</td>
</tr>
<tr>
<td>11.10</td>
<td>Convergence of $S_n f$ in Maximum Norm</td>
<td>109</td>
</tr>
<tr>
<td>11.11</td>
<td>Smoothness of $f(x)$ and Decay of $\hat{f}(k)$</td>
<td>110</td>
</tr>
<tr>
<td>11.12</td>
<td>Exponential Decay of $\hat{f}(k)$ and Analyticity of $f$</td>
<td>112</td>
</tr>
<tr>
<td>11.13</td>
<td>Divergence of $S_n f(0)$: Explicit Construction of $f$</td>
<td>114</td>
</tr>
<tr>
<td>11.14</td>
<td>Fourier Series and the Dirichlet Problem for Laplace’s Equation on the Unit Disk</td>
<td>116</td>
</tr>
<tr>
<td>12</td>
<td>Fourier Transformation</td>
<td>117</td>
</tr>
<tr>
<td>12.1</td>
<td>Motivation: Application to PDEs</td>
<td>117</td>
</tr>
<tr>
<td>12.2</td>
<td>The Fourier Transform of an $L_1$ Function</td>
<td>120</td>
</tr>
<tr>
<td>12.3</td>
<td>The Riemann–Lebesgue Lemma</td>
<td>121</td>
</tr>
<tr>
<td>12.4</td>
<td>The Fourier Transform on the Schwartz Space $S$</td>
<td>123</td>
</tr>
<tr>
<td>12.5</td>
<td>The Fourier Inversion Formula on the Schwartz Space: Preparations</td>
<td>124</td>
</tr>
<tr>
<td>12.6</td>
<td>The Fourier Inversion Formula on the Schwartz Space</td>
<td>127</td>
</tr>
<tr>
<td>12.7</td>
<td>Operators</td>
<td>128</td>
</tr>
<tr>
<td>12.8</td>
<td>Elementary Theory of Tempered Distributions</td>
<td>129</td>
</tr>
<tr>
<td>12.8.1</td>
<td>Ordinary Functions as Tempered Distributions</td>
<td>130</td>
</tr>
<tr>
<td>12.9</td>
<td>The Fourier Transform on $S'$: An Example Using Complex Variables</td>
<td>130</td>
</tr>
<tr>
<td>12.10</td>
<td>Decay of the Fourier Transform of $f$ and Analyticity of $f$</td>
<td>131</td>
</tr>
<tr>
<td>12.11</td>
<td>The Paley–Wiener Theorem</td>
<td>131</td>
</tr>
<tr>
<td>12.12</td>
<td>The Laplace Transform and Its Inversion</td>
<td>132</td>
</tr>
<tr>
<td>13</td>
<td>Growth and Zeros of Entire Functions</td>
<td>133</td>
</tr>
<tr>
<td>13.1</td>
<td>Jensen’s Formula</td>
<td>133</td>
</tr>
<tr>
<td>13.2</td>
<td>The Order of Growth</td>
<td>135</td>
</tr>
<tr>
<td>13.3</td>
<td>Zeros and Growth Estimates</td>
<td>136</td>
</tr>
<tr>
<td>13.4</td>
<td>Hadamard’s Factorization Theorem</td>
<td>138</td>
</tr>
<tr>
<td>13.5</td>
<td>Entire Functions of Non–Integer Order of Growth</td>
<td>141</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>13.6</td>
<td>Proof of Hadamard’s Factorization Theorem</td>
<td>141</td>
</tr>
<tr>
<td>14</td>
<td>The Prime Number Theorem</td>
<td>148</td>
</tr>
<tr>
<td>14.1</td>
<td>Functions</td>
<td>148</td>
</tr>
<tr>
<td>14.2</td>
<td>Reduction to Asymptotics of $\psi_1(x)$</td>
<td>150</td>
</tr>
<tr>
<td>14.3</td>
<td>Integral Representation of $\psi_1(x)$</td>
<td>153</td>
</tr>
<tr>
<td>14.4</td>
<td>Auxiliary Results</td>
<td>153</td>
</tr>
<tr>
<td>14.5</td>
<td>The $\zeta$-Function has no Zero on the Line $s = 1 + it$</td>
<td>155</td>
</tr>
<tr>
<td>15</td>
<td>Complex Variables and PDEs</td>
<td>158</td>
</tr>
<tr>
<td>15.1</td>
<td>2D Irrotational Euler Flows</td>
<td>158</td>
</tr>
<tr>
<td>15.2</td>
<td>Laplace Equation</td>
<td>163</td>
</tr>
<tr>
<td>15.2.1</td>
<td>Derivation of the Poisson Kernel via Complex Variables</td>
<td>163</td>
</tr>
<tr>
<td>15.2.2</td>
<td>Derivation of the Poisson Kernel via Separation of Variables</td>
<td>167</td>
</tr>
</tbody>
</table>
1 Notations

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{R} )</td>
<td>field of real numbers</td>
</tr>
<tr>
<td>( \mathbb{C} )</td>
<td>field of complex numbers</td>
</tr>
<tr>
<td>( D(z_0, R) )</td>
<td>{ ( z :</td>
</tr>
<tr>
<td>( \overline{D}(z_0, R) )</td>
<td>{ ( z :</td>
</tr>
<tr>
<td>( \partial D(z_0, R) )</td>
<td>{ ( z :</td>
</tr>
<tr>
<td>( \mathbb{D} )</td>
<td>( \mathbb{D} = D(0, 1) ) : open unit disk</td>
</tr>
<tr>
<td>( \mathbb{H} )</td>
<td>( \mathbb{H} = { z = a + bi : b } ) : open upper half-plane</td>
</tr>
<tr>
<td>( H(U) )</td>
<td>set of all holomorphic functions ( f : U \to \mathbb{C} ) where ( U \subset \mathbb{C} ) is open</td>
</tr>
<tr>
<td>( C(U) )</td>
<td>set of all continuous functions ( f : U \to \mathbb{C} ) where ( U \subset \mathbb{C} ) is any set</td>
</tr>
<tr>
<td>( \ln(x) )</td>
<td>for real positive ( x ): ( \ln(x) = \int_1^x \frac{ds}{s} )</td>
</tr>
<tr>
<td>( \log(z) )</td>
<td>complex logarithm</td>
</tr>
</tbody>
</table>

2 The Schwarz Lemma and \( \text{Aut}(\mathbb{D}) \)

2.1 Schwarz Lemma

Let \( \mathbb{D} = D(0, 1) \) denote the open unit disk. In Schwarz’ lemma one considers functions \( f \in H(\mathbb{D}) \) obeying \( f(0) = 0 \) and \( f(\mathbb{D}) \subset \overline{\mathbb{D}} \). Schwarz Lemma then says that the estimate \( |f(z)| \leq 1 \) can be sharpened.

We recall the maximum modulus theorem, which will be used in the proof of the Schwarz Lemma.

**Theorem 2.1 (Maximum Modulus Theorem)** Let \( U \subset \mathbb{C} \) denote an open, connected, and bounded set. Let \( g \in H(U) \cap C(\overline{U}) \) and set

\[
M = \max \{|g(z)| : z \in \partial U\}.
\]

Then

\[
|g(z)| < M \quad \text{for all} \quad z \in U
\]

unless \( g \) is constant.

**Theorem 2.2 (Schwarz Lemma)** Let \( f \in H(\mathbb{D}) \) satisfy

a) \( |f(z)| \leq 1 \) for all \( z \in \mathbb{D} \);

b) \( f(0) = 0 \).

Then \( |f(z)| \leq |z| \) for all \( z \in \mathbb{D} \) and \( |f'(0)| \leq 1 \). In addition, if

\[
|f(z_0)| = |z_0|
\]

for some \( z_0 \in \mathbb{D} \setminus \{0\} \) or if \( |f'(0)| = 1 \), then \( f \) is a rotation, i.e.,

\[
f(z) = \alpha z
\]

for some \( \alpha \in \mathbb{C} \) with \( |\alpha| = 1 \).
Proof: Set

\[ g(z) = \begin{cases} \frac{f(z)}{z}, & 0 < |z| < 1 \\ f'(0), & z = 0 \end{cases} \]

Since \( f(0) = 0 \) we have

\[ f(z) = \sum_{j=1}^{\infty} a_j z^j \quad \text{for} \quad |z| < 1 \]

and obtain that \( g \in H(\mathbb{D}) \). Let \( 0 < \varepsilon < 1 \) and consider \( g \) in the closed disk \( \bar{D}(0, 1-\varepsilon) \subset \mathbb{D} \). Since \( |f(z)| \leq 1 \) in \( \mathbb{D} \) one obtains that

\[ |g(z)| \leq \frac{1}{1-\varepsilon} \quad \text{for} \quad |z| = 1 - \varepsilon . \]

By the maximum modulus theorem we conclude that

\[ |g(z)| \leq \frac{1}{1-\varepsilon} \quad \text{for} \quad |z| \leq 1 - \varepsilon . \]

As \( \varepsilon \to 0 \) this yields:

\[ |g(z)| \leq 1 \quad \text{for} \quad |z| < 1 . \]

Therefore,

\[ |f(z)| \leq |z| \quad \text{for} \quad |z| < 1 \quad \text{and} \quad |f'(0)| \leq 1 . \]

Now assume that \( |f(z_0)| = |z_0| \) for some \( z_0 \in \mathbb{D}, z_0 \neq 0 \), or assume that \( |f'(0)| = 1 \). Then, using the definition of \( g \), we have \( |g(z_0)| = 1 \) or \( |g(0)| = 1 \). In both cases, the absolute value of \( g(z) \) attains a maximum at a point in \( \mathbb{D} \), which implies that \( g(z) \equiv \alpha \) is a constant, \( |\alpha| = 1 \). Therefore, \( f(z) = \alpha z \). \( \diamond \)

2.2 Biholomorphic Maps and Automorphisms

Terminology: A nonempty, open, connected subset \( \Omega \subset \mathbb{C} \) is called a region.

Let \( U, V \) be regions in \( \mathbb{C} \). A map \( f : U \to V \) is called biholomorphic if \( f \) is \( 1-1 \) and onto and \( f \) as well as \( f^{-1} \) are holomorphic. It is sufficient to assume that \( f : U \to V \) is holomorphic, \( 1-1 \) and onto. Then \( f^{-1} \) is automatically holomorphic, as we have proved earlier. We also recall that \( f'(z) \neq 0 \) for all \( z \in U \) if \( f \) is biholomorphic.

Definition: Let \( U \) be a region in \( \mathbb{C} \). If \( f : U \to U \) is \( 1-1 \), onto, and holomorphic (thus biholomorphic), then \( f \) is called an automorphism of \( U \). The set of all automorphisms of \( U \) is denoted by \( \text{Aut}(U) \). This set forms a group if the product of \( f, g \in \text{Aut}(U) \) is defined as the composition, \( f \circ g \).

Example 1: According to Homework 5, Problem 2, the set \( \text{Aut}(\mathbb{C}) \) consists of all functions \( f(z) \) of the form \( f(z) = az + b \) where \( a \neq 0 \).
2.3 The Automorphism Group of $\mathbb{D}$

We first consider some special Möbius transformations. Let $c \in \mathbb{C}$ and $|c| < 1$. Then set

$$\phi_c(z) = \frac{z - c}{1 - \overline{c}z}, \quad z \in \mathbb{D}.$$  

Clearly, $\phi_c \in H(D)$ for $|c| < 1$. Note that

$$\phi_c(z) \equiv z \quad \text{for} \quad c = 0.$$

We prove:

**Lemma 2.1**

a) $|\phi_c(z)| < 1$ for all $z \in \mathbb{D}$.

b) $\phi_c(\phi_{-c}(z)) = z$ for all $z \in \mathbb{D}$.

c) $\phi_c \in \text{Aut}(\mathbb{D})$ for $|c| < 1$.

**Proof:**

a) Clearly, $|\phi_c(z)| < 1$ is equivalent to

$$|1 - \overline{c}z|^2 > |z - c|^2,$$

which is equivalent to

$$(1 - \overline{c}z)(1 - cz) > (z - c)(\overline{z} - \overline{c}),$$

which is equivalent to

$$1 + |c|^2|z|^2 > |z|^2 + |c|^2. \quad (2.1)$$

Since

$$(1 - |c|)(1 - |z|) > 0$$

the inequality (2.1) holds.

b) With

$$\phi_c(w) = \frac{w - c}{1 - \overline{c}w}, \quad \phi_{-c}(z) = \frac{z + c}{1 + \overline{c}z}$$

we have

$$\phi_c(\phi_{-c}(z)) = \frac{\frac{z + c}{1 + \overline{c}z} - c}{1 - \overline{\frac{z + c}{1 + \overline{c}z}}} = \frac{z + c - c - z|c|^2}{1 + \overline{c}z - \overline{c}z - |c|^2} = z,$$

c) Since $\phi_c(\phi_{-c}(z)) = z$ for all $z \in \mathbb{D}$, we obtain that $\phi_c$ is onto and $\phi_{-c}$ is $1 - 1$.

Replacing $c$ by $-c$ shows that $\phi_c$ is a bijection of $\mathbb{D}$. \phi

Another set of automorphisms of $\mathbb{D}$ are the rotations $R_\alpha$ defined by
\[ R_\alpha(z) = \alpha z, \quad |z| < 1, \]

for \(|\alpha| = 1\). Clearly, \(R_\alpha \in \text{Aut}(\mathbb{D})\) and

\[ \phi_c \circ R_\alpha \in \text{Aut}(\mathbb{D}) \quad (2.2) \]

whenever \(|c| < 1, |\alpha| = 1\). The following theorem says that every \(f \in \text{Aut}(\mathbb{D})\) has the form (2.2).

**Theorem 2.3** Let \(f \in \text{Aut}(\mathbb{D})\) and let \(b = f(0)\). Then we have

\[ f = \phi_{-b} \circ R_\alpha = R_\alpha \circ \phi_{-\bar{b}\alpha} \quad (2.3) \]

for some \(\alpha\) with \(|\alpha| = 1\).

**Proof:** 1) First assume that \(f(0) = 0\) and let \(g = f^{-1}\). By Schwarz Lemma we conclude that \(|f'(0)| \leq 1\) and \(|g'(0)| \leq 1\). From \(f(g(z)) = z\) for all \(z \in \mathbb{D}\) we have

\[ f'(g(z))g'(z) = 1, \]

and, in particular,

\[ f'(0)g'(0) = 1. \]

It follows that \(|f'(0)| = 1\). Another application of Schwarz Lemma yields \(f(z) = \alpha z\). Equation (2.3) holds with \(b = 0\).

2) Let \(b := f(0)\) and consider \(h = \phi_b \circ f\). We have

\[ h(0) = \phi_b(b) = 0. \]

By Part 1) of the proof we conclude that \(h = R_\alpha\) for some \(\alpha\) with \(|\alpha| = 1\). Therefore, \(f = \phi_{-b} \circ R_\alpha\). In other words,

\[ f(z) = \phi_{-b}(\alpha z) = \phi_{-b} \circ R_\alpha(z). \]

3) The equations

\[
\begin{align*}
    f(z) &= \phi_{-b}(\alpha z) \\
    &= \frac{\alpha z + b}{1 + b\bar{\alpha}z} \\
    &= \frac{\alpha z + b\bar{\alpha}}{1 + b\bar{\alpha}z} \\
    &= \alpha \phi_{-\bar{b}\alpha}(z)
\end{align*}
\]

complete the proof of the theorem. \(\diamondsuit\)
2.4 The Schwarz–Pick Lemma

The automorphisms $\phi_c$ of $\mathbb{D}$, obtained for every $c$ with $|c| < 1$, allow to prove an extension of the Schwarz lemma. Note that

$$\phi_c(z) = \frac{z - c}{1 - \overline{c}z}, \quad \phi'_c(z) = \frac{1 - |c|^2}{(1 - \overline{c}z)^2}.$$ 

**Theorem 2.4** (Schwarz–Pick) Let $f : \mathbb{D} \to \mathbb{D}, f \in H(\mathbb{D})$. (It is neither assumed that $f$ is $1 - 1$ nor that $f$ is onto nor that $f(0) = 0$.)

1) For all $a \in \mathbb{D}$:

$$|f'(a)| \leq \frac{1 - |f(a)|^2}{1 - |a|^2} \quad (2.4)$$

(If $a = 0$ and $f(0) = 0$ this reduces to the estimate $|f'(0)| \leq 1$ of the Schwarz lemma.)

2) If $a_1, a_2 \in \mathbb{D}$ then

$$\frac{|f(a_2) - f(a_1)|}{1 - \overline{f(a_1)f(a_2)}} \leq \frac{|a_2 - a_1|}{1 - \overline{a_1a_2}}. \quad (2.5)$$

**Proof:** 1) Let $b = f(a)$ and set

$$F = \phi_b \circ f \circ \phi_{-a}.$$ 

Then we have $\phi_{-a}(0) = a$ and

$$F(0) = \phi_b(f(a)) = \phi_b(b) = 0.$$ 

The Schwarz lemma is applicable to $F$ and implies that

$$|F'(0)| \leq 1.$$ 

Further, by the chain rule,

$$F'(0) = \phi'_b(b)f'(a)\phi'_{-a}(0).$$

Here

$$\phi'_{-a}(0) = 1 - |a|^2$$

and

$$\phi'_b(b) = \frac{1 - |b|^2}{(1 - |b|^2)^2} = \frac{1}{1 - |b|^2}.$$ 

Therefore,
The estimate \(|F'(0)| \leq 1\) yields that

\[ |f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}. \]

This proves (2.4).

2) Let \(b_1 = f(a_1)\) and \(b_2 = f(a_2)\). Consider the function

\[ F = \phi_{b_1} \circ f \circ \phi_{-a_1}. \]

As above, we have \(F(0) = 0\) and the Schwarz lemma yields that \(|F(z)| \leq |z|\) for all \(z \in \mathbb{D}\). Setting

\[ w = \phi_{-a_1}(z), \quad z = \phi_{a_1}(w), \]

the estimate \(|F(z)| \leq |z|\) becomes

\[ |\phi_{b_1}(f(w))| \leq |\phi_{a_1}(w)|. \]

Using the definition of \(\phi_c\) this reads:

\[ \left| \frac{f(w) - b_1}{1 - \overline{b_1}f(w)} \right| \leq \left| \frac{w - a_1}{1 - \overline{a_1}w} \right|. \]

If we use this estimate for \(w = a_2\) we obtain (2.5). ♦

2.5 Remarks

The formula

\[ d(a, b) = \frac{|b - a|}{1 - \overline{a}b} \quad a, b \in \mathbb{D}, \]

defines the so-called pseudo–hyperbolic metric on \(\mathbb{D}\). (This is related to Poincaré geometry on \(\mathbb{D}\).) The second part of the Schwarz–Pick Lemma then says that

\[ d(f(a), f(b)) \leq d(a, b) \quad \text{for all} \quad a, b \in \mathbb{D} \]

if \(f \in H(\mathbb{D}), f : \mathbb{D} \to \mathbb{D}\). If \(f \in \text{Aut}(\mathbb{D})\) then one can apply the estimate to \(f\) and \(f^{-1}\) to obtain

\[ d(f(a), f(b)) = d(a, b) \quad \text{for all} \quad a, b \in \mathbb{D}. \quad (2.6) \]

In fact, one can prove a converse: If \(f \in H(\mathbb{D}), f : \mathbb{D} \to \mathbb{D},\) and if (2.6) holds, then \(f \in \text{Aut}(\mathbb{D})\).

The introduction of non–Euclidean geometries was historically very important. Recall: Euclid of Alexandria (\(\sim 365–300\) B.C.) and his Parallel Postulate, the famous fifth Postulate.
Immanuel Kant: *The concept of Euclidean space is by no means of empirical origin, but is an inevitable necessity of thought.* (Critique of Pure Reason, 1781)

Work of Carl Friedrich Gauss (1777–1855) and Janos Bolyai (1802–1860) on non-Euclidean geometry made it clear that Kant’s position was wrong. One has to distinguish between geometry as a mathematical subject and geometry of physical space, which is a subject of observation. Newtonian physics assumes Euclidean space and a time variable independent of space. To create the theory of general relativity, it was necessary to overcome the doctrine of Euclidean space.
3 Linear Fractional Transformations and the Riemann Sphere

3.1 The Riemann Sphere

It is often useful to compactify the complex plane $\mathbb{C}$ by formally adding the point $\infty$. We use the notation

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$ 

One can turn $\hat{\mathbb{C}}$ into a topological space by using an identification (this is nothing but a map which is 1-1 and onto) with the unit sphere in $\mathbb{R}^3$,

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$ 

This identification can be established as follows: Let $N = (0, 0, 1) \in S$ denote the north pole of $S$ and let

$$z = x + iy \in \mathbb{C}.$$ 

draw the straight line from $N$ to $(x, y, 0)$. It intersects $S$ in a unique point, which we call $\Pi(z)$, the stereographic projection of $z$ to $S$. The map $\Pi$ maps $\mathbb{C}$ bijectively onto $S \setminus \{N\}$:

$$\Pi : \mathbb{C} \rightarrow S \setminus \{N\}.$$ 

As $|z| \rightarrow \infty$, one obtains that $\Pi(z) \rightarrow N$ on $S$. It is therefore natural to extend $\Pi$ as a mapping from $\hat{\mathbb{C}}$ to $S$ by defining

$$\Pi(\infty) = N = (0, 0, 1).$$

In this way we have defined a bijection

$$\Pi : \hat{\mathbb{C}} \rightarrow S.$$ 

The set $S$ is a metric space if we use the Euclidean distance in $\mathbb{R}^3$ as distance between points in $S$. We know, then, what it means that a sequence $q_n \in S$ converges to $q \in S$. We know what the open and the closed sets in $S$ are etc. Using the map $\Pi$ we transform the concept of convergence etc. (or the metric) to $\hat{\mathbb{C}}$. If the point $\infty$ is not involved, these concepts agree with the standard concepts in $\mathbb{C}$, but they are now extended in a meaningful way to $\hat{\mathbb{C}}$. The space $\hat{\mathbb{C}}$ is called the one–point compactification of $\mathbb{C}$, or the Riemann sphere.

We mention that one can also consider $\hat{\mathbb{C}}$ as a complex manifold by introducing local coordinates near the point $\infty \in \hat{\mathbb{C}}$. If one has done this, then the notion of holomorphy of a function $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ makes sense.

3.2 Linear Fractional Transformations

Let
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det(A) \neq 0,

denote a nonsingular matrix in \( \mathbb{C}^{2\times2} \). Then A determines the linear fractional transformation

\[ \phi_A(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0. \quad (3.1) \]

(If one would allow \( \det(A) = 0 \) then the transformation \( \phi_A(z) \) would be constant, which is an uninteresting transformation.) The transformations (3.1) are also called Möbius transformations. Note that \( A \) and \( qA, q \neq 0 \), determine the same transformation, \( \phi_A = \phi_{qA} \).

It will be convenient to consider \( \phi_A \) as a bijection of \( \hat{\mathbb{C}} \) onto itself. Then one finds that \( \phi_A \) is an automorphism of \( \hat{\mathbb{C}} \).

**Case 1:** \( c = 0 \). In this case \( d \neq 0 \) and \( a \neq 0 \) and

\[ \phi_A(z) = \frac{a}{d} z + \frac{b}{d}. \]

The transformation \( \phi_A : \mathbb{C} \to \mathbb{C} \) is \( 1 - 1 \) and onto. As \( z_n \to \infty \), we have \( \phi_A(z_n) \to \infty \). Therefore, we set \( \phi_A(\infty) = \infty \) and obtain a continuous bijection of \( \mathbb{C} \). If one considers \( \hat{\mathbb{C}} \) as a one–dimensional complex manifold, then this bijection is holomorphic.

**Case 2:** \( c \neq 0 \). In this case, if \( z_0 = -d/c \), then \( \phi_A(z_0) \) is not defined as a complex number. If \( z_n \to z_0 = -d/c \) then \( \phi_A(z_n) \to \infty \). Therefore, we set

\[ \phi_A(-d/c) = \infty. \quad (3.2) \]

In this way, \( \phi_A \) is continuously extended to \( z_0 = -d/c \). Also, if \( z_n \to \infty \), then

\[ \phi_A(z_n) \to a/c. \]

Therefore, we set

\[ \phi_A(\infty) = a/c. \quad (3.3) \]

In this way, \( \phi_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is defined as a continuous map, which is holomorphic.

We claim that, with the additional definitions, (3.2) and (3.3), the map \( \phi_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is a bijection and its inverse is \( \phi_A^{-1} \). This claim follows from the following important lemma.

**Lemma 3.1** Let

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \]

denote nonsingular matrices in \( \mathbb{C}^{2\times2} \) and let \( C \) denote their product,

\[ C = AB = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}. \]
Then we have

\[ \phi_C = \phi_{AB} = \phi_A \circ \phi_B . \]

**Proof:** In all cases where the point \( \infty \) is not involved we have

\[ \phi_B(z) = \frac{\alpha z + \beta}{\gamma z + \delta} =: w \]

and

\[ \phi_A(w) = \frac{aw + b}{cw + d} = \frac{a(\alpha z + \beta) + b(\gamma z + \delta)}{c(\alpha z + \beta) + d(\gamma z + \delta)} = \frac{(ac + b\gamma)z + a\beta + b\delta}{(c\alpha + d\gamma)z + c\beta + d\delta} = \phi_C(z) \]

This proves that

\[ \phi_{AB}(z) = \phi_A \circ \phi_B(z) \]

when \( \infty \) is not involved. By continuous extension, the equation

\[ \phi_{AB}(z) = \phi_A \circ \phi_B(z), \quad z \in \hat{\mathbb{C}}, \]

follows. \( \diamond \)

### 3.3 The Automorphisms of the Riemann Sphere

Let \( M \) denote the group of all Möbius transformations \( \phi_A(z) \). With \( GL(2, \mathbb{C}) \) one denotes the group of all nonsingular matrices in \( \mathbb{C}^{2 \times 2} \). The previous lemma says that

\[ \Phi : \begin{cases} GL(2, \mathbb{C}) & \mapsto M \\ A & \mapsto \phi_A(z) \end{cases} \quad (3.4) \]

is a group homomorphism. Clearly, \( \Phi \) is onto.

**Lemma 3.2** We have

\[ M = \text{Aut}(\hat{\mathbb{C}}) , \]

thus every automorphism of \( \hat{\mathbb{C}} \) is a Möbius transformation.

**Proof:** Let \( f \in \text{Aut}(\hat{\mathbb{C}}) \). a) Assume that \( f(\infty) = \infty \). In this case, the restriction of \( f \) to \( \mathbb{C} \) is an automorphism of \( \mathbb{C} \). We have shown that

\[ f(z) = az + b, \quad a \neq 0 . \]
In particular, \( f \in M \).

b) Let \( f(\infty) = b, b \in \mathbb{C} \). Consider
\[
\phi(z) = \frac{1}{z - b}
\]
and define \( g = \phi \circ f \). Since \( g \) is an automorphism of \( \hat{\mathbb{C}} \) and \( g(\infty) = \infty \), we have \( g \in M \) and then \( f = \phi^{-1} \circ g \in M \).

Thus we have shown that
\[
\Phi : \{ GL(2, \mathbb{C}) \mapsto Aut(\hat{\mathbb{C}}) \}
\]
is a group epimorphism. (A homomorphism, which is onto, is called an epimorphism.)

By definition, the kernel \( ker(\Phi) \) of the epimorphism \( \Phi \) defined in (3.5) consists of all matrices \( A \in GL(2, \mathbb{C}) \) for which the Möbius transformation \( \phi_A(z) \) is the unit element of the group \( Aut(\hat{\mathbb{C}}) \), i.e., \( A \in ker(\Phi) \) if and only if \( \phi_A(z) \equiv z \).

By a simple result of group theory, one obtains that \( Aut(\hat{\mathbb{C}}) \) is isomorphic to the quotient group
\[
GL(2, \mathbb{C})/ker(\Phi) .
\]

It is not difficult to see that \( \phi_A(z) \equiv z \) if and only if \( A = aI \) for some \( a \in \mathbb{C}, a \neq 0 \). Thus,
\[
ker(\Phi) = \{ aI : a \in \mathbb{C}, a \neq 0 \} .
\]

If one identifies two matrices \( A, B \in GL(2, \mathbb{C}) \) if and only if \( A = qB \) for some \( q \in \mathbb{C} \), then one obtains the group \( GL(2, \mathbb{C})/ker(\Phi) \). This group is isomorphic to \( Aut(\hat{\mathbb{C}}) \).

With
\[
SL(2, \mathbb{C}) = \{ A \in GL(2, \mathbb{C}) : det(A) = 1 \}
\]
one denotes the special linear group of all complex \( 2 \times 2 \) matrices with determinant 1. Then
\[
\Phi_S : \{ SL(2, \mathbb{C}) \mapsto Aut(\hat{\mathbb{C}}) \}
\]
is an epimorphism and
\[
ker(\Phi_S) = \{ I, -I \} .
\]

Therefore, \( Aut(\hat{\mathbb{C}}) \) is isomorphic to
\[
SL(2, \mathbb{C})/\{ I, -I \} .
\]
3.4 A Remarkable Geometric Property of Linear Fractional Transformations

Let $C$ denote any circle or straight line in the $z$–plane. We claim that, if $\phi_A(z)$ is any linear fractional transformation, then the image of $C$ under the transformation

$$ z \rightarrow \phi_A(z) = w $$

is a circle or a straight line in the $w$–plane. Since $\phi_A$ is a composition of transformations of the form

$$ z \rightarrow az, \quad z \rightarrow z + b, \quad z \rightarrow \frac{1}{z}, $$

and since the statement is easily shown for transformations $z \rightarrow az$ and $z \rightarrow z + b$, we consider the transformation

$$ z = x + iy \rightarrow \frac{1}{z} = w = u + iv. $$

3.4.1 Analytical Description of Circles

The circle $C = C(z_0, r)$ centered at $z_0$ with radius $r > 0$ has the equation

$$ |z - z_0|^2 = r^2, $$

or,

$$ z\bar{z} - z_0\bar{z}_0 - \bar{z}z_0 + A = 0 $$

with

$$ A = z_0\bar{z}_0 - r^2, \quad r^2 = z_0\bar{z}_0 - A. $$

One obtains:

**Lemma 3.3** Let $z_0 \in \mathbb{C}$ and let $A \in \mathbb{R}$. Then the equation

$$ z\bar{z} - z_0\bar{z}_0 - \bar{z}z_0 + A = 0 $$

describes a circle if and only if

$$ z_0\bar{z}_0 - A > 0. \quad (3.7) $$

Assuming (3.7) to hold, the circle is centered at $z_0$ and has radius $r = \sqrt{z_0\bar{z}_0 - A}$. The circle passes through the point $z = 0$ if and only if $A = 0$. 

16
3.4.2 Circles under Reciprocation

Let $C = C(z_0, r)$ denote the circle centered at $z_0$ with radius $r > 0$. Set $A = z_0\bar{z}_0 - r^2$.

**Case 1:** The circle does not pass through $z = 0$, i.e., $A \neq 0$. We apply the transformation

$$T(z) = \frac{1}{z} = w$$

to the points of $C$. The points $w \in T(C)$ satisfy

$$\frac{1}{w\bar{w}} - \frac{\bar{z}_0}{w} = \frac{z_0}{w} + A = 0,$$

or,

$$1 - \bar{w}z_0 - wz_0 + Aw\bar{w} = 0,$$

or,

$$w\bar{w} - w\frac{z_0}{\bar{A}} - w\frac{\bar{z}_0}{\bar{A}} + \frac{1}{\bar{A}} = 0.$$

The last equation has the form

$$w\bar{w} - w\bar{w}_0 - \bar{w}w_0 + B = 0$$

with

$$w_0 = \frac{\bar{z}_0}{\bar{A}}, \quad B = \frac{1}{\bar{A}}.$$

We have

$$w_0\bar{w}_0 - B = \frac{z_0\bar{z}_0}{A^2} - \frac{1}{\bar{A}}$$

$$= \frac{1}{A^2} (z_0\bar{z}_0 - A)$$

$$> 0.$$

Using the previous lemma, we see that $T(C)$ is the circle centered at

$$w_0 = \frac{\bar{z}_0}{\bar{A}}$$

with radius

$$R = \frac{r}{|A|}.$$

**Case 2:** The circle passes through $z = 0$, i.e., $A = 0$. We apply the transformation

$$T(z) = \frac{1}{z} = w$$
to the points of \( \mathcal{C} \setminus \{0\} \). The points \( w = T(z) \) satisfy

\[
\frac{1}{w} - \frac{\bar{z}_0}{w} - \frac{z_0}{w} = 0 ,
\]
or,

\[
1 - \bar{w}z_0 - wz_0 = 0 .
\]

If

\[
w = x + iy, \quad z_0 = a + ib ,
\]
(with real \( x, y, a, b \)) then the above equation reads

\[
1 - (x - iy)(a - ib) - (x + iy)(a + ib) = 0 ,
\]
or,

\[
1 - 2ax + 2by = 0 .
\]

We see that \( T(\mathcal{C} \setminus \{0\}) \) is a straight line. If we let \( T(0) = \infty \), then \( T(\mathcal{C}) \) is a straight line together with \( w = \infty \).

### 3.4.3 Straight Lines under Reciprocation

Let \( \alpha, \beta, \gamma \) be real numbers, \( (\alpha, \beta) \neq (0, 0) \). Then the equation

\[
\alpha x + \beta y + \gamma = 0
\]
describes a straight line \( \mathcal{L} \).

**Case 1:** The line \( \mathcal{L} \) does not pass through \( z = 0 \), i.e., \( \gamma \neq 0 \). We rewrite the above equation as

\[
1 + \frac{\alpha}{\gamma} x + \frac{\beta}{\gamma} y = 0 ,
\]
or,

\[
1 - 2ax + 2by = 0
\]
with

\[
-2a = \frac{\alpha}{\gamma}, \quad 2b = \frac{\beta}{\gamma} .
\]

We can also write the equation for \( \mathcal{L} \) as

\[
1 - (x - iy)(a - ib) - (x + iy)(a + ib) = 0 .
\]

Setting

\[
z_0 = a + ib
\]
we see that the equation for \( \mathcal{L} \) is
$$1 - \bar{z}z_0 - zz_0 = 0 \quad \text{with} \quad z = x + iy.$$  

It is then clear that $T(L)$ consists of the points $w$ with  

$$1 - \frac{\bar{z}_0}{w} - \frac{z_0}{w} = 0,$$  

or,  

$$w\bar{w} - w\bar{z}_0 - \bar{w}z_0 = 0.$$  

Thus, $T(L)$ is a circle, centered at $z_0$, that passed through $w = 0$.

**Case 2:** The line $L$ passes through $z = 0$, i.e., $\gamma = 0$. We could proceed as above, but here it is simpler to use the parametric description  

$$z = te^{i\theta}, \quad t \in \mathbb{R},$$  

of $L$. The points of $T(L)$ are  

$$w = \frac{1}{t}e^{-i\theta}.$$  

We see that $T(L)$ is another straight line through the origin.

### 3.5 Example: The Cayley Transform in the Complex Plane

Consider the Möbius transforms  

$$F(z) = \frac{i - z}{i + z} = \frac{-z + i}{z + i}, \quad A = \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix},$$  

and  

$$G(w) = i \frac{1 - w}{1 + w} = \frac{-iw + i}{w + 1}, \quad B = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}.$$  

The map $F$ is called the Cayley transform.

We have $\det(A) = -2i$ and  

$$A^{-1} = \frac{1}{-2i} \begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}.$$  

We see that $B$ is a scalar multiple of $A^{-1}$ and, therefore, $G$ is the inverse of $F$ if $F$ and $G$ are considered as functions on $\hat{\mathbb{C}}$. Here we use the extensions  

$$F(-i) = \infty, \quad F(\infty) = -1, \quad G(-1) = \infty, \quad G(\infty) = -i.$$  

Let  

$$\mathbb{H} = \{ z = x + iy : y > 0 \}$$  

denote the open upper half–plane and recall that $\mathbb{D}$ denotes the open unit circle.
Remark: Let $\overline{\mathbb{H}} = \mathbb{H} \cup \partial \mathbb{H}$ denote the closed upper half-plane. We have seen that $F$ maps $\overline{\mathbb{H}}$ bijectively onto $\overline{\mathbb{D}} \setminus \{-1\}$ and the inverse map is $G$. It is clear that one cannot obtain a continuous bijection between $\overline{\mathbb{H}}$ and $\overline{\mathbb{D}}$ since $\overline{\mathbb{D}}$ is compact and $\overline{\mathbb{H}}$ is not compact. Any continuous image of $\overline{\mathbb{D}}$ is also compact.

However, if we use the topology of $\hat{\mathbb{C}}$, then $\overline{\mathbb{H}} \cup \{\infty\}$ is a compact compact subset of $\hat{\mathbb{C}}$ and

$$F : \overline{\mathbb{H}} \cup \{\infty\} \to \overline{\mathbb{D}}$$

is an isomorphism between compact sets.

3.6 The Cayley Transform of a Matrix

We have seen that

$$F : \mathbb{C} \setminus \{-i\} \to \mathbb{C} \setminus \{-1\}, \quad F(z) = \frac{i - z}{i + z},$$

is 1-1 and onto with inverse

$$G : \mathbb{C} \setminus \{-1\} \to \mathbb{C} \setminus \{-i\}, \quad G(w) = i \frac{1 - w}{1 + w}.$$
is called the Cayley transform of \( A \).

In analogy to (3.8) we have the following result:

**Lemma 3.5** If \( A = A^* \) then \( V := F(A) \) is unitary and \( V = F(A) \neq -I \).

**Proof:** We have

\[
V^* = (-iI + A)^{-1}(-iI - A) = (iI + A)(iI - A)^{-1},
\]

thus

\[
V^*V = (iI + A)(iI + A)^{-1} = I.
\]

Suppose that

\[
V = F(A) = (iI - A)(iI + A)^{-1} = -I.
\]

Then one obtains that

\[
iI - A = -iI - A,
\]

a contradiction. \( \diamond \)

**Remarks:** The Cayley transform plays a role in functional analysis. One can use it to transform even unbounded Hermitian operators to unitary operators. Unitary operators are obviously bounded, and it is often easier to study them. Then the back transform can give information about unbounded Hermitian operators. This is used for spectral theory of unbounded operators.
4 The Riemann Mapping Theorem

4.1 Statement and Overview of Proof

Recall that $D$ denotes the open unit disk. Recall that we know the group $\text{Aut}(D)$: First, if $|c| < 1$ then the Möbius transformation

$$\phi_c(z) = \frac{z - c}{1 - \bar{c}z}, \quad |z| < 1,$$

is an automorphism of $D$ with inverse $\phi_{-c}$. Second, if $|\alpha| = 1$ then the rotation

$$R_\alpha(z) = \alpha z, \quad |z| < 1,$$

is an automorphism of $D$ with inverse $R_{\bar{\alpha}}$. Third, if $f \in \text{Aut}(D)$ is arbitrary and $b = f(0)$ then

$$f = \phi_{-b} \circ R_{\alpha}$$

for some $\alpha$ with $|\alpha| = 1$.

The Riemann mapping theorem is the following remarkable result:

**Theorem 4.1** Let $U \subset \mathbb{C}$ be open and simply connected and let $U \neq \mathbb{C}$. Then there exists a biholomorphic map $f : U \to D$.

A sharper result, containing a uniqueness statement, is formulated in Theorem 4.2. If Theorem 4.1 is known, then we can prove Theorem 4.2 rather easily using our knowledge of $\text{Aut}(D)$.

**Theorem 4.2** Let $U \subset \mathbb{C}$ be open and simply connected and let $U \neq \mathbb{C}$. Let $P \in U$ be any fixed point in $U$. Then there exists a unique biholomorphic map $f : U \to D$ satisfying

$$f(P) = 0, \quad f'(P) > 0. \quad (4.1)$$

**Proof:** Existence of $f$: By Theorem 4.1 there exists a biholomorphic $F : U \to D$. Set $c = F(P)$ and recall that

$$\phi_c(z) = \frac{z - c}{1 - \bar{c}z}, \quad z \in D,$$

is an automorphism of $D$ with $\phi_c(c) = 0$. Set

$$g = \phi_c \circ F.$$

Then $g : U \to D$ is biholomorphic and $g(P) = 0$. Write

$$g'(0) = re^{i\gamma} \quad \text{where} \quad r > 0 \quad \text{and} \quad \gamma \in \mathbb{R},$$

and set

$$\alpha = e^{-i\gamma}.$$
Define \( f = R_\alpha \circ g \), thus
\[
  f(z) = \alpha g(z), \quad f'(z) = \alpha g'(z).
\]
We have
\[
  f'(0) = \alpha g'(0) = r > 0.
\]
The function
\[
  f = R_\alpha \circ \phi_c \circ F : U \to \mathbb{D}
\]
is biholomorphic and satisfies (4.1).

**Uniqueness of \( f \):** Suppose that \( f_1 \) and \( f_2 \) satisfy the conditions of the theorem and set \( h = f_1 \circ f_2^{-1} \). Then \( h \in Aut(\mathbb{D}) \) and \( h(0) = 0 \). It follows that \( h \) is a rotation, \( h(z) \equiv \alpha z, |\alpha| = 1 \). From
\[
  h \circ f_2 = f_1, \quad f_j(P) = 0,
\]
we obtain
\[
  h'(0)f_2'(P) = f_1'(P).
\]
Since \( f_j'(P) > 0 \) and \( |\alpha| = 1 \) it follows that \( h'(0) = \alpha = 1 \), yielding \( f_1 = f_2 \).

**Overview of the Proof of Theorem 4.1:**

a) Fix any point \( P \in U \) and let the set \( F \) consist of all functions \( f : U \to \mathbb{D} \) that have the following properties:
1) \( f \in H(U) \);
2) \( f \) is \( 1-1 \);
3) \( f(P) = 0 \).

We will prove by an explicit construction that \( F \) is not empty.

b) If \( f \in F \) then \(|f'(0)| > 0 \) since \( f \) is \( 1-1 \). Using Cauchy’s inequality, it is easy to see that
\[
  s := \sup\{|f'(P)| : f \in F}\]
is finite.

c) Using Montel’s theorem, one obtains the existence of a function \( f \in H(U) \) with
\[
  f(U) \subset \mathbb{D}, \quad f(P) = 0, \quad s = |f'(0)|.
\]
The function \( f \) is constructed as the normal limit of a sequence of functions in \( F \).

d) Using Hurwitz’s theorem, it follows that the function \( f \) obtained in c) is \( 1-1 \). Therefore, \( f \in F \). In other words, the supremum defining \( s \) is in fact a maximum.

e) An argument using the automorphisms of \( \mathbb{D} \) shows that \( f \) maps \( U \) onto \( \mathbb{D} \). If \( f \) is not onto, then we can construct a map \( g \in F \) with \(|g'(0)| > |f'(0)| = s \), in contradiction to the definition of \( s \).
4.2 The Theorem of Arzela–Ascoli

The Theorem of Arzela–Ascoli is an important result of real analysis. Montel’s Theorem, which we consider in the next section, is a complex version of the Theorem of Arzela–Ascoli.

Let $\Omega \subset \mathbb{R}^s$ be a compact set. With $C(\Omega)$ we denote the linear space of all continuous functions $u : \Omega \to \mathbb{R}$. Let

$$|u|_\infty = \max_x |u(x)|$$

denote the maximum norm of $u \in C(\Omega)$. We know that $C(\Omega)$ with $| \cdot |_\infty$ is a Banach space.

We will need the following version of the Arzela–Ascoli Theorem.

**Theorem 4.3** Let $\Omega$ be a compact subset of $\mathbb{R}^s$. Let $u_n \in C(\Omega)$ denote a sequence of functions with the following two properties:

1) For every $\varepsilon > 0$ there is a $\delta > 0$ so that for all $n \in \mathbb{N}$:

$$|u_n(x) - u_n(y)| < \varepsilon \quad \text{if} \quad |x - y| < \delta, \quad x, y \in \Omega.$$

(It is important that $\delta$ does not depend on $n$. This property is called equicontinuity of the sequence $u_n$.)

2) There is a constant $C > 0$ with

$$|u_n|_\infty \leq C \quad \text{for all} \quad n \in \mathbb{N}.$$

(This property is called uniform boundedness of the sequence $u_n$.)

Under these assumptions, there is a subsequence $u_{n_j}$ and a function $u \in C(\Omega)$ with

$$|u_{n_j} - u|_\infty \to 0 \quad \text{as} \quad n_j \to \infty.$$

**Proof:** 1. Using a diagonal sequence argument, we will show that the sequence $u_n$ has a subsequence $u_{j_k}$ which is a Cauchy sequence in the Banach space $(C(\Omega), | \cdot |_\infty)$.

2. A set $F_\varepsilon \subset \Omega$ is called an $\varepsilon$–net for $\Omega$ if for every $y \in \Omega$ there is $x \in F_\varepsilon$ with $|x - y| < \varepsilon$. Since $\Omega$ is bounded, it is easy to show that for every $\varepsilon > 0$ there exists a finite $\varepsilon$–net for $\Omega$. To see this, cover the set $\Omega$ with finitely many boxes of diameter $\varepsilon$ and, if a box has an intersection with $\Omega$, choose one point in the intersection as an element in the $\varepsilon$–net.

For every $n \in \mathbb{N}$ let $F_{1/n}$ denote a finite $\frac{1}{n}$–net for $\Omega$ and let

$$F = \cup_n F_{1/n}.$$

We enumerate the set $F$ by first listing the points in $F_1$, then the points in $F_{1/2}$, then the points in $F_{1/3}$ etc:

$$F = \{x_1, x_2, \ldots\}.$$
We will need the following property of $F$, which is more special than density of $F$ in $\Omega$: If $\delta > 0$ is given, then there is a finite $K = K(\delta)$ so that the points $x_1, \ldots, x_K \in F$

form a $\delta$–net for $\Omega$. This is clear from the construction of $F$.

3. In the following, $N_1, N_2$ etc. denote infinite subsets of $\mathbb{N}$. Since $u_n(x_1)$ is bounded, there is $N_1$ so that

$$u_n(x_1), \ n \in N_1,$$

is a convergent sequence of real numbers,

$$u_n(x_1) \to a_1, \ n \in N_1.$$

Since $u_n(x_2), n \in N_1$, is bounded, there is $N_2 \subset N_1$ so that

$$u_n(x_2), \ n \in N_2,$$

is a convergent sequence of real numbers,

$$u_n(x_2) \to a_2, \ n \in N_2.$$

Repeat this construction inductively. For every $k \in \mathbb{N}$ obtain a set $N_k$ with

$$N_k \subset N_{k-1} \subset \ldots \subset N_1 \subset \mathbb{N}$$

so that

$$u_n(x_k), \ n \in N_k,$$

is a convergent sequence of real numbers,

$$u_n(x_k) \to a_k, \ n \in N_k.$$

We now use a diagonal sequence argument: Let

$$N_k = \{ n_1^{(k)} < n_2^{(k)} < \ldots \}$$

and consider the diagonal sequence

$$j_k := n_k^{(k)}, \ k = 1, 2, \ldots$$

4. We claim that

$$u_{j_k}(x_\nu) \to a_\nu \text{ as } j_k \to \infty,$$

for every fixed $x_\nu \in F$. This is clear since the tail of the sequence

$$j_1 < j_2 < \ldots < j_\nu < j_{\nu+1} < \ldots$$

is a subsequence of $N_\nu$, and
\[ u_n(x_\nu) \to a_\nu, \quad n \in \mathbb{N}_\nu. \]

To summarize, we have shown that \( u_{jk}(x_\nu) \) converges for every \( x_\nu \in F \) as \( j_k \to \infty \). In particular, given any \( \varepsilon > 0 \) there is \( N(x_\nu, \varepsilon) \) with

\[ |u_{jk}(x_\nu) - u_{ji}(x_\nu)| < \frac{1}{3} \varepsilon \quad \text{for} \quad k, l \geq N(x_\nu, \varepsilon). \]

So far, we have only used that the sequence of functions \( u_n(x) \) is bounded for every \( x \in \Omega \).

5. We now show that the sequence \( u_{jk} \) is a Cauchy sequence in \( C(\Omega) \) w.r.t. \( |\cdot|_\infty \). Let \( \varepsilon > 0 \) be given. There is \( \delta > 0 \) with

\[ |u_n(x) - u_n(y)| < \frac{1}{3} \varepsilon \quad \text{if} \quad |x - y| < \delta. \]

If \( y \in \Omega \) and \( |y - x_\nu| < \delta \) and \( k, l \geq N(x_\nu, \varepsilon) \) then we have

\[ |u_{jk}(y) - u_{ji}(y)| \leq |u_{jk}(y) - u_{jk}(x_\nu)| + |u_{jk}(x_\nu) - u_{ji}(x_\nu)| + |u_{ji}(x_\nu) - u_{ji}(y)| \leq \frac{1}{3} \varepsilon + \frac{2}{3} \varepsilon = \varepsilon. \]

If \( \varepsilon > 0 \) is given, then there are finitely many points \( x_1, \ldots, x_K \in F \) so that for every \( y \in \Omega \) there is \( x_\nu \) with \( 1 \leq \nu \leq K \) and \( |y - x_\nu| < \delta \). If we set

\[ N(\varepsilon) = \max_{1 \leq \nu \leq K} N(x_\nu, \varepsilon) \]

then we have for \( k, l \geq N(\varepsilon) \):

\[ \max_y |u_{jk}(y) - u_{ji}(y)| < \varepsilon. \]

This proves the theorem. ⚫

**Extension:** It is clear that the theorem generalizes to sequences of functions \( f_n : \Omega \to \mathbb{R}^k \) for any finite \( k \).

### 4.3 Montel’s Theorem

Montel’s theorem is a result about families of functions \( f \in H(U) \) which are uniformly bounded on compact subsets of \( U \).

**Theorem 4.4 (Montel)** Let \( U \subset \mathbb{C} \) denote an open set and let \( \mathcal{F} \) denote a set of functions \( f \in H(U) \), i.e., \( \mathcal{F} \subset H(U) \). Assume that for every compact subset \( K \) of \( U \) there is a constant \( C_K \) with

\[ |f(z)| \leq C_K \quad \text{for all} \quad z \in K \quad \text{and} \quad \text{for all} \quad f \in \mathcal{F}. \]

Then, if \( f_n \in \mathcal{F} \) is a sequence, there is a subsequence \( f_{n_j} \) and a function \( f \in H(U) \) so that \( f_{n_j} \) converges normally to \( f \). (The limit function \( f \) generally does not belong to the set \( \mathcal{F} \).)
We will only need the following simpler version of the theorem.

**Theorem 4.5 (Montel’s theorem for a sequence, simple version)** Let $U \subset \mathbb{C}$ be an open set and let $f_n \in H(U)$ be a bounded sequence, i.e., there is a constant $M$ with

$$|f_n(z)| \leq M \quad \text{for all} \quad z \in U \quad \text{and} \quad n \in \mathbb{N}.$$ 

Then there is a function $f \in H(U)$ and a subsequence $f_{n_j}$ of $f_n$ with $f_{n_j} \to f$ normally in $U$. This means: If $K$ is any compact subset of $U$ and if $\varepsilon > 0$, then there exists an integer $J(K, \varepsilon)$ with

$$\max_{z \in K} |f_{n_j}(z) - f(z)| < \varepsilon \quad \text{for} \quad j \geq J(K, \varepsilon).$$

Essentially, Montel’s theorem follows from the Cauchy inequalities and the Arzela–Ascoli theorem. We now give a detailed proof. See Figure 4.1 for sets referred to in the proof.

1. First, let $K$ be any compact subset of $U$. We assume that $U^c = \mathbb{C} \setminus U$ is not empty. (If $U = \mathbb{C}$ then every $f_n$ is constant and the claim follows directly from Bolzano–Weierstrass.) Set

$$\delta = \inf_{z \in U^c} \min_{w \in K} |z - w|$$

$$= \min_{z \in U^c} \min_{w \in K} |z - w|.$$ 

(Here the infimum equals the minimum since $U^c$ is closed and large $z \in U^c$ play no role in determining $\delta$.) We have $\delta > 0$.

2. We want to show that the sequence $f_n$ is equicontinuous on $K$. For $z \in \mathbb{C}$ let

$$\text{dist}(z, K) = \min_{w \in K} |z - w|.$$ 

Define

$$L = \{z \in \mathbb{C} : \text{dist}(z, K) \leq \delta/4\}.$$
Clearly, $K \subset L \subset U$. If $v \in L$ then $\bar{D}(v, \delta/2) \subset U$. By Cauchy’s inequality we have for all $v \in L$:

$$|f'_n(v)| \leq \frac{M}{\delta/2} =: C_1.$$  

If $z, w \in K$ and $|z - w| \leq \delta/4$ then the line segment

$$\gamma(t) = tz + (1 - t)w, \quad 0 \leq t \leq 1,$$

lies in $L$ since $|\gamma(t) - w| \leq \delta/4$. We then have for $z, w \in K$ with $|z - w| \leq \delta/4$:

$$f_n(z) - f_n(w) = f_n(\gamma(1)) - f_n(\gamma(0))$$

$$= \int_0^1 f'_n(\gamma(t)) \gamma'(t) \, dt$$

$$= \int_0^1 f'_n(\gamma(t)) \, dt \, (z - w)$$

Therefore,

$$|f_n(z) - f_n(w)| \leq C_1 |z - w| \quad \text{for} \quad z, w \in K \quad \text{if} \quad |z - w| \leq \frac{\delta}{4}.$$  

It is clear that this estimate implies equicontinuity of the sequence $f_n$ on $K$. Thus we can apply the Arzela–Ascoli Theorem to the sequence $f_n$ on any compact subset $K$ of $U$.

3. Given $K \subset U$ as above, there exists a subsequence $f_{n_j}$ and a function $f \in C(K)$ with $\max_{x \in K} |f_{n_j}(x) - f(x)| \to 0$.

4. We now choose a sequence of compact sets $K_\nu$ in $U$ as follows. For $\nu = 1, 2, \ldots$ let

$$K_\nu = \left\{ z \in U : \text{dist}(z, U^c) \geq 1/\nu \right\} \cap \bar{D}(0, \nu).$$

It is clear that

$$K_1 \subset K_2 \subset \ldots \subset U \quad \text{and} \quad \cup_\nu K_\nu = U.$$  

Furthermore, if $K$ is any compact subset of $U$ then there is a sufficiently large $\nu$ with

$$K \subset K_\nu.$$  

The sequence $f_n$ has a subsequence $f_{n_n}, n \in \mathbb{N}_1$, which converges uniformly on $K_1$ to some $f \in C(K_1)$. Then consider the sequence $f_{n_n}, n \in \mathbb{N}_1$, on $K_2 \supset K_1$. There is a subsequence $f_{n_{n_{\nu}}}, n \in \mathbb{N}_2 \subset \mathbb{N}_1$, converging uniformly to some $g \in C(K_2)$. However, $g$ equals $f$ on $K_1$, and we may denote $g$ by $f$. We repeat the argument and obtain: For every $\nu$ there is $\mathbb{N}_\nu \subset \mathbb{N}_{\nu-1}$ so that $f_{n_n}, n \in \mathbb{N}_\nu$, converges uniformly on $K_\nu$ to $f \in C(K_\nu)$. Since
\[ \bigcup_{\nu} K_\nu = U \]
this process defines a continuous function \( f \) on \( U \).

5. If

\[ \mathbb{N}_k = \{ n_1^{(k)} < n_1^{(k)} < \ldots \} \]
then let

\[ j_k = n_k^{(k)}, \quad k = 1, 2, \ldots \]
In this way we obtain the subsequence

\[ f_{j_k} \]
of \( f_n \). A tail of \( f_{j_k} \) is a subsequence of \( f_n, n \in \mathbb{N}_\nu \). Therefore, for every fixed \( \nu \), the subsequence \( f_{j_k} \) converges uniformly on \( K_\nu \) to \( f \).

6. Finally, if \( K \) is an arbitrary compact subset of \( U \) then \( K \subset K_\nu \) for some large \( \nu \). Therefore, \( f_{j_k} \) converges uniformly on \( K \) to \( f \). The limit function \( f \) is holomorphic on \( U \) by Morera’s theorem. \( \diamond \)

### 4.4 Auxiliary Results on Logarithms and Square Roots

**Lemma 4.1** Let \( U \subset \mathbb{C} \) be open and simply connected; let \( f \in H(U) \). If \( f(z) \neq 0 \) for all \( z \in U \) then there exists a function \( h \in H(U) \) with

\[ e^{h(z)} = f(z), \quad z \in U. \]

**Proof:** Motivation: Suppose that \( e^{h(z)} = f(z) \). Then we have

\[ f'(z) = h'(z)f(z), \]
thus

\[ h'(z) = f'(z)/f(z). \]
This motivates to construct \( h(z) \) as a function with \( h' = f'/f \).

Fix \( z_0 \in U \) and let \( \Gamma_z \) denote a curve in \( U \) from \( z_0 \) to \( z \). Write

\[ f(z_0) = e^c \]
and define

\[ h(z) = c + \int_{\Gamma_z} \frac{f'(w)}{f(w)} \, dw, \quad z \in U. \]
Then we have \( h'(z) = f'(z)/f(z) \) and \( e^{h(z)} = e^c = f(z_0) \). Consider the function

\[ g(z) = f(z)e^{-h(z)}, \quad z \in U. \]
We have
\[ g(z_0) = 1 \]

and

\[
g'(z) = f'(z)e^{-h(z)} - f(z)e^{-h(z)}h'(z) \\
= e^{-h(z)}\left(f'(z) - f(z)h'(z)\right) \\
= 0
\]

It follows that \( g(z) \equiv 1 \), proving the lemma. ◦

**Remark:** Since \( e^{h(z)} = f(z) \) we may consider the function \( h(z) \) as a complex logarithm of \( f(z) \),

\[ h(z) = \log(f(z)) , \]

where \( \log(w) \) is any inverse of the exponential function defined on the range of \( f(z) \).

**Lemma 4.2** Let \( U \subset \mathbb{C} \) be open and simply connected; let \( f \in H(U) \). If \( f(z) \neq 0 \) for all \( z \in U \) then there exists a function \( g \in H(U) \) with

\[ f(z) = (g(z))^2, \quad z \in U . \]

**Proof:** Using the previous lemma, we can write

\[ f(z) = e^{h(z)} \]

and define

\[ g(z) = e^{h(z)/2} . \]

◊

4.5 Construction of a Map in \( \mathcal{F} \)

**Lemma 4.3** Let \( U \subset \mathbb{C} \) be open and simply connected. Further, let \( U \neq \mathbb{C} \). Let \( P \in U \) be arbitrary, but fixed. Then there exists \( f \in H(U) \) with \( f(U) \subset \mathbb{D} \), \( f \) is 1−1, and \( f(P) = 0 \).

**Proof:** If \( U \) is bounded then \( f(z) \) can be taken as a function of the form \( f(z) = az + b \). We treat this simple case first in Case a. In Case b we treat the general case.

**Case a)** Let \( U \) be bounded, \( U \subset D(0, R) \). Let

\[ a = \frac{1}{2R} \]

and set

\[ f(z) = a(z - P) . \]
Then, clearly, $f$ is 1−1 and $f(P) = 0$. If $z \in U$ then

$$|f(z)| = a|z - P| < 2Ra = 1.$$ 

This shows that $f$ maps $U$ into $\mathbb{D}$.

**Case b)** Let $U$ be an arbitrary open, simply connected subset of $\mathbb{C}$ and $U \neq \mathbb{C}$. Let $Q \in \mathbb{C} \setminus U$ and define

$$\phi(z) = z - Q, \quad z \in U.$$ 

Then $\phi \in H(U)$ is 1 − 1 and $\phi(z) \neq 0$ for all $z \in U$. By Lemma 4.2 there is $h \in H(U)$ with

$$h^2(z) = \phi(z), \quad z \in U.$$ 

We will show that there is an open disk $D(-b, r)$ which is contained in the complement of $h(U)$:

$$D(-b, r) \subset \left( \mathbb{C} \setminus h(U) \right).$$ 

Once this is done, the construction of $f$ will be easy.

Clearly, if $z_1, z_2 \in U$ are two distinct points, then

$$h(z_1) \neq -h(z_2) \quad \text{for} \quad z_1 \neq z_2 \quad (4.2)$$

since otherwise

$$h^2(z_1) = h^2(z_2)$$

contradicting $\phi(z_1) \neq \phi(z_2)$. Since $h \in H(U)$ is not constant, the set $h(U)$ is open. There exists $b \in \mathbb{C}$ and $r > 0$ with

$$D(b, r) \subset h(U).$$

We may assume that $b \neq 0$ and $0 < r < |b|$. We claim that the open disk $D(-b, r)$ lies in $\mathbb{C} \setminus h(U)$. If this is not the case, then let

$$q_1 = h(z_1) \in D(-b, r) \cap h(U), \quad z_1 \in U.$$ 

Set $q_2 = -q_1$. From

$$|q_1 + b| < r$$

obtain that

$$|q_2 - b| < r,$$

i.e., $q_2 \in D(b, r) \subset h(U)$. Therefore, $q_2 = h(z_2)$ for some $z_2 \in U$. Because of $0 < r < |b|$ the two disks

$$D(-b, r) \quad \text{and} \quad D(b, r)$$

are disjoint. This shows that
We then have
\[ q_2 = -q_1 \neq q_1. \]

Thus we have
\[ h(z_1) = q_1 \neq q_2 = h(z_2), \]

thus \( z_1 \neq z_2 \), but
\[ h(z_1) = q_1 = -q_2 = -h(z_2), \]
in contradiction to (4.2).

This contradiction proves that
\[ D(-b, r) \subset \left( \mathbb{C} \setminus h(U) \right). \]

In other words,
\[ |h(z) + b| \geq r \quad \text{for all} \quad z \in U. \]

Define
\[ f_1(z) = \frac{r}{2 \overline{h(z)} + b}. \quad z \in U. \]

Then \( f_1 \in H(U) \) is \( 1 - 1 \) and \( |f_1(z)| \leq \frac{1}{2} < 1 \). Let \( a := f_1(P) \) and recall that
\[ \phi_a(z) = \frac{z - a}{1 - \overline{a}z} \]
is an automorphism of \( \mathbb{D} \) with \( \phi_a(a) = 0 \). The function \( f = \phi_a \circ f_1 \) satisfies \( f(P) = 0 \). To summarize, we have constructed a function \( f \in H(U) \) which is \( 1 - 1 \) and satisfies \( f(U) \subset \mathbb{D} \) and \( f(P) = 0 \).

4.6 A Bound for \( |f'(0)| \)

Let \( f \in \mathcal{F} \), i.e., \( f \in H(U), f(U) \subset \mathbb{D}, f(P) = 0 \), and \( f \) is one-to-one. There is \( r > 0 \) with \( D(P, r) \subset U \). Let \( \gamma(t) = P + re^{it} \). We know that, for \( z \in D(P, r) \),
\[ f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw. \]

Differentiation yields
\[ f'(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - z)^2} \, dw. \]

For \( z = P \) one obtains the Cauchy inequality
\[ |f'(P)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(P + re^{it})|}{r^2} \, dt \]
\[ \leq \frac{1}{r}. \]
Here we have used the bound $|f(w)| \leq 1$, which follows from $f(U) \subset \mathbb{D}$.

We have shown:

**Lemma 4.4** The number $s$ defined by

$$s = \sup\{ |f'(P)| : f \in \mathcal{F} \}$$

satisfies $0 < s \leq \frac{1}{r}$ if $\bar{D}(P,r) \subset U$.

### 4.7 Application of the Theorems of Montel and Hurwitz

There is a sequence of functions $f_n \in \mathcal{F}$ with

$$s \geq |f'_n(P)| \geq s - \frac{1}{n}, \quad n = 1,2,\ldots$$

Since $f_n(U) \subset \mathbb{D}$ the functions $f_n$ are uniformly bounded on $U$. By Montel’s theorem (simple version), there is a subsequence $f_{n_j}$ and a function $f \in H(U)$ so that $f_{n_j}$ converges normally to $f$ on $U$. In particular, $f(P) = 0$ and $|f'(P)| = s$.

Also, $|f(z)| \leq 1$ for all $z \in U$. If $|f(z)| = 1$ for some $z \in U$ then $f$ is constant, contradicting $|f'(P)| = s > 0$. Therefore, $f(U) \subset \mathbb{D}$.

We now claim that $f$ is $1 - 1$. To prove this we will use Hurwitz Theorem:

**Theorem 4.6** (Hurwitz) Let $V \subset \mathbb{C}$ be open and connected. Let $h_j, h \in H(V)$ and assume that the sequence $h_j(z)$ converges normally to $h(z)$ in $V$. If

$$h_j(z) \neq 0 \quad \text{for all} \quad z \in V \quad \text{and} \quad \text{for all} \quad j = 1,2,\ldots$$

then either $h(z) \equiv 0$ or

$$h(z) \neq 0 \quad \text{for all} \quad z \in V .$$

**Proof:** Suppose that $P \in V$ is an isolated zero of $h(z)$. There exists $\varepsilon > 0$ so that

$$h(z) \neq 0 \quad \text{for} \quad 0 < |z - P| \leq \varepsilon .$$

If $\Gamma$ denotes the boundary curve of the disk $D(P,\varepsilon)$ then the positive integer

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{h'(z)}{h(z)} \, dz$$

is the order of the zero $P$ of $h(z)$. On the other hand,

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{h_j'(z)}{h_j(z)} \, dz = 0 \quad \text{for all} \quad j = 1,2,\ldots$$

One obtains a contradiction as $j \to \infty$. \bigstar

Recall that $f_n(z)$ denotes a sequence of 1–1 functions with

$$f_n \in H(U), \quad f_n(U) \subset \mathbb{D}, \quad f_n(P) = 0$$
and
\[ s \geq |f'_n(P)| \geq s - \frac{1}{n}, \quad n = 1, 2, \ldots \]

By Montel’s Theorem there exists a subsequence \( f_{n_j} \) and \( f \in H(U) \) so that \( f_{n_j} \) converges normally to \( f \) on \( U \). One obtains that \( |f'(P)| = s > 0 \).

We claim that \( f \) is 1–1 on \( U \). Suppose not. Then there are two points \( z_1, z_2 \in U \) with
\[ f(z_1) = f(z_2), \quad z_1 \neq z_2 . \]

Consider the sequence of functions
\[ h_j(z) = f_{n_j}(z) - f_{n_j}(z_2), \quad z \in V = U \setminus \{z_2\} . \]

The functions \( h_j \in H(V) \) converge normally in \( V \) to \( f(z) - f(z_2) \). Also, the functions \( h_j(z) \) have no zero in \( V \) since they are 1–1 on \( U \). By Hurwitz’s theorem, the limit function \( f(z) - f(z_2) \) either is identically zero on \( V \) or has no zero in \( V \). Since \( f(z) \) is not constant, we conclude that \( f(z) - f(z_2) \) has no zero in \( V \), contradicting the assumption \( f(z_1) = f(z_2) \).

4.8 Proof That \( f \) is Onto

We have shown that there is a function \( f \in \mathcal{F} \) with
\[ s = |f'(P)| \geq |g'(P)| \quad \text{for all} \quad g \in \mathcal{F}, \quad s > 0 . \quad (4.3) \]

We claim that \( f : U \to \mathbb{D} \) is onto. This claim then completes the proof of the Riemann Mapping Theorem.

We will show:

**Lemma 4.5** Let \( f \in \mathcal{F} \) and assume that \( f : U \to \mathbb{D} \) is not onto \( \mathbb{D} \). Then we can construct \( g \in \mathcal{F} \) with
\[ |g'(P)| > |f'(P)| . \]

**Proof:** With \( S : \mathbb{D} \to \mathbb{D} \) we denote the squaring function, i.e., \( S(v) = v^2, v \in \mathbb{D} \). Also, we recall that for \( |c| < 1 \) the function
\[ \phi_c(w) = \frac{w - c}{1 - \overline{c}w}, \quad |w| < 1 , \]
is an automorphism of \( \mathbb{D} \). Since \( f : U \to \mathbb{D} \) is not onto, we can choose
\[ a \in \mathbb{D} \setminus f(U) \]
and consider
\[ \phi(z) = \phi_a \circ f(z) \]
\[ = \frac{f(z) - a}{1 - \overline{a}f(z)}, \quad z \in U . \]
Since \( a \notin f(U) \) we have \( \phi(z) \neq 0 \) for all \( z \in U \). Also, \( \phi \in H(U) \) is \( 1 - 1 \). There is a function \( \psi \in H(U) \) which is \( 1 - 1 \) and satisfies
\[
\psi(z) \psi(z) = \phi(z) \quad \text{for all} \quad z \in U, \quad \psi(U) \subset \mathbb{D}.
\]
Define
\[
g(z) = \phi_{\psi(P)}(\psi(z)), \quad z \in U.
\]
Then \( g \in H(U), g \) is \( 1 - 1 \), \( g(U) \subset \mathbb{D} \), and \( g(P) = 0 \). Therefore, \( g \in F \). Also, if we abbreviate \( b = -\psi(P) \),
\[
\begin{align*}
\phi_b \circ g &= \psi \\
 S \circ \phi_b \circ g &= \psi \psi \\
 &= \phi \\
 &= \phi_a \circ f,
\end{align*}
\]
thus
\[
f = \phi_{-a} \circ S \circ \phi_b \circ g.
\]
We now set
\[
h = \phi_{-a} \circ S \circ \phi_b.
\]
Then the above relation says that
\[
f(z) = h(g(z)), \quad z \in U.
\]
We have \( h \in H(\mathbb{D}), h(\mathbb{D}) \subset \mathbb{D} \), and
\[
0 = f(P) = h(g(P)) = h(0).
\]
By Schwarz Lemma, we conclude that
\[
|h'(0)| < 1
\]
unless \( h \) is a rotation, \( h = R_\alpha \). However, if \( h = R_\alpha \), then the squaring function \( S \) would be an automorphism of \( \mathbb{D} \), which is obviously not the case. It follows that \( |h'(0)| < 1 \). Since
\[
f(z) = h(g(z)), \quad z \in U,
\]
we have
\[
f'(P) = h'(0)g'(P).
\]
This proves the lemma. \( \diamond \).
Figure 4.2: Biholomorphic mapping from upper half-plane onto unit disk

4.9 Examples of Biholomorphic Mappings

Recall that $\mathbb{D}$ denotes the open unit disk and $\mathbb{H}$ denotes the open upper half-plane.

**Example 1:** Let $P = a + ib, b > 0$, be a point in $\mathbb{H}$. Then $\bar{P} = a - ib$ is the reflection of $P$ w.r.t. the real axis. The function

$$g(z) = \frac{z - P}{z - \bar{P}}, \quad z \neq \bar{P},$$

maps $\mathbb{H}$ onto $\mathbb{D}$, maps the real line onto $\partial\mathbb{D}\setminus\{1\}$, and maps $\mathbb{C}\setminus\overline{\mathbb{H}}$ onto $\mathbb{C}\setminus\mathbb{D}$. In particular, $g$ is a biholomorphic mapping from $\mathbb{H}$ onto $\mathbb{D}$ with $g(P) = 0$. If you want to find a biholomorphic mapping $f : \mathbb{H} \to \mathbb{D}$ with $f(P) = 0$ and $f'(P) > 0$ then set

$$f(z) = e^{i\theta}g(z)$$

and determine $\theta$ so that $f'(P) > 0$.

**Example 2:** Let $H$ denote an open half-plane in $\mathbb{C}$. The boundary of $H$ is a straight line, $L = \partial H$. Let $P \in H$ and let $Q \in H^c$ denote the reflection of $P$ w.r.t. $L$. Then

$$g(z) = \frac{z - P}{z - Q}, \quad z \in H,$$

maps $H$ biholomorphically onto $\mathbb{D}$. 
5 An Introduction to Schwarz–Christoffel Formulas

Schwarz–Christoffel formulas give biholomorphic maps \( w = w(z) \) from the open upper half–plane \( \mathbb{H} \) onto regions enclosed by polygons. The real axis is mapped to the polygon. Typically, the maps \( w(z) \) are given in terms of integrals. Similar integrals also appear in the theory of elliptic functions.

We treat an example of a Schwarz–Christoffel map, which is Example 8, p. 350, from [Hahn, Epstein]. A more comprehensive treatment of Schwarz–Christoffel maps is given in [Hille] and in [Stein, Sharkarchi].

A clear understanding of general power functions \( z \to z^\alpha \) is helpful and will be reviewed first.

5.1 General Power Functions

Let

\[
U_0 = \mathbb{C} \setminus \{ z = iy : y \leq 0 \}
\]

denote a slit plane. Precisely, \( U_0 \) is the complex plane \( \mathbb{C} \) with the negative imaginary axis, including \( z = 0 \), removed. On \( U_0 \) we define the logarithm \( \log_0(z) \) as a holomorphic function as follows: If \( z \in U_0 \) then there is a unique argument

\[
\arg_0(z) = \theta \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2}
\]

so that

\[
z = |z|e^{i\theta} = e^{\ln |z| + i\theta}.
\]

We then set

\[
\log_0(z) = \ln |z| + i\theta, \quad z \in U_0,
\]

and call \( \log_0(z) \) the principle branch of the logarithm on \( U_0 \). This function \( \log_0(z) \) is holomorphic on \( U_0 \) and extends the natural logarithm \( \ln r = \int_1^r \frac{dx}{x} \), defined for positive real numbers \( r \), to the slit plane \( U_0 \).

**General Powers on \( U_0 \).** Let \( z \in U_0 \) and let \( \alpha \in \mathbb{C} \). We then set

\[
z^\alpha = e^{\alpha \log_0(z)} \tag{5.1}
\]

If \( \alpha \) is real and \( z \) is real and positive then the definition of \( z^\alpha \) by (5.1) agrees with the usual definition of \( z^\alpha \) in real analysis and \( z^\alpha > 0 \). For fixed \( \alpha \in \mathbb{R} \), the function \( z \to z^\alpha \) given in (5.1) extends the real analysis function \( x^\alpha \), defined for real positive \( x \), to the slit plane \( U_0 \). We call the function given in (5.1) the principle branch of the power function \( z^\alpha \) on \( U_0 \).

**Lemma 5.1** Let \( z \in U_0 \) and let \( \alpha \in \mathbb{R} \). Then we have

\[
|z^\alpha| = |z|^\alpha.
\]
Proof: From
\[ z = |z|e^{i\theta} = e^{\ln|z|}e^{i\theta} \]
we obtain
\[ z^\alpha = e^{\alpha \ln|z|}e^{i\alpha \theta}, \]
thus
\[ |z^\alpha| = e^{\alpha \ln|z|} = |z|\alpha. \]
\[ \diamond \]

The following simple examples show that one has to be careful when computing powers of powers.

1) Since \(-1 = e^{\pi i}\) we have
\[ (-1)^{-1} = (e^{\pi i})^{-1} = e^{-\pi i} = -1 = e^{\pi i}. \]

2) Therefore,
\[ \left((-1)^{-1}\right)^{2/3} = (e^{\pi i})^{2/3} = e^{2\pi i/3}. \]

3) On the other hand
\[ (-1)^{-2/3} = e^{-2\pi i/3}. \]

This shows that
\[ \left((-1)^{-1}\right)^{2/3} \neq (-1)^{-2/3}. \]

We see that, in general,
\[ (z^{-1})^\alpha \neq z^{-\alpha}. \]

5.2 An Example of a Schwarz–Christoffel Map

The following example is Example 8, p. 350, from [Hahn, Epstein]. It gives a good introduction to Schwarz–Christoffel formulas.

For real \(\beta\) let \(L_\beta\) denote the half-line
\[ L_\beta = \{z = \beta + iy : y \leq 0\} \]
and let
\[ V := \mathbb{C} \setminus \{L_{-1} \cup L_0 \cup L_1\}, \]
i.e., \(V\) is the plane \(\mathbb{C}\) with three half-lines removed.
Consider the function \(g : V \rightarrow \mathbb{C}\) given by
\[ g(\zeta) = e^{2\pi i/3} \cdot (\zeta + 1)^{-5/6} \cdot \zeta^{-1/2} \cdot (\zeta - 1)^{-2/3}, \quad \zeta \in V. \]
Figure 5.1: $|g(\zeta)| = |\zeta + 1|^{-5/6}|\zeta|^{-1/2}|1 - \zeta|^{-2/3}$

Figure 5.1 shows a rough graph of the function

$$|g(\zeta)| = |\zeta + 1|^{-5/6} \cdot |\zeta|^{-1/2} \cdot |\zeta - 1|^{-2/3}$$

for real values of $\zeta$.

The domain of definition of $g$ is the plane $\mathbb{C}$ with three half–lines removed. By removing the three half–lines, we can use the principle branches of the power functions on $U_0$ for the factors of $g(\zeta)$. It is then clear that $g \in H(V)$.

For the following, note that the domain $V$ is simply connected. If $z \in V$ then let $\Gamma_z$ denote a curve in $V$ from 0 to $z$ and set

$$w(z) = \int_{\Gamma_z} g(\zeta) \, d\zeta .$$

Since the integral does not depend on the choice of the curve $\Gamma_z$, but only depends on $z$, we write

$$\int_0^z = \int_{\Gamma_z} .$$

(Curve–independence of the integral holds though the starting point of $\Gamma_z$, the point $z_0 = 0$, lies on the boundary of $V$, but does not lie inside $V$. Note, however, that the singularity $\zeta^{-1/2}$ of $g(\zeta)$ at $\zeta = 0$ is integrable.)

We will now discuss the function $w(z)$ for $z$ on the real axis. We note that the singularities of the function $g(\zeta)$ are integrable and that $|g(\zeta)|$ decays like $|\zeta|^{-2}$ as $|\zeta| \to \infty$. Decay like the power $|\zeta|^{-2}$ occurs since

$$\frac{5}{6} + \frac{1}{2} + \frac{2}{3} = 2 . \quad (5.2)$$

Considering the arguments of the factors of $g(\zeta)$, we have for real $\zeta$:

$$g(\zeta) = |g(\zeta)| \cdot \begin{cases} e^{2\pi i/3} & \text{for } \zeta > 1 \\ 1 & \text{for } 0 < \zeta < 1 \\ e^{-\pi i/2} = -i & \text{for } -1 < \zeta < 0 \\ e^{2\pi i/3} & \text{for } \zeta < -1 \end{cases} \quad (5.3)$$

To obtain the result for $\zeta < -1$ note that
\[-ie^{-5\pi i/6} = e^{-\pi i (\frac{1}{2} + \frac{5}{6})} = e^{2\pi i/3}\]

since (see (5.2))

\[-\frac{1}{2} - \frac{5}{6} = \frac{2}{3} - 2.\]

We now consider the straight lines \(\Gamma_0, \ldots, \Gamma_3\) on the real axis of the \(z\)-plane (see Figure 5.2) and the image of \(\Gamma_j\) under the map \(z \rightarrow w(z) = \int_0^z g(\zeta) \, d\zeta\).

![Figure 5.2: Path of integration for \(g(\zeta)\)](image)

We start the consideration by determining \(w(z)\) for \(z \in \Gamma_2\). Note that \(g(z) = |g(z)|\) for \(z \in \Gamma_2\). The point \(w(z)\) moves from \(w(0) = 0\) to the finite positive value \(w(1)\) along the positive real axis. See Figure 5.3.

![Figure 5.3: \(w(z) = \int_0^z g(\zeta) d\zeta\)](image)

Next consider \(w(z)\) for \(z \in \Gamma_1\) and note that \(g(z) = -i|g(z)|\) for \(z \in \Gamma_1\). We have

\[w(z) = -\int_z^0 g(\zeta) \, d\zeta = i \int_z^0 |g(\zeta)| \, d\zeta .\]

Therefore, \(w(z)\) moves from \(w(0) = 0\) to the finite value \(w(-1)\) up the imaginary axis.

Next consider \(z > 1\). We have
\[ w(z) = \int_{1}^{z} g(\zeta) \, d\zeta + w(1) \, . \]

The point \( w(z) \) moves along the hypotenuse of the triangle in Figure 5.3.

Similarly, if \( z < -1 \), then

\[ w(z) = -\int_{z}^{-1} g(\zeta) \, d\zeta + w(-1) \, . \]

We see that \( w(z) \) moves along a direction parallel to the direction from 0 to \(-e^{2\pi i/3}\). To complete the picture, it is important to understand that

\[ w(-\infty) = w(\infty) \, . \]

To see this, consider the curve from \( z = -R \) to \( z = R \) along the real axis completed by the semi-circle \( \zeta(\theta) = R e^{i\theta}, 0 \leq \theta \leq \pi \). Denote this closed curve by \( \mathcal{C}_R \).

(To be completely correct, the straight line along the real axis should avoid the singularities of \( g(\zeta) \) at \( \zeta = -1, 0, 1 \) and one should consider a corresponding curve \( \mathcal{C}_{R,\varepsilon} \). However, since the singularities of \( g \) are integrable, one can let \( \varepsilon \to 0 \) without difficulty.)

By Cauchy’s theorem,

\[ \int_{\mathcal{C}_R} g(\zeta) \, d\zeta = 0 \, . \]

Since \( |g(\zeta)| \leq C|\zeta|^{-2} \) for large \( |\zeta| \), the integral along the semicircle is bounded by \( C/R \). Therefore,

\[ |w(R) - w(-R)| \leq C/R \, , \]

and we obtain that \( w(-\infty) = w(\infty) \) for \( R \to \infty \). These considerations prove that the image of the real axis under the map

\[ z \to w(z) = \int_{0}^{z} g(\zeta) \, d\zeta \]

is the triangle shown in Figure 5.3 where the point \( w(\infty) = w(-\infty) \) is removed from the hypotenuse.

Another good example is the map

\[ w(z) = \int_{0}^{z} (1 - \zeta^2)^{-1/2} \, d\zeta = \int_{0}^{z} (1 + \zeta)^{-1/2}(1 - \zeta)^{-1/2} \, d\zeta \, . \]

Here one should note that

\[ \int_{-1}^{1} \frac{d\zeta}{\sqrt{1 - \zeta^2}} = \arcsin(1) - \arcsin(-1) = \pi \, . \]
6 Meromorphic Functions with Prescribed Poles

6.1 Meromorphic Functions on \( \mathbb{C} \)

Recall that a function \( f \) is called meromorphic on \( \mathbb{C} \) if there is a finite or denumerable set \( D = \{ b_1, b_2, \ldots \} \subset \mathbb{C} \) so that

(a) \( f \) is holomorphic on \( \mathbb{C} \setminus D \);
(b) every point \( b_k \in D \) is a pole of \( f \).

The set \( D \) is finite or denumerable and does not have an accumulation point in \( \mathbb{C} \). (Suppose \( b \) is an accumulation point of \( D \). If \( b \in D \) then the point \( b \) is an accumulation point of poles and cannot be a pole itself. If \( b \in \mathbb{C} \setminus D \) then \( f \) is holomorphic in \( b \), and poles of \( f \) cannot accumulated at \( b \).)

If \( f \) has infinitely many poles \( b_k \) then we will order them so that

\[ |b_1| \leq |b_2| \leq \ldots \text{ and } |b_k| \to \infty \text{ as } k \to \infty. \]  

(6.1)

If \( f \) is meromorphic on \( \mathbb{C} \) we write \( f \in M(\mathbb{C}) \).

Example 1: \( f(z) = \frac{1}{\sin(\pi z)} \). It is easy to show that the pole set of \( f \) is the set \( \mathbb{Z} \) of all integers.

Example 2: Every rational function \( f(z) = p(z)/q(z) \) is meromorphic on \( \mathbb{C} \) with finitely many poles.

Example 3: The function \( f(z) = e^{1/z} \) is not meromorphic since \( z = 0 \) is not a pole, but an essential singularity.

Let \( f \in M(\mathbb{C}) \) and let \( b_k \) be a pole of \( f \). The Laurent expansion of \( f \) about \( b_k \) has the form

\[ f(z) = P_k((z - b_k)^{-1}) + g_k(z), \quad 0 < |z - b_k| < r_k. \]

Here \( P_k(w) \) is a polynomial without constant term and \( g_k \in H(D(b_k, r_k)) \). The function

\[ P_k((z - b_k)^{-1}), \quad z \in \mathbb{C} \setminus \{b_k\}, \]

is called the singular part of \( f \) at \( b_k \).

6.2 The Mittag–Leffler Theorem

Notation: If \( K \subset \mathbb{C} \) is compact and \( f \in C(K) \) then let

\[ |f|_K = \max\{|f(z)| : z \in K\}. \]

Definition: Let \( U \subset \mathbb{C} \) be open and let \( f_k \in H(U) \). The series

\[ \sum_{k=1}^{\infty} f_k(z) \]

(6.2)

converges normally in \( U \) if for every compact set \( K \subset U \) the series
\[ \sum_{k=1}^{\infty} |f_k|_K \]

converges.

Thus, normal convergence of a series (6.2) implies that the sequence of partial sums,
\[ s_n(z) = \sum_{k=1}^{n} f_k(z) \]
converges locally uniformly on \( U \) to a unique limit, \( f \in H(U) \). As usual, the limit is denoted by
\[ f(z) = \sum_{k=1}^{\infty} f_k(z) \, . \]

Also, note that in case of normal convergence of (6.2), the series converges absolutely. This implies that the terms of the series can be reordered without changing the value of the limit. In other words, the ordering of the terms \( f_k(z) \) is not important.

We also note that normal convergence of the series
\[ f(z) = \sum_{k=1}^{\infty} f_k(z) \]
implies normal convergence of
\[ f'(z) = \sum_{k=1}^{\infty} f'_k(z) \, . \]

**Theorem 6.1 (Mittag–Leffler)** Let \( b_1, b_2, \ldots \) denote an infinite sequence of distinct points in \( \mathbb{C} \) without accumulation point in \( \mathbb{C} \) and set \( \mathcal{D} = \{b_1, b_2, \ldots \} \). Further, let \( P_k(w) \) denote a sequence of polynomials without constant terms, i.e., \( P_k(0) = 0 \) for every \( k \). Then there is a function \( f \in M(\mathbb{C}) \) so that:

(a) the pole set of \( f \) equals \( \mathcal{D} \);

(b) the singular part of \( f \) at \( b_k \) equals \( P_k((z - b_k)^{-1}) \).

**Remark:** We cannot simply define
\[ f(z) = \sum_{k=1}^{\infty} P_k((z - b_k)^{-1}) \]
since, in general, the series does not converge. For example, let \( b_k = k \) and \( P_k(w) = w \). Then the above series is
\[ \sum_{k=1}^{\infty} \frac{1}{z - k} \, . \]
This series diverges for every \( z \).

**Proof of Theorem 6.1:** We first assume that \( z = 0 \) is not an element of \( D \). Then we may assume \( 0 < |b_1| \leq |b_2| \leq \ldots \) and \( |b_k| \to \infty \) as \( k \to \infty \). Let \( \sum_{k=1}^{\infty} c_k \) denote a convergent series of positive numbers, \( c_k > 0 \). For example, \( c_k = 1/k^2 \).

The function

\[
z \to P_k((z - b_k)^{-1})
\]

is singular only at \( z = b_k \). Thus we can write

\[
P_k((z - b_k)^{-1}) = \sum_{j=0}^{\infty} a_{jk} z^j, \quad |z| < |b_k|.
\]

Let

\[
Q_k(z) = \sum_{j=0}^{n_k} a_{jk} z^j
\]

where the integer \( n_k \) is chosen so large that

\[
\max_{|z| \leq |b_k|/2} \left| P_k((z - b_k)^{-1}) - Q_k(z) \right| \leq c_k.
\]  \hfill (6.3)

We claim that the series

\[
f(z) := \sum_{k=1}^{\infty} \left( P_k((z - b_k)^{-1}) - Q_k(z) \right), \quad z \in \left( \mathbb{C} \setminus D \right) =: U,
\]

converges normally in \( U \) and the limit \( f \) is a meromorphic function with the desired properties.

Set

\[
f_k(z) = P_k((z - b_k)^{-1}) - Q_k(z), \quad z \in \mathbb{C} \setminus \{b_k\}.
\]

Let \( K \subset U \) be any compact set. There exists \( R > 0 \) with

\[
K \subset D(0, R).
\]

Since \( |b_k| \to \infty \) there exists a positive integer \( k_0(R) \) with

\[
|b_k| \geq 2R \quad \text{for} \quad k \geq k_0(R).
\]

If \( k \geq k_0(R) \) then, by (6.3),

\[
|f_k|_K \leq |f_k|_{D(0,R)} \leq c_k.
\]

Therefore,

\[
\sum_{k=1}^{\infty} |f_k|_K < \infty.
\]
This proves normal convergence of the series

\[ f(z) := \sum_{k=1}^{\infty} f_k(z) \quad \text{in} \quad U. \]

We must show that the pole set of \( f \) is precisely \( \mathcal{D} \) and that

\[ P_k((z-b_k)^{-1}) \]

is the singular part of \( f \) at \( b_k \). To this end, fix any \( R > 0 \). For \( z \in U \cap D(0, R) \) we have

\[ f(z) = \sum_{k=1}^{k_0(R)} f_k(z) + \sum_{k>k_0(R)} f_k(z) =: f^I(z) + f^{II}(z). \]

Note that the decomposition \( f = f^I + f^{II} \) depends on \( R \). The first part, \( f^I(z) \), is meromorphic in \( \mathbb{C} \). In fact, \( f^I \) is a rational function. In the disk \( D(0, R) \) the function \( f^I \) has poles precisely at those \( b_k \) which lie in \( D(0, R) \) and at each \( b_k \) the singular part of \( f^I \) is \( P_k((z-b_k)^{-1}) \). The second function, \( f^{II}(z) \), is holomorphic in \( D(0, R) \) because every function \( f_k(z), k > k_0(R) \), is holomorphic in \( D(0, 2R) \) and the series defining \( f^{II} \) converges uniformly on \( \overline{D}(0, R) \).

Since \( R > 0 \) was arbitrary, it is shown that \( f \) has the desired properties.

So far we have assumed that \( b_1 \neq 0 \). If \( b_1 = 0 \) we just add the term

\[ P_1(1/z) \]

to the constructed function.

Remark: An important technical point of the proof is that the decomposition, \( f = f^I + f^{II} \), depends on \( R \), and we consider it in \( D(0, R) \). In this way, for each finite \( R \), one only considers functions with finitely poles.

### 6.3 An Example

As before, let \( \mathcal{D} = \{b_1, b_2, \ldots \} \) denote an infinite set in \( \mathbb{C} \) without accumulation point in \( \mathbb{C} \). For simplicity, let \( 0 < |b_1| \leq |b_2| \leq \ldots \).

Assume that \( P_k(w) = w \) for all \( k \), i.e., we want to construct a meromorphic functions with simple poles and residue 1 at each \( b_k \). Assume that

\[ \sum_k \frac{1}{|b_k|} = \infty, \quad \sum_k \frac{1}{|b_k|^2} < \infty, \]

where the sum is taken over all \( b_k \neq 0 \).

For \( b_k \neq 0 \) we expand \( P_k((z-b_k)^{-1}) \) about \( z = 0 \):

\[ P_k((z-b_k)^{-1}) = \frac{1}{z-b_k} = -\frac{1}{b_k} + \mathcal{O}(z). \]

If we take
we obtain

\[ f_k(z) = \frac{1}{z - b_k} - Q_k(z) \]
\[ = \frac{1}{z - b_k} + \frac{1}{b_k} \]
\[ = \frac{1}{(z - b_k)b_k} \]

If \(|z| \leq R\) and \(|b_k| \geq 2R\) then we have

\[ |z - b_k| \geq |b_k| - |z| \geq \frac{1}{2} |b_k| , \]

and, therefore,

\[ |f_k|_{D(0,R)} \leq \frac{2R}{|b_k|^2} . \]

It follows that the series

\[ f(z) = \sum_{k=1}^{\infty} \left( \frac{1}{z - b_k} + \frac{1}{b_k} \right) , \quad z \in \mathbb{C} \setminus D =: U , \tag{6.4} \]

converges normally in \(U\) to a meromorphic function \(f(z)\). The function \(f(z)\) has a simple pole at each \(b_k\) with residue 1; the function \(f(z)\) has no other pole. We also know that we can differentiate (6.4) arbitrarily often term by term.

Now assume that we want to construct a meromorphic function \(f(z)\) with poles precisely at the integers and residue 1 at each integer. According to the above, such a function is

\[ f(z) = \frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z - n} + \frac{1}{n} \right) . \]
7 Infinite Products

7.1 Infinite Products of Complex Numbers

Let $q_1, q_2, \ldots$ denote a sequence of complex numbers. We want to define what it means that the infinite product

$$\prod_{j=1}^{\infty} q_j$$

converges and what the value of this product is if it converges. Naively, and in analogy to series, one considers the sequence of finite products,

$$p_n = \prod_{j=1}^{n} q_j ,$$

for $n = 1, 2, \ldots$ and calls the infinite product convergent to $p$ if $p_n \to p$. This leads to Definition 7.1 below. Most authors define convergence of an infinite product more restrictively. Using the more restrictive definition, the theorems about infinite products become simpler to formulate.

7.1.1 Two Definitions of Convergence

Let

$$q_j, \quad j = 1, 2, \ldots$$

denote a sequence of complex numbers. We can form the finite products

$$p_n = \prod_{j=1}^{n} q_j$$

for $n = 1, 2, \ldots$. In correspondence with infinite series, we define:

**Definition 7.1:** The infinite product

$$\prod_{j=1}^{\infty} q_j$$

converges (in the simple sense) if the sequence of partial products $p_n$ converges as $n \to \infty$. If $p_n \to p$ as $n \to \infty$ then we write

$$\prod_{j=1}^{\infty} q_j = p \quad (in \ the \ simple \ sense)$$

and call $p$ the value of the infinite product.

This definition is used, for example, in [Stein, Sharkarchi].

If one of the factors $q_j$ in the infinite product is zero then, using the above definition, the infinite product $\prod_j q_j$ always converges to zero (in the simple sense). Other authors prefer a more restrictive definition of convergence for infinite products. The following definition is used by [Greene, Krantz] and others.

**Definition 7.2:** The infinite product

$$\prod_{j=1}^{\infty} q_j$$

converges if the following holds:
1) At most finitely many of the $q_j$ are equal to zero.
2) If $N_0 > 0$ is so large that $q_j \neq 0$ for $j > N_0$ then

$$\lim_{N \to \infty} \prod_{j=N_0+1}^N q_j$$

exists and is non–zero. If these two conditions are met, then the value of the
infinite product is the number

$$\left( \prod_{j=1}^{N_0} q_j \right) \cdot \left( \lim_{N \to \infty} \prod_{j=N_0+1}^N q_j \right).$$

It is easy to show that the value of the product does not depend on the
choice of $N_0$. Here we assume that $q_j \neq 0$ for $j > N_0$.

The difference between the two definitions is not profound. Like most au-
thors, we prefer here to work with Definition 7.2. Then the formulation of
many theorems becomes simpler. For example, later we will consider products
of holomorphic functions,

$$p(z) = \prod_{j=1}^\infty \left( 1 + a_j(z) \right), \quad a_j \in H(U).$$

Then, using Definition 7.2, the product $p(z)$ is zero at some $z = z_0$ if and only
if at least one factor $1 + a_j(z)$ is zero at $z = z_0$.

7.1.2 Examples

Example 1: According to Definition 7.1, the infinite product

$$\prod_{j=0}^\infty j = 0 \cdot 1 \cdot 2 \cdot \ldots$$

converges to zero (in the simple sense). According to Definition 7.2, this product
diverges because

$$\prod_{j=1}^n j = n!$$

does not converge to a finite limit as $n \to \infty$.

Example 2: According to Definition 7.1, the infinite product

$$\prod_{j=1}^\infty \left( \frac{1}{2} \right)^j$$

converges to zero (in the simple sense). According to Definition 7.2, this product
diverges because

$$\prod_{j=1}^n \left( \frac{1}{2} \right)^j \to 0 \quad \text{as} \quad n \to \infty,$$

but no factor is zero.

Example 3: Let

$$q_j = 1 - \frac{1}{j^2}, \quad j = 2, 3, \ldots$$

Consider the infinite product
\[
\Pi_{j=2}^\infty \left(1 - \frac{1}{j^2}\right) = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \ldots
\]  \quad \text{(7.1)}

Here the partial products are

\[
p_n = \frac{1}{2^2}(2^2 - 1)\frac{1}{3^2}(3^2 - 1) \ldots \frac{1}{n^2}(n^2 - 1)
\]
\[
= \frac{1}{n!n!} (1 \cdot 3)(2 \cdot 4) \ldots ((n - 1)(n + 1))
\]
\[
= \frac{1}{n!n!} (n - 1)! \frac{1}{2} (n + 1)!
\]
\[
= \frac{1}{n!} \frac{n+1}{n}
\]

It follows that \(p_n \to \frac{1}{2}\). The infinite product (7.1) converges to \(\frac{1}{2}\), using either definition.

**Example 4:** Let

\[
q_j = 1 - \frac{1}{j}, \quad j = 2, 3, \ldots
\]

Consider the infinite product

\[
\Pi_{j=2}^\infty \left(1 - \frac{1}{j}\right) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \ldots
\]  \quad \text{(7.2)}

Here the partial products are

\[
p_n = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \ldots \left(1 - \frac{1}{n}\right)
\]
\[
= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \ldots \frac{n-1}{n}
\]
\[
= \frac{1}{n}.
\]

It follows that \(p_n \to 0\). According to Definition 7.1, the infinite product (7.2) converges to zero, in the simple sense. According to Definition 7.2, the infinite product diverges since \(p_n \to 0\), but no factor is zero.

**Example 5:** We claim that the infinite product

\[
\Pi_{j=1}^\infty \left(1 + \frac{1}{j}\right)
\]

also diverges. We have

\[
p_n = \Pi_{j=1}^n \left(1 + \frac{1}{j}\right)
\]
\[
= \Pi_{j=1}^n \left(\frac{j+1}{j}\right)
\]
\[
= \frac{2 \cdot 3 \ldots n+1}{1 \cdot 2 \ldots n}
\]
\[
= n+1
\]
Since $p_n \to \infty$ as $n \to \infty$ the infinite product diverges.

Example 6: According to [Remmert, Classical Topics in Complex Function Theory], infinite products first appeared in 1579 in the work of F. Vieta. He gave the formula

\[
\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2}}} \cdots}
\]

Let us understand the result. The infinite product can be written as

\[\Pi_{j=2}^{\infty} q_j\]

with

\[q_2 = \sqrt{\frac{1}{2}}\]

and

\[q_{j+1} = \sqrt{\frac{1}{2} + \frac{1}{2} q_j}\]

for $j = 2, 3, \ldots$

Thus, the $q_j$ obey the recursion

\[q_{j+1}^2 = \frac{1}{2} + \frac{1}{2} q_j, \quad q_2 = \sqrt{\frac{1}{2}}\]

Recall that

\[
\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \\
= 2 \cos^2 \alpha - 1,
\]

thus

\[\cos^2 \alpha = \frac{1}{2} + \frac{1}{2} \cos 2\alpha.
\]

We can also write this as

\[\cos^2 \beta = \frac{1}{2} + \frac{1}{2} \cos \beta.
\]

Since

\[q_2 = \sqrt{\frac{1}{2}} = \cos(\pi/2^2)\]

it follows that

\[q_j = \cos(\pi/2^j), \quad j = 2, 3, \ldots\]

This formula for $q_j$ makes the determination of the infinite product manageable.
Recall that
\[ \sin 2\theta = 2 \sin \theta \cos \theta , \]
thus
\[ \cos \theta = \frac{1}{2} \frac{\sin 2\theta}{\sin \theta} . \]
We then obtain
\[ q_j = \frac{1}{2} \frac{\sin(\pi/2^{j-1})}{\sin(\pi/2^j)} , \]
thus
\[
q_2 q_3 \ldots q_n = \frac{1}{2^{n-1}} \frac{\sin(\pi/2)}{\sin(\pi/2^2)} \frac{\sin(\pi/2^2)}{\sin(\pi/2^3)} \ldots \frac{\sin(\pi/2^{n-1})}{\sin(\pi/2^n)}
= \frac{1}{2^{n-1}} \frac{\sin(\pi/2)}{\sin(\pi/2^n)}
= \frac{2}{\pi} \frac{\pi/2^n}{\sin(\pi/2^n)}
\]
Since
\[ \frac{t}{\sin t} \rightarrow 1 \quad \text{as} \quad t \rightarrow 0 \]

it follows that the partial products
\[ p_n = q_2 q_3 \ldots q_n \]
converge to \( 2/\pi \) as \( n \rightarrow \infty \). Using Definition 7.1 or 7.2, Vieta’s product converges to \( 2/\pi \).

7.2 Infinite Products of Numbers: Convergence Theory

Recall the Cauchy convergence criterion for a sequence \( p_n \) of complex numbers: The sequence \( p_n \) converges if and only if for every \( \varepsilon > 0 \) there exists \( J = J(\varepsilon) \in \mathbb{N} \) so that
\[ |p_n - p_m| < \varepsilon \quad \text{for} \quad n > m \geq J(\varepsilon) . \]

**Lemma 7.1 (Cauchy Criterion)** The infinite product
\[ \prod_{j=1}^{\infty} q_j \]
converges in the sense of Definition 7.2 if and only if for all \( \varepsilon > 0 \) there is \( J = J(\varepsilon) \in \mathbb{N} \) so that
\[ \left| \prod_{j=m+1}^{n} q_j - 1 \right| < \varepsilon \quad \text{for} \quad n > m \geq J(\varepsilon) . \]
Proof: 1) Assume that (7.3) converges. There exists \( J_0 \in \mathbb{N} \) so that \( q_j \neq 0 \) for \( j \geq J_0 \). Set
\[
p_n = \Pi_{j=J_0}^n q_j \quad \text{for} \quad n \geq J_0.
\]
By assumption,
\[
p_n \to L \quad \text{as} \quad n \to \infty, \quad L \neq 0.
\]
For \( n > m \geq J_0 \) we have
\[
\frac{p_n}{p_m} = \Pi_{j=m+1}^n q_j.
\]
If \( \varepsilon_0 > 0 \) is given, then there exists \( K(\varepsilon_0) \geq J_0 \) with
\[
|p_n - L| < \varepsilon_0 \quad \text{and} \quad |p_m - L| < \varepsilon_0 \quad \text{for} \quad n > m \geq K(\varepsilon_0).
\]
Therefore,
\[
\frac{p_n}{p_m} = \frac{L + \eta_n}{L + \eta_m} = \frac{1 + \eta_n/L}{1 + \eta_m/L} = 1 + \eta
\]
with
\[
|\eta_n| < \varepsilon_0 \quad \text{and} \quad |\eta_m| < \varepsilon_0.
\]
If \( \varepsilon > 0 \) is given and \( \varepsilon_0 > 0 \) is chosen small enough, then
\[
\left| \frac{p_n}{p_m} - 1 \right| = |\eta| < \varepsilon.
\]
This proves (7.4).

2) Conversely, assume that for all \( \varepsilon > 0 \) there exists \( J(\varepsilon) \in \mathbb{N} \) so that (7.4) holds. First, let \( J_0 = J(1/2) \). Then the estimate (7.4) with \( \varepsilon = 1/2 \) implies that \( q_j \neq 0 \) for \( j > J_0 \). Set
\[
p_n = \Pi_{j=J_0+1}^n q_j \quad \text{for} \quad n > J_0.
\]
We must show that the sequence \( p_n \) converges to a non-zero limit. From
\[
|p_n - 1| = \left| \Pi_{j=J_0+1}^n q_j - 1 \right| < \frac{1}{2} \quad \text{for} \quad n > J_0
\]
it follows that the sequence \( p_n \) is bounded; in fact, \( |p_n| < \frac{3}{2} \).

Let \( \varepsilon > 0 \) be given and let
\[
\left| \frac{p_n}{p_m} - 1 \right| = \left| \Pi_{j=m+1}^n q_j - 1 \right| < \varepsilon \quad \text{for} \quad n > m \geq J(\varepsilon) \geq J_0.
\]
This estimate implies that
\[
|p_n - p_m| = \left| \frac{p_n}{p_m} - 1 \right| |p_m| < \varepsilon \cdot \frac{3}{2} \quad \text{for} \quad n > m \geq J(\varepsilon).
\]
Therefore, the sequence \( p_n \) converges, \( p_n \to L \).
The estimate $|p_n - 1| < \frac{1}{2}$ proved above implies that $L \neq 0$. ♦

Using the notation of the above proof, we have

$$|q_{j+1} - 1| = \left| \frac{p_{j+1}}{p_j} - 1 \right| < \varepsilon \quad \text{for} \quad j \geq J(\varepsilon) .$$

This proves the following:

**Lemma 7.2** If

$$\Pi_{j=1}^{\infty} q_j$$

converges, then

$$q_j \to 1 \quad \text{as} \quad j \to \infty .$$

The previous lemma suggests to write the factors $q_j$ in an infinite product

$$\Pi_{j=1}^{\infty} q_j$$

in the form

$$q_j = 1 + a_j .$$

If the infinite product $\Pi(1 + a_j)$ converges, then $a_j \to 0$. The converse is not true, of course, as the example

$$\Pi_{j=2}^{\infty} \left( 1 - \frac{1}{j} \right)$$

shows.

We will need the following simple estimates for the real exponential function.

**Lemma 7.3** We have

$$1 + x \leq e^x \quad \text{for} \quad x \geq 0 \quad (7.5)$$

and

$$e^{x/2} \leq 1 + x \quad \text{for} \quad 0 \leq x \leq 2 . \quad (7.6)$$

**Proof:** The second assertion follows from $e < 3$ and convexity of the exponential function. ♦

The following theorem relates the convergence of an infinite product to the absolute convergence of a series.

**Theorem 7.1** Let $a_j$ denote a sequence of complex numbers. Then $\sum_{j=1}^{\infty} |a_j| < \infty$ if and only if

$$\Pi_{j=1}^{\infty} (1 + |a_j|)$$

converges.
**Proof:** a) First assume that

\[ S := \sum_{j=1}^{\infty} |a_j| < \infty . \]

Then let

\[ p_n = \Pi_{j=1}^{n} (1 + |a_j|) \]

and obtain

\[ p_n \leq e^{\sum_{j=1}^{n} |a_j|} \leq e^S < \infty . \]

Because of

\[ 1 \leq p_1 \leq p_2 \leq \ldots \leq p_n \leq e^S < \infty , \]

convergence

\[ p_n \to p, \quad p \geq 1 , \]

follows.

b) Assume that

\[ \Pi_{j=1}^{\infty} (1 + |a_j|) \]

converges. Since \(|a_j| \to 0\) we have

\[ |a_j| \leq 2 \quad \text{for} \quad j > J , \]

thus

\[ e^{\frac{|a_j|}{2}} \leq 1 + |a_j| \quad \text{for} \quad j > J . \]

Setting

\[ S_1 = \lim_{n \to \infty} \Pi_{j=J+1}^{n} (1 + |a_j|) \]

we obtain:

\[ \exp \left( \frac{1}{2} \sum_{j=J+1}^{n} |a_j| \right) \leq \Pi_{j=J+1}^{n} (1 + |a_j|) \leq S_1 < \infty \]

for all \( n \geq J + 1 \). This yields the bound

\[ \sum_{j=J+1}^{n} |a_j| \leq 2 \ln S_1 < \infty , \]

which implies convergence of the series \( \sum |a_j| \).

It is less clear how to relate convergence of the series

\[ \sum a_j \]
to convergence of the product

$$\Pi(1 + a_j) .$$

However, convergence of the series $\sum |a_j|$ does imply convergence of the product $\Pi(1 + a_j)$. We will formulate this as the next theorem.

**Theorem 7.2** Let $a_j$ be a sequence of complex numbers. If $\sum_{j=1}^{\infty} |a_j| < \infty$ then

$$\Pi_{j=1}^{\infty} (1 + a_j)$$

converges.

The proof uses the following lemma.

**Lemma 7.4** Let $a_1, \ldots, a_n \in \mathbb{C}$ and set

$$p_n = \Pi_{j=1}^{n} (1 + a_j), \quad q_n = \Pi_{j=1}^{n} (1 + |a_j|) .$$

Then the bound

$$|p_n - 1| \leq q_n - 1$$

holds.

**Proof:** We have

$$p_n = (1 + a_1) \ldots (1 + a_n)$$

$$= 1 + \sum a_{i_1} \ldots a_{i_r}$$

where the sum is taken over all indices $i_1, \ldots, i_r$ with

$$1 \leq i_1 < \ldots < i_r \leq n .$$

Therefore,

$$|p_n - 1| \leq \sum |a_{i_1} \ldots a_{i_r}|$$

$$= q_n - 1$$

This proves the lemma. ◻

**Proof of Theorem 7.2:** 1) We have $a_j \neq -1$ for $j \geq J$. For simplicity of notation, assume that $a_j \neq -1$ for all $j \geq 1$. Then define

$$p_n = \Pi_{j=1}^{n} (1 + a_j), \quad q_n = \Pi_{j=1}^{n} (1 + |a_j|) .$$

We have, for $n > m \geq 1$: 55
\[ |p_n - p_m| = |p_m| \left| \frac{p_n}{p_m} - 1 \right| \]
\[ = |p_m| \left| \prod_{j=m+1}^{n} (1 + a_j) - 1 \right| \]
\[ \leq |q_m| \left| \prod_{j=m+1}^{n} (1 + |a_j|) - 1 \right| \]
\[ = |q_m| \left( \frac{q_n}{q_m} - 1 \right) \]
\[ = |q_n - q_m|. \]

Therefore, if \( \varepsilon > 0 \) is given, there exists \( N(\varepsilon) \) with
\[ |p_n - p_m| \leq |q_n - q_m| < \varepsilon \quad \text{if} \quad n > m \geq N(\varepsilon). \]

Here, to estimate \( |q_n - q_m| \), we use Theorem 7.1. This proves that \( p_n \) converges to a limit \( L \) as \( n \to \infty \).

Fixing \( p_m \) with \( p_m \neq 0 \) and choosing \( \varepsilon = \frac{|p_m|}{2} \) in the above estimate yields that
\[ |p_n - p_m| < \frac{|p_m|}{2} \]
for all large \( n \). This implies that \( L = \lim p_n \) cannot be zero. \( \diamond \)

### 7.3 Infinite Products of Functions

Looking carefully at the proofs of the previous section, one obtains uniformity of convergence of a product of functions
\[ \Pi(1 + f_j(z)) \]
if the corresponding series \( \sum |f_j(z)| \) converges uniformly. We formulate such a result in the following lemma.

**Lemma 7.5** Let \( K \subset \mathbb{C} \) denote a compact set and let \( f_j : K \to \mathbb{C} \) denote a sequence of continuous functions with
\[ \sum_{j=1}^{\infty} |f_j|_K < \infty \quad \text{where} \quad |f_j|_K := \max_{z \in K} |f_j(z)|. \]

Then the sequence of functions
\[ p_n(z) = \prod_{j=1}^{n} (1 + f_j(z)), \quad n = 1, 2 \ldots \]
converges uniformly on \( K \).

**Proof:** Fix \( z \in K \) and set
\[ p_n(z) = \Pi_{j=1}^{n} (1 + f_j(z)) \]
\[ q_n(z) = \Pi_{j=1}^{n} (1 + |f_j(z)|) \]
\[ Q_n = \Pi_{j=1}^{n} (1 + |f_j|_K) \]

By Theorem 11.1, the sequence of numbers \( Q_n \) converges.

We have, for \( n > m \geq 1 \):

\[
|p_n(z) - p_m(z)| \leq |q_n(z) - q_m(z)|
= |q_m(z)||\Pi_{j=m+1}^{n}(1 + |f_j(z)|) - 1|
\leq Q_m|\Pi_{j=m+1}^{n}(1 + |f_j|_K) - 1|
= Q_m\left| \frac{Q_n}{Q_m} - 1 \right|
= |Q_n - Q_m|.
\]

Therefore, if \( \varepsilon > 0 \) is given, there is \( N(\varepsilon) \) with

\[
|p_n(z) - p_m(z)| \leq |Q_n - Q_m| < \varepsilon \quad \text{for} \quad n > m \geq N(\varepsilon).
\]

Here \( N(\varepsilon) \) does not depend on \( z \). This proves the lemma. \( \diamond \)

The following result is the workhorse for convergence of infinite products of holomorphic functions.

**Theorem 7.3 (Main Theorem on Convergence of Infinite Products of Holomorphic Functions)** Assume that \( U \) is an open subset of \( \mathbb{C} \) and let \( f_j \in H(U) \) denote a sequence of holomorphic functions on \( U \). Assume that the series \( \sum_{j=1}^{\infty} |f_j|_K \) converges for every compact subset \( K \) of \( U \). Then the following holds:

1) For every \( z \in U \) the infinite product

\[ \Pi_{j=1}^{\infty} (1 + f_j(z)) =: F(z) \]

converges and defines a function \( F \in H(U) \).

2) The sequence of holomorphic functions

\[ F_N(z) = \Pi_{j=1}^{N} (1 + f_j(z)), \quad z \in U, \]

converges normally in \( U \) to \( F(z) \).

3) For every \( z_0 \in U \) the function \( F(z) \) has a zero at \( z_0 \) if and only if one of the factors \( q_j(z) = 1 + f_j(z) \) has a zero at \( z_0 \). Furthermore, the multiplicity of \( z_0 \) as a zero of \( F(z) \) is the finite sum (over \( j \)) of the multiplicities of the zero \( z_0 \) of the factors \( q_j(z) \).

4) If \( F(z) \neq 0 \) then

\[
\frac{F'(z)}{F(z)} = \sum_{j=1}^{\infty} \frac{f_j'(z)}{1 + f_j(z)}.
\]
Here the series converges normally in the set

\[ U_0 = \{z \in U : F(z) \neq 0\} . \]

**Proof:** 1) and 2) The convergence of the infinite product for each fixed \( z \in U \) follows from Theorem 7.2. Fix \( z_0 \in U \) and let \( K := \bar{D}(z_0, \varepsilon) \subset U \). Let \( |f_j|_K \leq \frac{1}{2} \) for \( j > J \). Write

\[ F_N(z) = \Pi_{j=1}^J (1 + f_j(z)) \cdot \Pi_{j=J+1}^N (1 + f_j(z)) =: F_J(z) \cdot G_N(z) . \] (7.7)

Note that the function \( G_N(z) \) has no zero in \( K \). The first finite product, \( F_J(z) \), is a function in \( H(U) \). The sequence \( G_N(z) \) converges uniformly on \( K \) to a function \( G \in C(K) \). This follows from the previous lemma. We then have \( G \in H(D(z_0, \varepsilon)) \). The sequence \( F_N \) converges uniformly on \( K \) to \( F \). It follows that \( F \in H(D(z_0, \varepsilon)) \). Since \( z_0 \in U \) was arbitrary, we have shown that \( F \in H(U) \).

3) In the factorization (7.7) let \( N \to \infty \) to obtain that

\[ F(z) = F_J(z) \cdot G(z), \quad z \in D(z_0, \varepsilon) . \]

Here \( G \) is holomorphic and nowhere zero in \( D(z_0, \varepsilon) \). This implies 3).

4) For a product of three functions \( q_j(z) \),

\[ Q = q_1 q_2 q_3 \]

we have

\[ Q'(z) = q_1 q_2 q_3 + q_1 q_2' q_3 + q_1 q_2 q_3' . \]

Therefore, if \( Q(z) \neq 0 \),

\[ \frac{Q'(z)}{Q(z)} = \sum_{j=1}^3 q_j'(z) q_j(z) . \]

It is clear that this generalizes to any finite sum and we have for \( F(z) \neq 0 \):

\[ \frac{F'_N(z)}{F_N(z)} = \sum_{j=1}^N \frac{f'_j(z)}{1 + f_j(z)} . \]

Letting \( N \to \infty \) we obtain that

\[ \frac{F'(z)}{F(z)} = \sum_{j=1}^\infty \frac{f'_j(z)}{1 + f_j(z)} . \]

It is not difficult to prove that the series converges normally in \( U_0 \). (Homework) \( \diamond \)
7.4 Example: The Product Formula for the Sine Function

We recall: \( \pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}, \quad z \in \mathbb{C} \setminus \mathbb{Z} . \quad (7.8) \)

We want to use this to prove:

\[
\frac{1}{\pi} \sin(\pi z) = z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right), \quad z \in \mathbb{C} . \quad (7.9)
\]

Let

\[
f_n(z) = \frac{-z^2}{n^2}, \quad n \in \mathbb{N} .
\]

If \( K \) is a compact subset of \( \mathbb{C} \) then \( K \subset \bar{D}(0, R) \) for some \( R > 0 \). We then have

\[
|f_n|_K \leq \frac{R^2}{n^2}.
\]

Since \( \sum n^{-2} < \infty \) we obtain \( \sum |f_n|_K < \infty \) and, by the previous theorem, convergence of the infinite product \( \prod_n (1 + f_n(z)) \) follows. The function

\[
P(z) = z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right)
\]

is entire and \( P(z) \) has a simple zero at each integer. Also, for \( z \in \mathbb{C} \setminus \mathbb{Z} \),

\[
\frac{P'(z)}{P(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \pi \cot(\pi z) .
\]

If we set

\[
G(z) = \frac{1}{\pi} \sin(\pi z)
\]

then we also have, from calculus,

\[
\frac{G'(z)}{G(z)} = \pi \cot(\pi z), \quad z \in \mathbb{C} \setminus \mathbb{Z} .
\]

The function \( P/G \) has a removable singularity at each integer. We have

\[
\left( \frac{P}{G} \right)' = \frac{P'G - PG'}{G^2} = \frac{P'}{G} \cdot \left( \frac{P'}{P} - \frac{G'}{G} \right) = 0 .
\]

Therefore, \( P/G = c = \text{const} . \) Considering

\[
\lim_{z \to 0} \frac{P(z)}{z} = 1 = \lim_{z \to 0} \frac{G(z)}{z},
\]

we conclude that \( c = 1 \), i.e., \( P(z) \equiv G(z) \). This proves the product formula (7.9).

\[\footnote{Using residue calculus for \( \int_{\gamma_n} \frac{\cot(\pi z)}{(\zeta - z)^2} \, d\zeta \), \( z \in \mathbb{C} \setminus \mathbb{Z} \), where \( \gamma_n \) is a sequence of growing rectangles, we have seen that \( \sum_{j=-\infty}^{\infty} \frac{1}{\pi(j^2 \pi)} = \frac{\pi}{\sin^2(\pi z)} \). Then the formula (7.8) follows by integration.} \]
7.5 Euler’s Constant

We define here Euler’s constant $\gamma$, also called the Euler–Mascheroni constant. It occurs frequently when one discusses the $\Gamma$–function. A numerical value is

$$\gamma = 0.57721566\ldots$$

It is unknown whether $\gamma$ is rational or irrational.

Lemma 7.6  For $n = 1, 2, \ldots$ let

$$\gamma_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n)$$

$$\delta_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n+1)$$

We claim that

$$\delta_n < \delta_{n+1} < \gamma_{n+1} < \gamma_n .$$

The sequences $\gamma_n$ and $\delta_n$ converge to the same limit $\gamma$, called Euler’s constant.

Proof: We have

$$\gamma_{n+1} - \gamma_n = \frac{1}{n+1} + \ln(n) - \ln(n+1)$$

$$= \frac{1}{n+1} - \int_n^{n+1} \frac{dx}{x} .$$

For $n < x < n+1$ we have

$$\frac{1}{n+1} < \frac{1}{x} < \frac{1}{n} ,$$

thus

$$\gamma_{n+1} - \gamma_n < 0 .$$

Similarly,

$$\delta_{n+1} - \delta_n = \frac{1}{n+1} + \ln(n+1) - \ln(n+2)$$

$$= \frac{1}{n+1} - \int_{n+1}^{n+2} \frac{dx}{x} .$$

For $n+1 < x < n+2$ we have

$$\frac{1}{n+2} < \frac{1}{x} < \frac{1}{n+1} ,$$

thus

$$\delta_{n+1} - \delta_n > 0 .$$
Also,

\[ \gamma_n - \delta_n = \ln (n + 1) - \ln (n) = \ln \left( 1 + \frac{1}{n} \right) > 0. \]

Since \( \gamma_n - \delta_n \to 0 \) as \( n \to \infty \), the lemma is proved. \( \Diamond \)

7.6 The Gauss Formula for \( \Gamma(z) \) and Weierstrass’ Product Formula for \( 1/\Gamma(z) \)

Recall that

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad \text{Re} \, z > 0. \]

Splitting the integral as

\[ \int_0^\infty = \int_0^1 \ldots + \int_1^\infty \ldots =: g(z) + h(z) \]

we obtain \( h \in H(\mathbb{C}) \) and use the series

\[ e^{-t} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} t^j \]

to obtain

\[ g(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_0^1 t^{z-1+j} \, dt = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{1}{z+j}. \]

In this way we obtain a formula for the \( \Gamma \)-function valid for

\[ z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} =: U, \]

namely,

\[ \Gamma(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{1}{z+j} + \int_1^{\infty} t^{z-1} e^{-t} \, dt, \quad z \in U. \]

It is clear that \( \Gamma \in H(U) \). The function \( \Gamma \) has a simple pole at each \( n \in \{0, -1, -2, \ldots\} \) and

\[ \text{Res}(\Gamma, -j) = \frac{(-1)^j}{j!} \quad \text{for} \quad j = 0, 1, 2, \ldots \]

We also recall the reflection formula

\[ \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \in \mathbb{C} \setminus \mathbb{Z}, \]
which implies that $\Gamma$ has no zero. Therefore, the function
\[
\Delta(z) := \frac{1}{\Gamma(z)}
\]
is entire. The function $\Delta(z)$ has a simple zero at each $n \in \{0, -1, -2, \ldots\}$ and has no other zeros. The zeros of $\Delta(z)$ clearly show up in the following product formula.

**Theorem 7.4 (Weierstrass)** We have
\[
\Delta(z) = \frac{1}{\Gamma(z)} = e^{\gamma z} \cdot z \cdot \prod_{j=1}^{\infty} \left(1 + \frac{z}{j}\right) e^{-z/j}, \quad z \in \mathbb{C},
\]
where $\gamma$ is Euler’s constant,
\[
\gamma = \lim_{n \to \infty} \gamma_n = 0.57721\ldots
\]
with
\[
\gamma_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n).
\]

**Convergence of the infinite product in (7.10):** We have
\[
e^{-z/j} = 1 - \frac{z}{j} + \eta_j(z), \quad |\eta_j(z)| \leq C(R)j^{-2} \quad \text{for} \quad |z| \leq R.
\]
We then have
\[
\left(1 + \frac{z}{j}\right) e^{-z/j} = 1 - \frac{z^2}{j^2} + \tilde{\eta}_j(z)
\]
with
\[
|\tilde{\eta}_j(z)| \leq \tilde{C}(R)j^{-2} \quad \text{for} \quad |z| \leq R.
\]
It is then clear that the infinite product in (7.10) converges uniformly for $|z| \leq R$, where $R > 0$ is arbitrary. Therefore,
\[
Q(z) := e^{\gamma z} \cdot z \cdot \prod_{j=1}^{\infty} \left(1 + \frac{z}{j}\right) e^{-z/j}
\]
is an entire function. This function has a simple zero at each $n \in \{0, -1, -2, \ldots\}$ and has no other zeros.

We want to prove that $Q(z) = \Delta(z)$. To do this, we will prove the equality for $0 < x = z \leq 1$ and then apply the identity theorem.

Recall that $\phi(x) = \ln(\Gamma(x))$, $x > 0$, satisfies $\phi''(x) > 0$ for $x > 0$. (See Math 561.)

**Lemma 7.7** Let $\phi : (0, \infty) \to \mathbb{R}$ be a $C^2$ function with $\phi''(x) > 0$ for all $x > 0$. If $0 < a < b < c$ then we have
\[
\frac{\phi(b) - \phi(a)}{b-a} \leq \frac{\phi(c) - \phi(b)}{c-b}.
\]
and
\[ \frac{\phi(b) - \phi(a)}{b - a} \leq \frac{\phi(c) - \phi(a)}{c - a} . \]

**Proof:** The first estimate follows from the mean value theorem. To prove the second estimate, consider the function
\[ h(s) = \frac{\phi(s) - \phi(a)}{s - a} \quad \text{for} \quad s > a . \]

Differentiation yields
\[ h'(s) = \frac{1}{s - a} \left( \frac{\phi'(s) - \phi(a)}{s - a} \right) . \]

Then, using the mean value theorem, we find that \( h'(s) \geq 0 \) for \( s > a \). ∎

**Application:** We let \( \phi(x) = \ln \Gamma(x) \), \( x > 0 \). Let \( n \) be an integer, \( n \geq 2 \), and let \( 0 < x \leq 1 \). We have
\[ 0 < n - 1 < n < n + x \leq n + 1 \]
and obtain
\[ \frac{\phi(n) - \phi(n - 1)}{n - (n - 1)} \leq \frac{\phi(n + x) - \phi(n)}{n + x - n} \leq \frac{\phi(n + 1) - \phi(n)}{n + 1 - n} . \]

Since
\[ \phi(n) - \phi(n - 1) = \ln \frac{\Gamma(n)}{\Gamma(n - 1)} = \ln(n - 1) \]
we obtain
\[ x \ln(n - 1) \leq \ln \left( \frac{\Gamma(n + x)}{\Gamma(n)} \right) \leq x \ln n . \]

Therefore,
\[ (n - 1)^x \leq \frac{\Gamma(n + x)}{\Gamma(n)} \leq n^x , \]
thus
\[ (n - 1)! (n - 1)^x \leq \Gamma(n + x) \leq (n - 1)! n^x . \]

From the functional equation \( \Gamma(z + 1) = z \Gamma(z) \) we have
\[ \Gamma(n + x) = (x + n - 1)(x + n - 2) \cdots (x + 1)x \Gamma(x) . \]

This yields the bounds
\[ \frac{(n - 1)! (n - 1)^x}{x(x + 1) \cdots (x + n - 1)} \leq \Gamma(x) \leq \frac{(n - 1)! n^x}{x(x + 1) \cdots (x + n - 1)} . \]

In the lower bound, we may replace \( n - 1 \) by \( n \). Then we have
\[
\frac{n!n^x}{x(x+1)\ldots(x+n)} \leq \Gamma(x) \leq \frac{n!n^x}{x(x+1)\ldots(x+n)} \cdot \frac{x+n}{n}.
\]

Denote
\[
A_n = \frac{n!n^x}{x(x+1)\ldots(x+n)}.
\]

We have shown that
\[
A_n \leq \Gamma(x) \leq A_n \cdot \frac{x+n}{n}.
\]

Equivalently,
\[
\frac{n}{n+x} \Gamma(x) \leq A_n \leq \Gamma(x).
\]

We obtain:

**Lemma 7.8** Let \(0 < x \leq 1\). Then we have
\[
\Gamma(x) = \lim_{n \to \infty} \frac{n!n^x}{x(x+1)\ldots(x+n)}.
\]

Define, for \(z \in \mathbb{C}\),
\[
h_n(z) = \frac{z(z+1)\ldots(z+n)}{n!n^z} = e^{-\ln n} z \cdot z \cdot \left(1 + \frac{z}{1}\right) \left(1 + \frac{z}{2}\right) \ldots \left(1 + \frac{z}{n}\right)
\]
\[
= e^{\ln z} \cdot z \cdot \left(1 + \frac{z}{1}\right) e^{-z/1} \left(1 + \frac{z}{2}\right) e^{-z/2} \ldots \left(1 + \frac{z}{n}\right) e^{-z/n}
\]

The above lemma implies that, for \(0 < x \leq 1\):
\[
\Gamma(x) = \lim_{n \to \infty} \frac{1}{h_n(x)}.
\]

Also,
\[
h_n(z) \to Q(z)
\]

normally in \(\mathbb{C}\) where \(Q(z)\) is defined in (7.11). Therefore,
\[
\frac{1}{h_n(z)} \to \frac{1}{Q(z)}
\]

normally in \(U\). By (7.12) we obtain that
\[
\Gamma(x) = \frac{1}{Q(x)} \quad \text{for} \quad 0 < x \leq 1.
\]

The identity theorem then yields that
\[ \Gamma(z) = \frac{1}{Q(z)} \quad \text{for} \quad z \in U. \]

This proves the Weierstrass’ product formula for \( \Delta(z) = 1/\Gamma(z) \) and the following Gauss’ formula for \( \Gamma(z) \):

**Theorem 7.5 (Gauss)** We have

\[ \Gamma(z) = \lim_{n \to \infty} \frac{n! \, n^z}{z(z+1) \cdots (z+n)} \]

with normal convergence in \( U = \mathbb{C} \setminus \{0, -1, -2, \ldots\} \).

### 7.6.1 A Formula for \( (\Gamma' / \Gamma)' \)

Let \( U = \mathbb{C} \setminus \{0, -1, -2, \ldots\} \) and set

\[ f_n(z) = n! e^{(\ln n)z} \prod_{j=0}^{n-1} (z+j)^{-1}, \quad z \in U. \]

By Gauss’ formula we have \( f_n(z) \to \Gamma(z) \) normally in \( U \). We have

\[ \frac{f'_n(z)}{f_n(z)} = \ln n - \sum_{j=0}^{n} (z+j)^{-1} \]

and

\[ \frac{d}{dz} \frac{f'_n(z)}{f_n(z)} = \sum_{j=0}^{n} (z+j)^{-2}. \]

For \( n \to \infty \) obtain:

\[ \frac{d}{dz} \frac{\Gamma'(z)}{\Gamma(z)} = \sum_{j=0}^{\infty} (z+j)^{-2}. \]

Recall that we have used the inequality \( \phi''(x) > 0 \) for \( 0 < x < \infty \) where \( \phi(x) = \ln \Gamma(x) \). We now have shown that

\[ \phi''(x) = \sum_{j=0}^{\infty} \frac{1}{(x+j)^2}, \quad 0 < x < \infty. \]

### 7.7 Entire Functions with Prescribed Zeros

If \( a_1, \ldots, a_n \in \mathbb{C} \) then the polynomial

\[ p(z) = (a_1 - z) \cdots (a_n - z) \]

vanishes precisely at \( a_1, \ldots, a_n \). Now let \( a_1, a_2, \ldots \) denote an infinite sequence in \( \mathbb{C} \) without accumulation point in \( \mathbb{C} \), i.e., \( |a_n| \to \infty \). We want to construct an entire function \( f(z) \) with \( f(z) = 0 \) if and only if \( z \in \{a_1, a_2, \ldots\} \). We allow repetitions in the \( a_j \). Then, if a number \( a \) appears \( q \) times among the \( a_j \), we
want \( f(z) \) to have a zero of order precisely \( q \) at \( a \). We will show below how to construct such a function \( f(z) \).

**Remarks on Non–Uniqueness.** If \( f(z) \) vanishes precisely at the \( a_j \) (to the correct order) then any function

\[
g(z) = f(z)e^{h(z)}, \quad h \in H(\mathbb{C}),
\]

has the same property. We show:

**Lemma 7.9** Let \( a_j \) be a sequence in \( \mathbb{C} \) without accumulation point in \( \mathbb{C} \). Let \( f(z) \) and \( g(z) \) be entire functions that vanish precisely at the \( a_j \). If \( a \) appears \( q \) times in the sequence, then let \( f(z) \) and \( g(z) \) have a zero of order \( q \) at \( a \). Under these assumptions, there is \( h \in H(\mathbb{C}) \) with

\[
g(z) = f(z)e^{h(z)}, \quad z \in \mathbb{C}.
\]

**Proof:** The quotient

\[
Q(z) = \frac{f(z)}{g(z)}, \quad z \in \mathbb{C} \setminus \{a_1, a_2, \ldots\},
\]

is bounded near every \( a_j \). By Riemann’s removability theorem, \( Q(z) \) extends to an entire function. Also, \( Q(z) \neq 0 \) for all \( z \). Therefore, \( Q(z) \) has the form \( Q(z) = e^{h(z)} \) where \( h(z) \) is entire.

**7.7.1 Construction of \( f(z) \); Motivation**

One might try to construct \( f(z) \) as the infinite product

\[
(a_1 - z)(a_2 - z) \cdots
\]

However, since \( |a_j| \to \infty \), this product never converges.

Somewhat smarter is the following: Let us assume that

\[
a_1 = a_2 = \ldots = a_m = 0 \quad \text{and} \quad a_j \neq 0 \quad \text{for} \quad j > m.
\]

Then try the infinite product

\[
z^m \left(1 - \frac{z}{a_{m+1}}\right)\left(1 - \frac{z}{a_{m+2}}\right) \cdots \tag{7.13}
\]

This works if

\[
\sum_{j>m} \frac{1}{|a_j|} < \infty.
\]

Under this assumption the above infinite product defines an entire function with the desired property.

For example, it suffices that

\[
|a_j| \geq c_j^{1+\varepsilon}, \quad j \geq J,
\]
for some $c > 0$ and $\varepsilon > 0$. However, if

$$|a_j| \sim j \quad \text{or} \quad |a_j| \sim \sqrt{j},$$

the infinite product will typically not converge.

Weierstrass’ idea was to use extra factors in the infinite product (7.13), which ensure convergence, but do not introduce new zeros.

### 7.7.2 Weierstrass’ Canonical Factors

Let us assume $a_j \neq 0$ for all $j$, for simplicity. Instead of the infinite product

$$\left(1 - \frac{z}{a_1}\right)\left(1 - \frac{z}{a_2}\right) \cdots \tag{7.14}$$

try

$$E\left(\frac{z}{a_1}\right)E\left(\frac{z}{a_2}\right) \cdots \tag{7.15}$$

where $E(z)$ is an entire function of the form

$$E(z) = (1 - z)\phi(z).$$

Here $\phi(z)$ should be suitably chosen, with $\phi(z) \neq 0$ for all $z$. Then one obtains

$$E\left(\frac{z}{a_1}\right)E\left(\frac{z}{a_2}\right) \cdots = \left(1 - \frac{z}{a_1}\right)\phi\left(\frac{z}{a_1}\right)\left(1 - \frac{z}{a_2}\right)\phi\left(\frac{z}{a_2}\right) \cdots \tag{7.16}$$

The factors $\phi(z/a_j)$ should make the product converge without introducing new zeros. Note that, as $|a_j| \to \infty$, the argument $z/a_j$ of $\phi$ converges to zero. We then want to construct $\phi(z)$ with

$$(1 - z)\phi(z) \sim 1 \quad \text{for} \quad |z| < \varepsilon$$

to enhance convergence of the product. However, if we would require $(1 - z)\phi(z) = 1$, then $\phi(z)$ would become singular at $z = 1$, and we do not obtain an entire function.

We want to construct $\phi(z)$ so that

a) $(1 - z)\phi(z) = 1 + O(z^{k+1})$ for $z \sim 0;

b) $\phi$ is entire.

c) $\phi(z) \neq 0$ for all $z \in \mathbb{C}$.

To ensure that $\phi(z)$ never vanishes, let us construct $\phi(z)$ in the form

$$\phi(z) = e^{h(z)}.$$

We then have the requirement, for small $|z|$, 

$$e^{h(z)} = \phi(z) = \frac{1}{1 - z} + O(z^{k+1}),$$

or,
\[ h(z) = \log \left( \frac{1}{1-z} + O(z^{k+1}) \right) = -\log(1-z) + O(z^{k+1}) \, . \]

The second equation holds since \( \log \left( \frac{1}{1-z} + \varepsilon \right) = -\log(1-z) + O(\varepsilon) \) for \( z \sim 0 \).

Set \( r(z) = -\log(1-z) \). Then we have \( r(0) = 0 \) and

\[ r'(z) = \frac{1}{1-z} = 1 + z + z^2 + \ldots \]

Therefore,

\[ r(z) = -\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots . \]

This leads to the following requirement for \( h(z) \):

\[ h(z) = h_k(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots + O(z^{k+1}) \, . \]

Therefore, we set

\[ h_k(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots + \frac{z^k}{k} \, . \]

This leads to the following definition of the Weierstrass’ canonical factors.

**Definition:** For \( k = 0, 1, \ldots \) define \( E_k(z) \) by

\[ E_0(z) = 1 - z, \quad E_k(z) = (1-z) \exp\left( z + \frac{z^2}{2} + \ldots + \frac{z^k}{k} \right) \text{ for } k \geq 1 \, . \]

Clearly, \( E_k(z) \) vanishes only at \( z = 1 \) and has a simple zero at \( z = 1 \). Therefore, the function \( z \to E_k(z/a_j) \) vanishes to first order at \( z = a_j \) and vanishes nowhere else. We will show now that \( |1 - E_k(z)| \) vanishes to order \( k+1 \) at \( z = 0 \). In this sense, \( E_k(z) \sim 1 \) to order \( k+1 \) at \( z = 0 \).

**Lemma 7.10** There is a constant \( C > 0 \), independent of \( k \) and \( z \), with

\[ |1 - E_k(z)| \leq C|z|^{k+1} \text{ for } |z| \leq \frac{1}{2} \, . \]

One can choose \( C = 2(e - 1) \).

**Proof:** In the following let \( |z| \leq \frac{1}{2} \). Let

\[ h_k(z) = z + \frac{z^2}{2} + \ldots + \frac{z^k}{k} \, . \]

We have

\[
E_k(z) = (1-z) \exp(h_k(z)) \\
= \exp \left( \log(1-z) + h_k(z) \right) \\
= e^{w}
\]
with
\[ w = \log(1 - z) + h_k(z). \]

Since
\[ \log(1 - z) = -\sum_{j=1}^{\infty} \frac{z^j}{j}, \]
we have
\[ w = -\sum_{j=k+1}^{\infty} \frac{z^j}{j}. \]

Therefore,
\[
|w| \leq |z|^{k+1} \sum_{j=k+1}^{\infty} \frac{1}{j} |z|^{j-k-1}
\leq |z|^{k+1} \left(1 + \frac{1}{2} + \frac{1}{4} + \ldots\right)
= 2|z|^{k+1}.
\]

In particular, \(|w| \leq 1\) for \(|z| \leq \frac{1}{2}\). Thus,
\[
|1 - E_k(z)| = |1 - e^w|
= \left|w + w^2/2! + w^3/3! + \ldots\right|
\leq |w| \left(1 + 1/2! + 1/3! + \ldots\right)
= |w|(e - 1)
\leq 2(e - 1)|z|^{k+1}.
\]

Remark: It will be useful below that the constant \(C\) in the previous lemma does not depend on \(k\). This is not completely obvious from the construction, which only yields \(1 - E_k(z) = \mathcal{O}(z^{k+1})\).

Theorem 7.6 Let \(a_j\) denote a sequence of complex numbers with \(|a_j| \to \infty\) as \(j \to \infty\). Assume that
\[
a_1 = \ldots = a_m = 0 < |a_{m+1}| \leq |a_{m+2}| \leq \ldots
\]

Then the infinite product
\[ f(z) = z^m \prod_{j=m+1}^{\infty} E_j(\frac{z}{a_j}), \quad z \in \mathbb{C}, \]
defines an entire function \(f(z)\) which has zeros precisely at the \(a_j\). The multiplicity of a zero \(a_n\) of \(f(z)\) equals the number of occurrences of \(a_n\) in the sequence \(a_j\).
Proof: Assume $m = 0$. Fix $R > 0$ and note that $|a_j| \geq 2R$ for $j \geq j_0(R)$. If $|z| \leq R$ then

$$\left| \frac{z}{a_j} \right| \leq \frac{1}{2} \text{ for } j \geq j_0(R).$$

Therefore, by Lemma 7.10,

$$|1 - E_j(z/a_j)| \leq C \frac{|z|^{j+1}}{|a_j|^{j+1}} \leq C 2^{-(j+1)}.$$

If one writes

$$E_j(z/a_j) = 1 + f_j(z)$$

then

$$\sum_{j=1}^{\infty} |f_j|_{\infty,D(0,R)} < \infty.$$

The claim now follows from Theorem 7.3. ☐

In the previous theorem, the index $j$ of $E_j$ increases unboundedly. It is of interest to give a condition under which this index can be chosen constant.

**Theorem 7.7** Let $a_j$ denote a sequence of complex numbers with $|a_j| \to \infty$ as $j \to \infty$. Assume that

$$a_1 = \ldots = a_m = 0 < |a_{m+1}| \leq |a_{m+2}| \leq \ldots$$

Also, assume that

$$\sum_{j=m+1}^{\infty} \frac{1}{|a_j|^s} < \infty$$

for some $s > 0$. If $k$ is an integer with $s \leq k + 1$ then the infinite product

$$f(z) = z^m \prod_{j=m+1}^{\infty} E_k(z/a_j), \quad z \in \mathbb{C},$$

defines an entire function $f(z)$ which has zeros precisely at the $a_j$. The multiplicity of a zero $a$ of $f(z)$ equals the number of occurrences of $a$ in the sequence $a_j$.

Proof: Assume $m = 0$. Fix $R > 0$ and note that $|a_j| \geq 2R$ for $j \geq j_0(R)$. If $|z| \leq R$ then

$$\left| \frac{z}{a_j} \right| \leq \frac{1}{2} \text{ for } j \geq j_0(R).$$

Therefore, by Lemma 7.10,

$$|1 - E_k(z/a_j)| \leq C \frac{|z|^{k+1}}{|a_j|^{k+1}} \leq C(R,k) \frac{1}{|a_j|^{k+1}}.$$
The claim now follows from

\[ \sum_{j=m+1}^{\infty} \frac{1}{|\alpha_j|^{k+1}} < \infty \]

and Theorem 7.3. \(\diamondsuit\)
8 The Bernoulli Numbers and Applications

8.1 The Bernoulli Numbers

The function $g(z)$ defined by

$$g(z) = \frac{z}{(e^z - 1)} \quad \text{for} \quad 0 < |z| < 2\pi, \quad g(0) = 1,$$

is holomorphic in $D(0, 2\pi)$. We write its Taylor series as

$$g(z) = \sum_{\nu=0}^{\infty} \frac{B_{\nu}}{\nu!} z^\nu, \quad |z| < 2\pi, \quad (8.1)$$

where the numbers $B_\nu$ are, by definition, the Bernoulli numbers. Since

$$g(z) = \frac{1}{1 + \frac{1}{2}z + \frac{1}{6}z^2 + \ldots} = 1 - \frac{1}{2}z + \ldots$$

it follows that

$$B_0 = 1, \quad B_1 = -\frac{1}{2}.$$ 

Lemma 8.1 The function

$$h(z) = g(z) + \frac{z}{2}$$

is even. Consequently,

$$B_\nu = 0 \quad \text{for} \quad \nu \geq 3, \quad \nu \text{ odd}.$$ 

Proof: We must show that

$$g(-z) - \frac{z}{2} = g(z) + \frac{z}{2},$$

i.e.,

$$g(-z) - g(z) = z.$$ 

With $a = e^z$ we have

$$g(-z) - g(z) = \frac{-z}{e^{-z} - 1} - \frac{z}{e^z - 1} = z \left( \frac{-1}{a - 1} - \frac{1}{a - 1} \right)$$

and
\[
\frac{-a}{1-a} - \frac{1}{a-1} = \frac{1-a}{1-a} = 1.
\]

One can compute the Bernoulli numbers easily using a recursion. We claim:

**Lemma 8.2** For \( n \geq 1 \) we have

\[
\sum_{\nu=0}^{n} \binom{n+1}{\nu} B_{\nu} = 0.
\]

**Proof:** We have, for \( 0 < |z| < 2\pi \):

\[
1 = \frac{e^z - 1}{z} \cdot \frac{z}{e^z - 1} = \left( \sum_{\mu=0}^{\infty} \frac{z^\mu}{(\mu+1)!} \right) \cdot \left( \sum_{\nu=0}^{\infty} \frac{B_{\nu} z^\nu}{\nu!} \right) = \sum_{\mu,\nu=0}^{\infty} \frac{B_{\nu}}{\nu!(\mu+1)!} z^{\mu+\nu} = \sum_{n=0}^{\infty} \left( \sum_{\nu=0}^{n} \frac{B_{\nu}}{\nu!(n+1-\nu)!} \right) z^n
\]

Since

\[
\binom{n+1}{\nu} = \frac{(n+1)!}{\nu!(n+1-\nu)!}
\]

the lemma is proved. \( \diamond \)

Using Pascal’s triangle, we can compute the binomial coefficients. Then, using the previous lemma and \( B_0 = 1 \) we obtain:

For \( n = 1 \):

\[
B_0 + 2B_1 = 0, \quad \text{thus} \quad B_1 = -\frac{1}{2}.
\]

For \( n = 2 \):

\[
B_0 + 3B_1 + 3B_2 = 0, \quad \text{thus} \quad B_2 = \frac{1}{6}.
\]

For \( n = 3 \):

\[
B_0 + 4B_1 + 6B_2 + 4B_3 = 0, \quad \text{thus} \quad B_3 = 0.
\]

For \( n = 4 \):

\[
B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 = 0, \quad \text{thus} \quad B_4 = -\frac{1}{30}.
\]

Continuing this process, one obtains the following non–zero Bernoulli numbers:
\[ B_6 = \frac{1}{42} \]
\[ B_8 = -\frac{1}{30} \]
\[ B_{10} = \frac{5}{66} \]
\[ B_{12} = -\frac{691}{2730} \]
\[ B_{14} = \frac{7}{6} \]

etc.

**Remark:** The sequence \(|B_{2\nu}|\) is unbounded since otherwise the series (8.1) would have a finite radius of convergence. We will see below that \((-1)^{\nu+1}B_{2\nu} > 0\). Thus, the sign pattern observed for \(B_2\) to \(B_{14}\) continuous.

### 8.2 The Taylor Series for \(z \cot z\) in Terms of Bernoulli Numbers

Recall that

\[
g(w) = \frac{w}{e^w - 1} = \sum_{\nu=0}^{\infty} \frac{B_{\nu}}{\nu!} w^\nu.
\]

We now express the Taylor series for \(z \cot z\) about \(z = 0\) in terms of Bernoulli numbers. Note that

\[
\cos z = \frac{1}{2}(e^{iz} + e^{-iz})
\]
\[
\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})
\]
\[
\cot z = i \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}
\]
\[
= i \frac{1 + e^{-2iz}}{1 - e^{-2iz}}
\]
\[
= i \frac{1 - e^{-2iz} + 2e^{-2iz}}{1 - e^{-2iz}}
\]
\[
= i \left(1 + \frac{2}{e^{2iz} - 1}\right)
\]

Therefore,

\[
\cot z = i + \frac{1}{z} \cdot \frac{2iz}{e^{2iz} - 1},
\]

thus
\[ z \cot z = iz + g(2iz) \]
\[ = iz + 1 - \frac{1}{2}(2iz) + \sum_{\nu=2}^{\infty} \frac{B_{\nu}}{\nu!} (2iz)^{\nu} \]
\[ = 1 + \sum_{\nu=1}^{\infty} (-1)^{\nu} \frac{4^{\nu}}{(2\nu)!} B_{2\nu} z^{2\nu} . \]

We substitute \( \pi z \) for \( z \) and summarize this:

**Lemma 8.3** If \( B_{\nu} \) denotes the sequence of the Bernoulli numbers, then we have for \( |z| < 1 \):

\[
\pi z \cot(\pi z) = 1 + \sum_{\nu=1}^{\infty} (-1)^{\nu} \frac{(2\pi)^{2\nu}}{(2\nu)!} B_{2\nu} z^{2\nu} .
\] (8.2)

### 8.3 The Mittag–Leffler Expansion of \( \pi z \cot(\pi z) \)

One can derive the following Mittag–Leffler expansion (see, for example, [Hahn, Epstein, p. 214]). One can also use residue calculus; see Math 561. Note that the function on the left side and the function on the right side of the following formula both have a simple pole at each integer.

\[
\pi \cot(\pi z) = \frac{1}{z} + 2\sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} .
\]

Therefore,

\[
\pi z \cot(\pi z) = 1 - 2\sum_{n=1}^{\infty} \frac{z^2}{n^2 - z^2} .
\]

Here, for \( |z| < 1 \):

\[
\frac{z^2}{n^2 - z^2} = \frac{(z/n)^2}{1 - (z/n)^2} = \sum_{m=1}^{\infty} \left( \frac{z}{n} \right)^{2m} .
\]

Therefore,

\[
\pi z \cot(\pi z) = 1 - 2\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \frac{z}{n} \right)^{2m} .
\] (8.3)

\[
= 1 - 2\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{z}{n} \right)^{2m} .
\] (8.4)

\[
= 1 - 2\sum_{m=1}^{\infty} \zeta(2m) z^{2m} .
\] (8.5)
8.4 The Values of $\zeta(2m)$

Comparing the expressions (8.5) and (8.2), we obtain the following result about the values of the Riemann $\zeta$–function at even integers. This result was already known to Euler in 1734.

**Theorem 8.1** For $m = 1, 2, \ldots$:

$$\zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{1}{2} (-1)^{m+1} \left( \frac{(2\pi)^2 m}{(2m)!} \right) B_{2m}. \quad (8.6)$$

**Remark:** Since, clearly, $\zeta(2m) > 0$ we obtain that $(-1)^{m+1} B_{2m} > 0$.

**Examples:**

For $m = 1$ we have $B_2 = \frac{1}{6}$, thus

$$\zeta(2) = \frac{(2\pi)^2}{2 \cdot 2} \cdot \frac{1}{6} = \frac{\pi^2}{6}.$$

For $m = 2$ we have $B_4 = -\frac{1}{30}$, thus

$$\zeta(4) = \frac{(2\pi)^4}{2 \cdot 4!} \cdot \frac{1}{30} = \frac{\pi^4}{90}.$$

For $m = 3$ we have $B_6 = \frac{1}{42}$, thus

$$\zeta(6) = \frac{(2\pi)^6}{2 \cdot 6!} \cdot \frac{1}{42} = \frac{\pi^6}{945}.$$

For $m = 4$ one obtains

$$\zeta(8) = \frac{\pi^8}{9450}.$$

**Remark:** According to [Temme, p. 6], no nice formula for $\zeta(2m+1)$ seems to be known if $m$ is an integer.

8.5 Sums of Powers and Bernoulli Numbers

It is not difficult to show the following formulae by induction in $n$:

$$\sum_{j=1}^{n-1} j = \frac{1}{2} n^2 - \frac{1}{2} n$$

$$\sum_{j=1}^{n-1} j^2 = \frac{1}{3} n^3 - \frac{1}{2} n^2 + \frac{1}{6} n$$

$$\sum_{j=1}^{n-1} j^3 = \frac{1}{4} n^4 - \frac{1}{2} n^3 + \frac{1}{4} n^2$$

The formulae follow the pattern:
\[
\sum_{j=1}^{n-1} j^k = \frac{1}{k+1} n^{k+1} - \frac{1}{2} n^k + \ldots + B_k n,
\]
but it is not obvious how the general formula should read.

Define the sum
\[
S_k(n-1) = \sum_{j=0}^{n-1} j^k
\]
where \(k = 0, 1, 2, 3, \ldots\) and \(n = 1, 2, 3, \ldots\). We claim that, for every fixed integer \(k \geq 0\), the sum \(S_k(n-1)\) is a polynomial
\[
\Phi_k(n)
\]
of degree \(k + 1\) in the variable \(n\) and that the coefficients of \(\Phi_k(n)\) can be obtained in terms of Bernoulli numbers. Precisely:

**Theorem 8.2** For every integer \(k \geq 0\), let \(\Phi_k\) denote the polynomial
\[
\Phi_k(n) = \frac{1}{k+1} \sum_{\mu=0}^{k} \binom{k+1}{\mu} B_\mu n^{k+1-\mu}.
\]
Then we have
\[
S_k(n-1) = \Phi_k(n) \quad \text{for all} \quad n = 1, 2, \ldots.
\]

**Remark:** Writing out a few terms of \(\Phi_k(n)\), the theorem says that
\[
S_k(n-1) = \frac{1}{k+1} n^{k+1} - \frac{1}{2} n^k + \frac{1}{k+1} \left(\begin{array}{c} k+1 \\ 2 \end{array}\right) B_2 n^{k-1} + \ldots + B_k n.
\]

**Proof of Theorem:** The trick is to write the finite geometric sum
\[
E_n(w) = 1 + e^w + e^{2w} + \ldots + e^{(n-1)w}
\]
in two ways and then to compare coefficients. We have
\[
E_n(w) = \sum_{j=0}^{n-1} e^{jw} = \sum_{j=0}^{n-1} \sum_{k=0}^{\infty} \frac{j^k}{k!} w^k = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n-1} j^k \right) \frac{1}{k!} w^k = \sum_{k=0}^{\infty} \frac{1}{k!} S_k(n-1) w^k
\]
On the other hand, we have

\[
E_n(w) = \frac{e^{nw} - 1}{e^w - 1} = \frac{w}{e^w - 1} \cdot e^{nw} - 1
\]

\[
= \left( \sum_{\mu=0}^{\infty} B_\mu \mu! w^\mu \right) \cdot \left( \sum_{\lambda=0}^{\infty} \frac{n^{\lambda+1}}{(\lambda+1)!} w^\lambda \right)
\]

\[
= \sum_{k=0}^{\infty} \left( \sum_{\mu+\lambda=k} \frac{B_\mu}{\mu!(\lambda+1)!} n^{\lambda+1} \right) w^k
\]

Comparison yields that

\[
S_k(n-1) = \sum_{\mu+\lambda=k} \frac{k!}{\mu!(\lambda+1)!} B_\mu n^{\lambda+1}
\]

\[
= \frac{1}{k+1} \sum_{\mu=0}^{k} \frac{(k+1)!}{\mu!(k+1-\mu)!} B_\mu n^{k+1-\mu}
\]

This proves the claim since

\[
\binom{k+1}{\mu} = \frac{(k+1)!}{\mu!(k+1-\mu)!}
\]

\[\blacksquare\]
9 The Riemann Zeta–Function

9.1 Definition for $\Re z > 1$

Let $z = s + it$. If $s > 1$ then the formula

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \quad (9.1)$$

defines $\zeta(z)$. For every $n = 1, 2, \ldots$ the function

$$z \to n^{-z} = e^{-z \ln n}$$
is entire. Therefore, for $N = 1, 2, \ldots$, the finite sum

$$S_N(z) = \sum_{n=1}^{N} n^{-z}$$
is an entire function. The sequence $S_N(z)$ converges uniformly for

$$\Re z \geq 1 + \delta > 1.$$ Thus, (9.1) defines a holomorphic function in

$$U_1 = \{ z : \Re z > 1 \}.$$ 

9.2 Simple Bounds for $s > 1$

Let $n$ be a positive integer and let $s > 1$. Then we have

$$(n + 1)^{-s} \leq x^{-s} \leq n^{-s} \quad \text{for } n \leq x \leq n + 1.$$ Therefore,

$$\sum_{n=1}^{\infty} (n + 1)^{-s} \leq \int_{1}^{\infty} \frac{dx}{x^s} \leq \sum_{n=1}^{\infty} n^{-s}.$$ This says that

$$\zeta(s) - 1 \leq \frac{1}{s - 1} \leq \zeta(s) \quad \text{for } s > 1.$$ In other words,

$$\frac{1}{s - 1} \leq \zeta(s) \leq \frac{s}{s - 1} \quad \text{for } s > 1.$$ The lower bound $\zeta(s) > 1$ for $s > 1$ is also obvious.
9.3 Meromorphic Continuation of $\zeta(z)$ to $\Re z > 0$

Let

$$U_0 = \{ z : \Re z > 0 \} .$$

We will show that $\zeta(z)$ can be continued as a holomorphic function defined in $U_0 \setminus \{1\}$. Also, $\zeta(z)$ has a simple pole at $z = 1$ with residue equal to 1. It turns out that $\zeta(z) - 1/(z-1)$ is in fact an entire function, but we will not show this here since our analytic continuation is only valid for $\Re z > 0$.

**Notation:** If $x \in \mathbb{R}$ then there is a unique integer $n$ with $n \leq x < n + 1$. We then write

$$\{x\} = x - n, \quad x = n + \{x\} = [x] + \{x\}$$

and call $\{x\}$ the fractional part and $[x] = n$ the integer part of $x$.

We use the function $[x]$ to derive an integral representation for $\zeta(z)$.

The following holds for all $z \in \mathbb{C}$:

$$\sum_{n=1}^{N} n \left( n^{-z} - (n+1)^{-z} \right) = 1 - 2^{-z} + 2 \left( 2^{-z} - 3^{-z} \right) + \ldots + N \left( N^{-z} - (N+1)^{-z} \right)$$

$$= 1 + 2^{-z} + 3^{-z} + \ldots + N^{-z} - N(N+1)^{-z}$$

Here, for all $z \in \mathbb{C}$:

$$n^{-z} - (n+1)^{-z} = -x^{-z} \bigg|_{x=n}^{x=n+1}
= z \int_{n}^{n+1} x^{-z-1} \, dx$$

Multiply the last equation by $n$ and sum over $n$ from 1 to $N$ to obtain:

$$z \sum_{n=1}^{N} \int_{n}^{n+1} [x] x^{-z-1} \, dx = \sum_{n=1}^{N} n \left( n^{-z} - (n+1)^{-z} \right)$$

$$= \sum_{n=1}^{N} n^{-z} - N(N+1)^{-z}$$

Thus we have for all $z \in \mathbb{C}$ and all $N = 1, 2, \ldots$:

$$z \int_{1}^{N+1} [x] x^{-z-1} \, dx = \sum_{n=1}^{N} n^{-z} - N(N+1)^{-z} .$$

In this equation we now assume $\Re z > 1$ and let $N \to \infty$. This yields the integral representation

$$z \int_{1}^{\infty} [x] x^{-z-1} \, dx = \zeta(z), \quad \Re z > 1 .$$
If we replace the integer part of \( x \), denoted by \([x]\), by \( x \) itself, then we can evaluate the integral. For \( \text{Re} \, z > 0 \):

\[
z \int_1^\infty x \cdot x^{-z-1} \, dx = z \int_1^\infty x^{-z} \, dx = \frac{z}{z-1}.
\]

Taking the difference between the last two equations and noting that \( x - [x] = \{x\} \) we have

\[
z \int_1^\infty (x - [x]) x^{-z-1} \, dx = \frac{z}{z-1} - \zeta(z),
\]

thus

\[
\zeta(z) = \frac{z}{z-1} - z \int_1^\infty \{x\} x^{-z-1} \, dx, \quad \text{Re} \, z > 1.
\] (9.2)

Here the function \( z/(z-1) \) is holomorphic in \( \mathbb{C} \setminus \{1\} \). In (9.2), we have identified the singular behavior of \( \zeta(z) \) at \( z = 1 \).

Since \( \{x\} \) is bounded, the integral

\[
H(z) = \int_1^\infty \{x\} x^{-z-1} \, dx
\]

defines a holomorphic function of \( z \) for \( \text{Re} \, z > 0 \). We obtain that

\[
\frac{z}{z-1} - z \int_1^\infty \{x\} x^{-z-1} \, dx
\] (9.3)

is holomorphic in \( U_0 \setminus \{1\} \) and, by (9.2), agrees with \( \zeta(z) \) for \( \text{Re} \, z > 1 \). Thus, the holomorphic extension of \( \zeta(z) \) to \( U_0 \setminus \{1\} \) exists and is given by

\[
\zeta(z) = \frac{z}{z-1} - z \int_1^\infty \{x\} x^{-z-1} \, dx
\]

\[
= \frac{1}{z-1} + F(z)
\]

with

\[
F(z) = 1 - z \int_1^\infty \{x\} x^{-z-1} \, dx.
\] (9.4)

**Lemma 9.1** The function \( \zeta(z) \) is meromorphic in \( U_0 = \{z : \text{Re} \, z > 0\} \). It has a simple pole at \( z = 1 \) with residue equal to 1. It has no other poles in \( U_0 \).

**Lemma 9.2** Let \( F(z) \) be defined in (9.4). Then we have

\[
F(1) = \gamma
\]

where \( \gamma = \lim_{n \to \infty} \left( \sum_{j=1}^n \frac{1}{j} - \ln n \right) \) is Euler’s constant.
Proof: We have $F(1) = 1 - \lim_{N \to \infty} J_N$ with

\[
J_N = \int_1^{N+1} \{x\} x^{-2} \, dx
\]

\[
= \sum_{n=1}^{N} \int_n^{n+1} (x - n) x^{-2} \, dx
\]

\[
= \int_1^{N+1} \frac{dx}{x} - \sum_{n=1}^{N} \int_n^{n+1} \frac{dx}{x^2}
\]

\[
= \ln(N + 1) - \sum_{n=1}^{N} \frac{1}{n + 1}
\]

\[
= \ln(N + 1) - \sum_{n=2}^{N+1} \frac{1}{n}
\]

\[
= \ln(N + 1) - \sum_{n=1}^{N+1} \frac{1}{n} + 1
\]

Therefore,

\[
F(1) = \lim_{N \to \infty} \left( \sum_{n=1}^{N+1} \frac{1}{n} - \ln(N + 1) \right) = \gamma.
\]

9.4 Analytic Continuation of $\zeta(z)$ to $\mathbb{C} \setminus \{1\}$

We claim that

\[
\Gamma(z)\zeta(z) = \int_0^\infty \frac{x^{z-1}}{e^x - 1} \, dx \quad \text{for} \quad \Re z > 1.
\]

This follows from the geometric sum formula

\[
\frac{1}{e^x - 1} = \sum_{n=1}^\infty e^{-nx}
\]

and

\[
\Gamma(z)\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z} \int_0^\infty e^{-t} t^{z-1} \, dt \quad (t = nx)
\]

\[
= \sum_{n=1}^\infty \int_0^\infty e^{-nx} x^{z-1} \, dx
\]

\[
= \int_0^\infty \sum_{n=1}^\infty e^{-nx} x^{z-1} \, dx
\]

\[
= \int_0^\infty x^{z-1} \frac{dx}{e^x - 1}
\]

82
For $\Re z > 1$ we can write
\[
\Gamma(z) \zeta(z) = \int_0^1 \frac{x^{z-1}}{e^x - 1} \, dx + \int_1^\infty \frac{x^{z-1}}{e^x - 1} \, dx =: I_1(z) + I_2(z).
\]
Here the second integral, $I_2(z)$, defines an entire function of $z$. To discuss the first integral, recall that
\[
\frac{x}{e^x - 1} = \sum_{m=0}^\infty \frac{B_m}{m!} x^m, \quad |x| < 2\pi,
\]
where $B_0, B_1, \ldots$ is the sequence of Bernoulli numbers. \footnote{Recall that $B_m = 0$ if $m$ is odd and $m \geq 3$. Also, by Hadamard’s formula, $\limsup_{m \to \infty} |B_m/m!|^{1/m} = \frac{1}{2\pi}$.}

For $\Re z > 1$ we have
\[
\int_0^1 x^{m+z-2} \, dx = \frac{1}{m+z-1}, \quad m = 0, 1, \ldots
\]
and one obtains that
\[
I_1(z) = \int_0^1 \frac{x^{z-1}}{e^x - 1} \, dx
= \sum_{m=0}^\infty \frac{B_m}{m!} \cdot \int_0^1 x^{m+z-2} \, dx
= \sum_{m=0}^\infty \frac{B_m}{m!} \cdot \frac{1}{z+m-1}, \quad \Re z > 1.
\]

For $m = 0, 1, 2, \ldots$ the functions
\[
\frac{1}{z-(1-m)}
\]
have poles at
\[
1, 0, -1, -2, -3, \ldots
\]
However, since $B_m = 0$ for $m = 3, 5, 7, \ldots$ the functions
\[
\frac{B_m}{m!} \cdot \frac{1}{z-(1-m)}
\]
have poles at
\[
1, 0, -1, -3, -5, \ldots
\]
Since
\[
\frac{|B_m|}{m!} \leq \frac{1}{2m} \quad \text{for} \quad m \geq m_0,
\]
it follows that
\[
I_1(z) = \sum_{m=0}^\infty \frac{B_m}{m!} \cdot \frac{1}{z-(1-m)}, \quad z \in \mathbb{C} \setminus \{1, 0, -1, -3, -5, \ldots\}
\]
is an analytic function with simple poles at

$$1, 0, -1, -3, -5, \ldots$$

We have

$$\zeta(z) = \frac{1}{\Gamma(z)} I_1(z) + \frac{1}{\Gamma(z)} I_2(z) \quad \text{for} \quad \Re z > 1$$

and, since

$$\Delta(z) = \frac{1}{\Gamma(z)}$$

is zero at

$$0, -1, -2, -3, \ldots$$

the term

$$\frac{1}{\Gamma(z)} I_1(z) = \Delta(z) I_1(z)$$

is an analytic function in $\mathbb{C} \setminus \{1\}$. One obtains that

$$\zeta(z) = \Delta(z) I_1(z) + \Delta(z) I_2(z), \quad z \in \mathbb{C} \setminus \{1\},$$

with

$$\Delta(z) = \frac{1}{\Gamma(z)}, \quad \Delta(z) \text{ entire}$$

$$I_1(z) = \sum_{m=0}^{\infty} B_m \frac{1}{m! \, z + m - 1}, \quad z \in \mathbb{C} \setminus \{1, 0, -1, -3, -5, \ldots\}$$

$$I_2(z) = \int_{1}^{\infty} \frac{x^{z-1}}{e^x - 1} \, dx, \quad I_2(z) \text{ entire}.$$

The function $\Delta(z) I_2(z)$ is the product of two entire functions. The function $\Delta(z) I_1(z)$ is analytic in $\mathbb{C} \setminus \{1\}$ with a simple pole at $z = 1$. Recall that $\Delta(z) = 1/\Gamma(z)$ has a simple zero at

$$n = 0, -1, -2, -3, -4, \ldots$$

and $I_1(z)$ has a simple pole at

$$n = 1, 0, -1, -3, -5 \ldots$$

Therefore

$$\zeta(z) = \Delta(z) \left( I_1(z) + I_2(z) \right)$$

has a simple zero at

$$n = -2, -4, -6, \ldots$$

These zeros are called the trivial zeros of the zeta–function.
9.5 Euler’s Product Formula for $\zeta(z)$

Let $p_j, j = 1, 2, \ldots$ denote the sequence of prime numbers, i.e.,

$$p_1 = 2, \quad p_2 = 3, \quad p_3 = 5, \ldots$$

Euler’s product formula for the $\zeta$–functions says that

$$\zeta(z) = \prod_{j=1}^{\infty} \frac{1}{1 - \frac{1}{p_j^{-z}}} \quad \text{for} \quad \text{Re} \ z > 1 .$$

Let us first prove convergence of the infinite product.

**Lemma 9.3** The infinite product

$$P(z) = \prod_{j=1}^{\infty} \frac{1}{1 - \frac{1}{p_j^{-z}}} \quad \text{for} \quad \text{Re} \ z > 1$$

converges and defines a holomorphic function for $\text{Re} \ z > 1$. The function $P(z)$ has no zero in $U_1 = \{z : \text{Re} \ z > 1\}$.

**Proof:** Since $p_j \geq 2$ we have

$$|p_j^{-z}| \leq \frac{1}{2} \quad \text{for} \quad \text{Re} \ z > 1 .$$

For $|\varepsilon| \leq \frac{1}{2}$ we have

$$\frac{1}{1 - \varepsilon} = 1 + \varepsilon(1 + \varepsilon + \varepsilon^2 + \ldots) = 1 + R(\varepsilon)$$

with

$$|R(\varepsilon)| \leq 2|\varepsilon| \quad \text{for} \quad |\varepsilon| \leq \frac{1}{2} .$$

Therefore, if $\text{Re} \ z = 1 + \eta > 1$, then

$$\frac{1}{1 - p_j^{-z}} = 1 + f_j(z)$$

with

$$|f_j(z)| \leq \frac{2}{|p_j|^{1+\eta}} \leq \frac{2}{j^{1+\eta}} .$$

Using Theorem 7.3, we obtain convergence of the infinite product

$$P(z) = \prod_{j=1}^{\infty} \frac{1}{1 - \frac{1}{p_j^{-z}}} \quad \text{for} \quad \text{Re} \ z > 1 ,$$

the function $P(z)$ is holomorphic for $\text{Re} \ z > 1$ and

$$P(z) \neq 0 \quad \text{for} \quad \text{Re} \ z > 1 .$$

$\diamondsuit$
An intuitive, but somewhat imprecise argument for the equation

$$\zeta(z) = \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-z}}$$

for \( \text{Re } z > 1 \) is the following: We have

$$\frac{1}{1 - p_j^{-z}} = 1 + p_j^{-z} + p_j^{-2z} + \ldots$$

and

$$\frac{1}{1 - p_1^{-z}} \cdot \frac{1}{1 - p_2^{-z}} \cdot \ldots = (1 + p_1^{-z} + p_1^{-2z} + \ldots) \cdot (1 + p_2^{-z} + p_2^{-2z} + \ldots) \cdot \ldots$$

$$= \sum \left( p_1^{k_1} \cdots p_J^{k_J} \right)^{-z}$$

where the sum is taken over all \( J = 1, 2, \ldots \) and nonnegative integers \( k_1, \ldots, k_J \).

By the Fundamental Theorem of Arithmetic, every \( n = 1, 2, \ldots \) has a unique representation as a product of prime powers,

$$n = p_1^{k_1} \cdots p_J^{k_J}$$

and, therefore, the above sum equals

$$\sum_{n=1}^{\infty} n^{-z} = \zeta(z), \quad \text{Re } z > 1 .$$

The argument is not rigorous since we multiplied infinitely many series somewhat carelessly.

We now give a rigorous argument and fix

$$J, K \in \mathbb{N}, \quad s > 1 .$$

We have, for every \( j = 1, 2, \ldots \)

$$\frac{1}{1 - p_j^{-s}} = 1 + p_j^{-s} + p_j^{-2s} + \ldots$$

$$\geq \sum_{k=0}^{K} p_j^{-ks} .$$

Taking the finite product over \( j = 1, \ldots, J \) yields

$$\prod_{j=1}^{J} \frac{1}{1 - p_j^{-s}} \geq \prod_{j=1}^{J} \left( \sum_{k=0}^{K} p_j^{-ks} \right)$$

$$= \sum_{0 \leq k \leq K} \left( p_1^{k_1} \cdots p_J^{k_J} \right)^{-s}$$

$$= \sum_{n \in S(J,K)} n^{-s}$$
Here $S(J, K)$ is the set of all positive integers $n$ of the form

$$n = p_1^{k_1} \cdots p_J^{k_J} \quad \text{with} \quad 0 \leq k_\nu \leq K \quad \text{for} \quad \nu = 1, \ldots, J.$$  

Since the above product is finite, we can let $K \to \infty$ and obtain

$$\Pi_{j=1}^J \frac{1}{1 - p_j^{-s}} = \sum_{n \in S(J)} n^{-s}$$

where $S(J)$ consists of all positive integers $n$ of the form

$$n = p_1^{k_1} \cdots p_J^{k_J} \quad \text{with} \quad k_\nu \in \{0, 1, \ldots\} \quad \text{for} \quad \nu = 1, \ldots, J.$$  

If $1 \leq n \leq J$ then $n \in S(J)$. Therefore,

$$\sum_{n=1}^J n^{-s} \leq \sum_{n \in S(J)} n^{-s} \leq \zeta(s), \quad s > 1.$$  

Taking the limit as $J \to \infty$ and recalling that

$$\Pi_{j=1}^J \frac{1}{1 - p_j^{-s}} = \sum_{n \in S(J)} n^{-s}$$

we have shown that

$$\Pi_{j=1}^\infty \frac{1}{1 - p_j^{-s}} = \sum_{n=1}^\infty n^{-s} = \zeta(s), \quad s > 1.$$  

Since $P(z)$ and $\zeta(z)$ are holomorphic for $\text{Re} \ z > 1$ we have shown:

**Theorem 9.1** For all $z$ with $\text{Re} \ z > 1$ we have

$$\Pi_{j=1}^\infty \frac{1}{1 - p_j^{-z}} = \sum_{n=1}^\infty n^{-z} = \zeta(z).$$

The function $\zeta(z)$ is zero–free in $\text{Re} \ z > 1$.

### 9.6 The Sum $\sum \frac{1}{p}$ and the Prime Number Theorem

**Theorem 9.2** (Euler) If $p_1, p_2, \ldots$ denotes the sequence of primes, then

$$\sum_{j=1}^\infty \frac{1}{p_j} = \infty.$$  

**Proof:** Suppose that

$$\sum_{j=1}^\infty \frac{1}{p_j} = L < \infty.$$  

Then let $s > 1$ and recall that
\[
\frac{1}{1 - p_j^{-s}} = 1 + f_j(s) \quad \text{with} \quad 0 < f_j(s) \leq 2p_j^{-s} \leq \frac{2}{p_j}.
\]

Therefore, for any finite \( J \),

\[
\Pi_{j=1}^{J} \frac{1}{1 - p_j^{-s}} \leq \Pi_{j=1}^{J} \left(1 + \frac{2}{p_j}\right) \\
\leq \Pi_{j=1}^{J} e^{2/p_j} \\
= \exp\left(2\left(\frac{1}{p_1} + \ldots + \frac{1}{p_J}\right)\right) \\
\leq e^{2L}.
\]

In this estimate we can let \( J \to \infty \) and obtain

\[
\zeta(s) \leq e^{2L} \quad \text{for} \quad s > 1.
\]

However, we know that

\[
\frac{1}{s - 1} \leq \zeta(s) \quad \text{for} \quad s > 1
\]

and obtain a contradiction as \( s \to 1^+ \).

It is clear that \( p_j \gg j \) for all large \( j \). One can ask how fast, in comparison with \( j \), the sequence \( p_j \) grows as \( j \to \infty \). As a consequence of the previous theorem, the following Corollary says that \( p_j \) cannot grow as fast as \( j^{1+\varepsilon} \) if \( \varepsilon > 0 \).

**Corollary:** If \( \varepsilon > 0 \) then numbers \( c_\varepsilon > 0 \) and \( J \in \mathbb{N} \) with

\[
c_\varepsilon j^{1+\varepsilon} \leq p_j \quad \text{for} \quad j \geq J
\]

do not exist.

**Proof:** Otherwise one would obtain

\[
c_\varepsilon \frac{1}{p_j} \leq \frac{1}{j^{1+\varepsilon}} \quad \text{for} \quad j \geq J,
\]

in contradiction to the previous theorem.

Instead of comparing \( p_j \) with \( j^{1+\varepsilon} \), it is common to introduce the function

\[
\pi(x) = \#\{p_j : p_j \leq x\}, \quad x \geq 1,
\]

and to compare the growth of \( \pi(x) \) with the growth of \( x \) as \( x \to \infty \). Here \( \pi(x) \) is the number of primes \( \leq x \).

Roughly speaking, Theorem 9.2 says that the sequence \( p_j \) of primes grows rather slowly, slower than the sequence

\[
j^{1+\varepsilon}, \quad j = 1, 2, \ldots
\]

for any \( \varepsilon > 0 \) since
Since the sequence \( p_j \) grows only 'a little' faster than the sequence \( j \) (there are 'many' primes), the function \( \pi(x) \) goes to infinity only 'a little' slower than \( x \). Thus, one expects that

\[
\frac{\pi(x)}{x} \rightarrow \infty.
\]

goes to zero as \( x \rightarrow \infty \), but only quite slowly. In fact, one can prove that

\[
\frac{\pi(x)}{x} \sim \frac{1}{\ln x} \quad \text{as} \quad x \rightarrow \infty.
\]

Precisely:

**Theorem 9.3 (The Prime Number Theorem)**

\[
\lim_{x \rightarrow \infty} \frac{\ln(x) \pi(x)}{x} = 1.
\]

### 9.7 A Functional Equation

This section needs details about the theta function.

For \( t > 0 \) one defines the theta function by

\[
\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}.
\]

Using the Poisson summation formula one finds that

\[
\theta(t) = t^{-1/2} \theta(1/t), \quad t > 0.
\]  

**Lemma 9.4** For \( \text{Re} \ s > 1 \) we have

\[
\pi^{-s/2} \Gamma(s) \zeta(s) = \frac{1}{2} \int_0^\infty t^{s-1} \left( \theta(t) - 1 \right) dt.
\]

**Proof:** We have

\[
\frac{1}{2} (\theta(t) - 1) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}.
\]

Multiply by \( t^{s-1} \) and integrate:

\[
\frac{1}{2} \int_0^\infty t^{s-1} \left( \theta(t) - 1 \right) dt = \sum_{n=1}^{\infty} \int_0^\infty t^{s-1} e^{-\pi n^2 t} dt.
\]

Using the substitution

\[
\pi n^2 t = q, \quad dt = (\pi n^2)^{-1} dq, \quad t^{s-1} = (\pi n^2)^{1-s} q^{s-1},
\]

89
one finds that

\[
\int_0^\infty t^{s/2-1}e^{-\pi n^2 t} dt = (\pi n^2)^{-s/2} \int_0^\infty q^{s/2-1}e^{-q} dq \\
= (\pi n^2)^{-s/2}\Gamma(s/2) \\
= \pi^{-s/2}n^{-s}\Gamma(s/2)
\]

Summation over \( n \) proves the lemma. \( \diamond \)

The function

\[
\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)
\]

is meromorphic in \( \mathbb{C} \) with simple poles at \( s = 1 \) (from \( \zeta(s) \)) and at \( s = 0 \) (from \( \Gamma(s/2) \)). The simple poles of \( \Gamma(s/2) \) at \( s = -2, -4, -6, \ldots \) are compensated by simple zeros of \( \zeta(s) \) at these poles. Thus, \( \xi(s) \) has no poles except at \( s = 0 \) and \( s = 1 \).

**Theorem 9.4** The function \( \xi(s) \) satisfies

\[
\xi(s) = \xi(s - 1) \quad \text{for} \quad s \in \mathbb{C} \setminus \{0, 1\}.
\]

**Proof:** See [Stein, Shakarchi, Theorem 2.3]. The proof is based on the previous lemma and the functional equation (9.5). \( \diamond \)

One obtains:

\[
\zeta(s) = \pi^{s-\frac{1}{2}}\Delta(s/2)\Gamma((1-s)/2)\zeta(1-s)
\]

with

\[
\Delta(s/2) = \frac{1}{\Gamma(s/2)}.
\]

For \( \text{Re} \, s < 0 \) we know that \( \zeta(1-s) \neq 0 \) and \( \Gamma((1-s)/2) \neq 0 \). Also, \( \Delta(s/2) \) has a zero precisely at \( s = -2, -4, -6, \ldots \) if \( \text{Re} \, s < 0 \).

We have shown:

**Theorem 9.5** a) If \( \text{Re} \, s > 1 \) then \( \zeta(s) \neq 0 \) by Euler’s product formula.

b) If \( \text{Re} \, s < 0 \) then \( \zeta(s) = 0 \) precisely for \( s = -2, -4, -6, \ldots \). These so-called trivial zeros of \( \zeta \) are all simple.

**9.8 Growth Estimates for the \( \zeta \)-Function**

We use the notation

\[
s = \sigma + it \quad \text{where} \quad \sigma \in \mathbb{R} \quad \text{and} \quad t \in \mathbb{R}.
\]

For \( n = 1, 2, \ldots \) we define the entire function

\[
\delta_n(s) = \int_n^{n+1} (n^{-s} - x^{-s}) \, dx = n^{-s} - \int_n^{n+1} x^{-s} \, dx.
\]
Clearly,

\[ n^{-s} = \int_{n}^{n+1} x^{-s} \, dx + \delta_n(s). \]

If \( \sigma = \text{Re} \, s > 1 \) we sum over all \( n = 1, 2, \ldots \) and obtain

\[
\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \int_{1}^{\infty} x^{-s} \, dx + \sum_{n=1}^{\infty} \delta_n(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} \delta_n(s)
\]

The following estimates for the functions \( \delta_n(s) \) will imply that the equation

\[
\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} \delta_n(s) \tag{9.6}
\]

holds for all \( s \neq 1 \) with \( \sigma = \text{Re} \, s > 0 \).

**Lemma 9.5** If \( s = \sigma + it \) and \( \sigma > 0 \) then

\[
|\delta_n(s)| \leq \frac{2}{n^\sigma} \tag{9.7}
\]

and

\[
|\delta_n(s)| \leq \frac{|s|}{n^{1+\sigma}}. \tag{9.8}
\]

**Proof:** a) We have \( |n^{-s}| = n^{-\sigma} \) and

\[
|x^{-s}| = x^{-\sigma} \leq n^{-\sigma} \quad \text{for} \quad x \geq n.
\]

The estimate (9.7) follows.

b) Set \( f(x) = x^{-s} \) and note that

\[
\delta_n(s) = f(n) - \int_{n}^{n+1} f(x) \, dx = \int_{n}^{n+1} (f(n) - f(x)) \, dx
\]

We have \( f'(x) = -sx^{-s-1} \) and

\[
|f'(x)| = |s|x^{-(s+1)} \leq \frac{|s|}{n^{\sigma+1}} \quad \text{for} \quad x \geq n.
\]

The estimate
\[|f(n) - f(x)| \leq \frac{|s|}{n^{\sigma + 1}} \text{ for } x \geq n\]

follows, which yields (9.8). \(\Box\)

**Lemma 9.6** For every \(0 < \varepsilon \leq 1\) there is a constant \(C_\varepsilon > 0\) so that

\[|\zeta(s)| \leq C_\varepsilon|t|^\varepsilon \text{ for } 1 \leq \sigma \leq 2, \quad |t| \geq 1.

Here \(s = \sigma + it\).

**Proof:** The formula (9.6) yields

\[|\zeta(s)| \leq \frac{1}{|s - 1|} + \sum_{n=1}^{\infty} |\delta_n(s)|\]

where

\[|s - 1|^2 = (\sigma - 1)^2 + t^2 \geq 1,

thus \(\frac{1}{|s - 1|} \leq 1\). Using the estimates for \(|\delta_n(s)|\) from the previous lemma, we obtain

\[|\delta_n(s)| = |\delta_n(s)|^\varepsilon |\delta_n(s)|^{1-\varepsilon} \leq \frac{|s|^{\varepsilon}}{n^{\varepsilon(\sigma + 1)}} \cdot \frac{2}{n(1-\varepsilon)\sigma} = \frac{2|s|^\varepsilon}{n^{\sigma + \varepsilon}} \leq \frac{2|s|^\varepsilon}{n^{1+\varepsilon}} \text{ (since } \sigma \geq 1)\]

Here

\[|s|^2 = \sigma^2 + t^2 \leq 4 + t^2 \leq 5t^2,

thus

\[|s|^\varepsilon \leq \sqrt{5}|t|^\varepsilon .\]

We obtain:

\[|\zeta(s)| \leq 1 + 2\sqrt{5}|t|^\varepsilon \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \leq 1 + 2\sqrt{5} \zeta(\varepsilon)|t|^\varepsilon \leq \left(1 + 2\sqrt{5} \zeta(\varepsilon)\right)|t|^\varepsilon

This proves the lemma. \(\Box\)

We will also need a growth bound for \(|\zeta(s)|\) for \(1 - \varepsilon \leq \sigma \leq 1\).
Lemma 9.7 For every $0 < \varepsilon \leq \frac{1}{2}$ there is a constant $C_\varepsilon > 0$ so that
\[|\zeta(s)| \leq C_\varepsilon |t|^{\varepsilon}\quad \text{for} \quad 1 - \frac{\varepsilon}{2} \leq \sigma \leq 1, \quad |t| \geq 1.\]
Here $s = \sigma + it$.

Proof: Proceeding as in the proof of the previous lemma, we have
\[
|\delta_n(s)| = |\delta_n(s)|^\varepsilon |\delta_n'(s)|^{1-\varepsilon} \\
\leq \frac{2|s|^{\varepsilon}}{n^{\sigma+\varepsilon}} \\
\leq \frac{2|s|^{\varepsilon}}{n^{1+\varepsilon/2}} \quad (\text{since } \sigma \geq 1 - \varepsilon/2)
\]
The claim follows as in the proof of the previous lemma. \(\diamondsuit\)

We can now bound the growth of $\zeta'(1 + it)$ for $|t| \to \infty$ using a Cauchy inequality.

Recall: If $f \in H(D(z_0, R))$ and $0 < r < \varepsilon$ then
\[|f'(z_0)| \leq \frac{1}{r} \max\{|f(z)| : |z - z_0| = r\}.\]

Apply this estimate to the $\zeta$–function with $z_0 = 1 + it$ and $r = \varepsilon$. Using the estimates of the two previous lemmas, one obtains that
\[|\zeta'(1 + it)| \leq \frac{1}{\varepsilon} C_\varepsilon |t|^{\varepsilon}.\]

Lemma 9.8 For every $0 < \varepsilon \leq \frac{1}{2}$ there is a constant $C_\varepsilon > 0$ so that
\[|\zeta'(1 + it)| \leq C_\varepsilon |t|^{\varepsilon} \quad \text{for} \quad |t| \geq 2.\]

We also need a bound for $1/\zeta(1 + it)$ as $|t| \to \infty$. 

93
10 Analytic Continuation

10.1 Analytic Continuation Using the Cauchy Riemann Equations

Discussion of the $2\pi$–periodic Cauchy problem for the Cauchy–Riemann equations. It is ill–posed.

10.2 Exponential Decay of Fourier Coefficients and the Strip of Analyticity

10.3 The Schwarz Reflection Principle

10.4 Examples for Analytic Continuation

10.5 Riemann Surfaces: Intuitive Approach

10.6 Riemann Surfaces: Germs, Sheafs, and Fibers
11 Fourier Series

11.1 Convergence Results: Overview

Let $X$ denote the linear space of all continuous functions $f : [0, 1] \rightarrow \mathbb{C}$ with $f(0) = f(1)$. (Some of the results that we discuss hold for more general functions than functions in $X$.)

For $f, g \in X$ we use the $L_2$–inner product,

$$(f, g)_{L_2} = \int_0^1 \bar{f}(x)g(x) \, dx ,$$

the $L_2$ norm,

$$\|f\|_{L_2} = (f, f)^{1/2}_{L_2} ,$$

and the maximum norm,

$$|f|_\infty = \max_x |f(x)| .$$

The sequence of functions

$$\phi_k(x) = e^{2\pi ikx}, \quad k \in \mathbb{Z} ,$$

is orthonormal in $X$, i.e.,

$$(\phi_j, \phi_k)_{L_2} = \delta_{jk} .$$

Let $f \in X$ be given by a series,

$$f(x) = \sum_k a_k \phi_k(x) .$$

If the series converges uniformly, then one obtains that

$$a_k = (\phi_k, f)_{L_2} .$$

This motivates the definition of the Fourier coefficients: If $f \in L_1(0, 1)$ then set

$$\hat{f}(k) = (\phi_k, f)_{L_2}, \quad k \in \mathbb{Z} .$$

The series

$$\sum_k \hat{f}(k)\phi_k(x)$$

is called the Fourier series of $f$. Basic questions are: Under what assumptions on $f$ does the Fourier series converge? In which sense does it converge (e.g., pointwise, or uniformly, or in $L_2$–norm)? If the series converges, does it converge to $f$?

With
we denote the $n$–th partial sum of the Fourier series of $f$.
A reasonable question is: Given $f \in X$, is it true that
\[ S_n f(x) \to f(x) \quad \text{as} \quad n \to \infty \]
for all $x \in [0,1]$? In 1873, du Bois–Raymond proved that the answer is No, in general. It is difficult to construct an explicit example. However, using the Principle of Uniform Boundedness of functional analysis, one can show rather easily that a function $f \in X$ exists for which the sequence of numbers $S_n f(0)$ is unbounded.

Let $f \in X$. The sequence of arithmetic means,
\[ \sigma_n f(x) = \frac{1}{n+1} \sum_{k=0}^{n} S_k f(x) , \]
can be shown to converge uniformly to $f$. These means are the so–called Cesaro means of the partial sums $S_n f(x)$.

**Theorem 11.1** Let $f \in X$. Then we have
\[ |f - \sigma_n f|_\infty \to 0 \quad \text{as} \quad n \to \infty . \]

We will prove this important result below.

Let $T_n = \text{span}\{\phi_k : -n \leq n \leq n\}$ and let
\[ T = \bigcup_n T_n . \]
The functions in $T$ are called trigonometric polynomials.

Using Theorem 11.1 it is easy to prove:

**Theorem 11.2** (Weierstrass) The space $T$ of trigonometric polynomials is dense in $X$ with respect to $| \cdot |_\infty$.

From Weierstrass’ theorem it follows rather easily that the Fourier series $S_n f$ converges to $f$ w.r.t. $\| \cdot \|_{L_2}$:

**Theorem 11.3** Let $f \in X$ (or, more generally, let $f \in L_2(0,1)$). Then we have
\[ \| f - S_n f \|_{L_2} \to 0 \quad \text{as} \quad n \to \infty . \]
11.2 The Dirichlet Kernel

Let $f \in X$. We have

$$\hat{f}(k) = \int_{0}^{1} e^{-2\pi i ky} f(y) \, dy$$

and

$$S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k)e^{2\pi ikx}$$

$$= \int_{0}^{1} \left( \sum_{k=-n}^{n} e^{2\pi ik(x-y)} \right) f(y) \, dy$$

$$= \int_{0}^{1} D_n(x-y) f(y) \, dy$$

where

$$D_n(t) = \sum_{k=-n}^{n} e^{2\pi ikt}, \quad t \in \mathbb{R}, \quad (11.1)$$

is called the Dirichlet kernel.

**Lemma 11.1** The Dirichlet kernel (11.1) has the following properties:

1. $D_n \in C^\infty(\mathbb{R})$;
2. $D_n(t) \equiv D_n(t+1)$;
3. $\int_{0}^{1} D_n(t) \, dt = 1$;
4. $D_n(t) = \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)}$;
5. $L_n := \int_{0}^{1} |D_n(t)| \, dt = \frac{4}{\pi^2} \ln n + O(1) \quad \text{as} \quad n \to \infty.$

**Proof:** We prove 4. using the geometric sum formula. Let $q = e^{2\pi i t} \neq 1$. We have
\[ D_n(t) = q^{-n} \sum_{j=0}^{2n} q^j \]
\[ = q^{-n} \frac{q^{2n+1} - 1}{q - 1} \]
\[ = \frac{q^{n+1/2} - q^{-n-1/2}}{q^{1/2} - q^{-1/2}} \]
\[ = \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)} \]

Property 5 will be shown below in Lemma 11.4. ∗

Note that, because of Property 4,
\[ D_n(t) \sim 2n + 1 \quad \text{for} \quad t \sim 0. \]

A plot of \( D_n(t) \) shows that \( D_n(t), -\frac{1}{2} \leq t \leq \frac{1}{2}, \) is concentrated near \( t = 0 \) for large \( n \). Together with Property 3, this makes it plausible that
\[ S_n f(x) = \int_0^1 D_n(x - y) f(y) \, dy \sim f(x) \]
for large \( n \). In fact, convergence \( S_n f(x) \to f(x) \) holds under suitable assumptions on \( f \). However, the oscillatory nature of \( D_n(t) \) makes a convergence analysis as \( n \to \infty \) difficult. In fact, in Section 11.8 we will use Property 5 to show existence of a function \( f \in X \) for which \( S_n f(0) \) diverges.

11.3 The Fejér Kernel

Let \( f \in X \). We have
\[ \sigma_n f(x) = \int_0^1 F_n(x - y) f(y) \, dy \]
where \( F_n(t) \) is the Fejér kernel:

\[
F_n(t) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(t) .
\]

The following result shows that \( F_n(t) \geq 0 \). This crucial inequality will allow us to prove Theorem 11.1.

**Lemma 11.2** We have

\[
F_n(t) = \frac{1}{n+1} \frac{\sin^2(\pi(n+1)t)}{\sin^2(\pi t)} .
\]

**Proof:** Using the formula

\[ D_k(t) = \frac{\sin(\pi(2k+1)t)}{\sin(\pi t)} \]

we must show that

\[
\sum_{k=0}^{n} \sin(\pi(2k+1)t) = \frac{\sin^2(\pi(n+1)t)}{\sin(\pi t)} .
\] (11.2)

We will prove this equation using again the geometric sum formula. Let

\[ r = e^{\pi it} \quad \text{and} \quad q = r^2 . \]

We have

\[
2i \sin(\pi(2k+1)t) = e^{i\pi(2k+1)t} - e^{-i\pi(2k+1)t} = rq^k - \frac{1}{r} q^{-k} .
\]

Therefore,

\[
2i \sum_{k=0}^{n} \sin(\pi(2k+1)t) = r \frac{q^{n+1} - 1}{q - 1} - \frac{1}{r} \frac{q^{-n-1} - 1}{q^{-1} - 1} \quad \text{(recall that } q = r^2) .
\]

\[
= \frac{q^{n+1} - 1}{r - r^{-1}} - \frac{q^{-n-1} - 1}{r^{-1} - r} .
\]

\[
= \frac{r^{2(n+1)} - 2 + r^{-2(n+1)}}{r - r^{-1}} .
\]

Here the numerator is (recall that \( r = e^{\pi it} \)):

\[
N = \left( r^{n+1} - r^{-n-1} \right)^2 = (2i)^2 \sin^2(\pi(n+1)t)
\]

and we obtain
\begin{align*}
\sum_{k=0}^{n} \sin(\pi(2k+1)t) &= 2i \cdot \frac{\sin^2(\pi(n+1)t)}{r - r^{-1}}.
\end{align*}

Since

\[ r - r^{-1} = 2i \sin(\pi t) \]

the equation (11.2) follows and the lemma is proved. ◦

\section*{11.4 Convergence of $\sigma_n f$ in Maximum Norm}

Let $f \in X$. We extend $f$ as a 1-periodic continuous function defined for all $x \in \mathbb{R}$.

Below we will use that

\[ \int_0^1 F_n(t) \, dt = 1. \]

Furthermore, if $0 < \delta \leq \frac{1}{2}$, then we have

\[ \sin(\pi t) \geq \delta > 0 \quad \text{for} \quad \delta \leq t \leq \frac{1}{2}. \]

Therefore,

\[ F_n(t) \leq \frac{1}{n+1} \frac{1}{\delta^2} \quad \text{for} \quad \delta \leq t \leq \frac{1}{2}. \quad (11.3) \]

We have

\begin{align*}
\sigma_n f(x) &= \int_0^1 F_n(x - y)f(y) \, dy \\
&= \int_0^1 F_n(t)f(x - t) \, dt \\
&= \int_{-1/2}^{1/2} F_n(t)f(x - t) \, dt.
\end{align*}
and

\[ f(x) - \sigma_n f(x) = \int_{-1/2}^{1/2} F_n(t)(f(x) - f(x-t)) dt. \]

First fix an arbitrary (small) \( \delta > 0 \). We have

\[
|f(x) - \sigma_n f(x)| \leq \int_{|t| \leq \delta} F_n(t)|f(x) - f(x-t)| dt + \int_{\delta \leq |t| \leq 1/2} F_n(t)|f(x) - f(x-t)| dt
\]

\[
\leq M_\delta + 4|f|_\infty \int_{\delta}^{1/2} F_n(t) dt \tag{11.4}
\]

with

\[ M_\delta = \max\{|f(x) - f(x-t)| : x \in \mathbb{R}, |t| \leq \delta\}. \]

Note that \( M_\delta \to 0 \) as \( \delta \to 0 \) since \( f \) is uniformly continuous. The integral in (11.4) is bounded by \( 1/(\delta^2(n + 1)) \). If \( \varepsilon > 0 \) is given, we can find \( \delta > 0 \) with \( M_\delta \leq \varepsilon/2 \) and then have

\[ 4|f|_\infty \int_{\delta}^{1/2} F_n(t) dt \leq \frac{\varepsilon}{2} \quad \text{for} \quad n \geq N(\varepsilon). \]

This proves Theorem 11.1. \( \diamond \)

11.5 Weierstrass’ Approximation Theorem for Trigonometric Polynomials

Since \( \sigma_n f \in T_n \) it is clear that Theorem 11.2 follows from Theorem 11.1.

11.6 Convergence of \( S_n f \) in \( L_2 \)

Let \( f \in X \). Recall that

\[ S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k)e^{2\pi i k x} \]

denotes the \( n \)-th partial sum of the Fourier series of \( f \). It is clear that \( S_n f \in T_n \). The following result says that \( S_n f \) is the unique best approximation to \( f \) in \( T_n \) w.r.t. the \( L_2 \)-norm.

**Theorem 11.4** Let \( f \in X \). If \( g \in T_n \) is arbitrary and \( g \neq S_n f \), then we have

\[ \|f - S_n f\|_{L_2} < \|f - g\|_{L_2}. \]

**Proof:** We have

\[ 0 = (e^{2\pi i j x}, f - S_n f)_{L_2} \quad \text{for} \quad |j| \leq n. \]

Therefore,
\[ 0 = (h, f - S_n f)_{L_2} \text{ for all } h \in \mathcal{T}_n. \]

If \( h \in \mathcal{T}_n \) is arbitrary, \( h \neq 0 \), then

\[
\|f - S_n f - h\|_{L_2}^2 = \|f - S_n f\|_{L_2}^2 + \|h\|_{L_2}^2 > \|f - S_n f\|_{L_2}^2
\]

which proves the theorem. \( \diamond \)

To prove Theorem 11.3 for \( f \in L_2(0,1) \) we will need the following result:

**Lemma 11.3**

a) The space \( X \) is dense in \( L_2(0,1) \) with respect to the \( L_2 \)-norm.

b) If \( g \in L_2(0,1) \) is arbitrary, then \( \|S_n g\|_{L_2} \leq \|g\|_{L_2} \).

**Proof:**

a) Step functions are dense in \( L_2 \) (integration theory). Then, every step function can be approximated in \( L_2 \)-norm by an element in \( X \) by ‘rounding the corners’ and enforcing periodicity.

b) Let \( g \in L_2(0,1) \) be arbitrary. By construction of \( S_n g \) we have

\[
(\phi_j, g - S_n g)_{L_2} = 0 \quad \text{for} \quad |j| \leq n.
\]

This says that the approximation error, \( \eta_n := g - S_n g \), is orthogonal to the space of trigonometric polynomials of degree \( \leq n \), i.e., to \( \mathcal{T}_n \). In particular, \( \eta_n = g - S_n g \) is orthogonal to \( S_n g \). Therefore,

\[
\|g\|_{L_2}^2 = (g - S_n g + S_n g, g - S_n g + S_n g)_{L_2}
\]

\[
= (\eta_n + S_n g, \eta_n + S_n g)_{L_2}
\]

\[
= \|\eta_n\|_{L_2}^2 + \|S_n g\|_{L_2}^2.
\]

\( \diamond \)

**Proof of Theorem 11.3:** First let \( f \in X \). By Theorem 11.1, given any \( \varepsilon > 0 \), there is \( g \in \mathcal{T} \) with

\[ |f - g|_{\infty} < \varepsilon. \]

If \( g \in \mathcal{T}_N \) then we have for all \( n \geq N \):

\[ \|f - S_n f\|_{L_2} \leq \|f - g\|_{L_2} \leq |f - g|_{\infty} < \varepsilon. \]

This proves that \( \|f - S_n f\|_{L_2} \to 0 \) as \( n \to \infty \) if \( f \in X \).

Second, let \( f \in L_2(0,1) \). Given \( \varepsilon > 0 \) there is \( f^\varepsilon \in X \) with

\[ \|f - f^\varepsilon\|_{L_2} < \varepsilon. \]

We have

\[ \|f - S_n f\|_{L_2} \leq \|f - f^\varepsilon\|_{L_2} + \|f^\varepsilon - S_n f^\varepsilon\|_{L_2} + \|S_n f^\varepsilon - S_n f\|_{L_2}. \]
The last term is
\[ \|S_n(f^\varepsilon - f)\|_{L^2} \leq \|f^\varepsilon - f\|_{L^2} < \varepsilon . \]

It follows that
\[ \|f - S_n f\|_{L^2} \leq 2\varepsilon + \|f^\varepsilon - S_n f^\varepsilon\|_{L^2} . \]

Therefore, if \( n \geq N(\varepsilon) \),
\[ \|f - S_n f\|_{L^2} \leq 3\varepsilon . \]

### 11.7 The Lebesgue Constant of the Dirichlet Kernel

Recall that
\[ D_n(t) = \frac{\sin(\pi(2n + 1)t)}{\sin(\pi t)} \]
denotes the Dirichlet kernel.

**Lemma 11.4** There is a constant \( c > 0 \) with
\[ L_n := \int_{-1/2}^{1/2} |D_n(t)| \, dt \geq c \ln n . \]

More precisely,
\[ L_n = \frac{4}{\pi^2} \ln n + O(1) . \]

The number \( L_n \) is called the Lebesgue constant of the Dirichlet kernel \( D_n(t) \).

**Proof:** 1) For \( 0 < t \leq \varepsilon \) we have
\[ 0 < \sin(\pi t) = \pi t(1 + O(t^2)) , \]
thus
\[ 0 < \frac{1}{\sin(\pi t)} = \frac{1}{\pi t}(1 + O(t^2)) . \]

This yields that
\[ \frac{1}{\sin(\pi t)} = \frac{1}{\pi t} + O(1) \text{ for } 0 < t \leq \frac{1}{2} . \]

2) We have
\( L_n = 2 \int_0^{1/2} |D_n(t)| \, dt \)

\[
= 2 \int_0^{1/2} \frac{\sin(\pi t(2n + 1))}{\pi t} \, dt + \mathcal{O}(1) \quad \text{(set } t(2n + 1) = s) \]

\[
= \frac{2}{\pi} \int_0^{n + \frac{1}{2}} \frac{1}{s} |\sin(\pi s)| \, ds + \mathcal{O}(1) \]

\[
= \frac{2}{\pi} \sum_{k=0}^{n-1} \int_k^{k+1} \frac{1}{s} |\sin(\pi s)| \, ds + \mathcal{O}(1) \]

\[
= \frac{2}{\pi} \int_0^1 |\sin(\pi s)| \sum_{k=1}^{n-1} \frac{1}{s + k} \, ds + \mathcal{O}(1) \]

For \( 0 \leq s \leq 1 \) and \( k \geq 1 \) we have

\[
\frac{1}{1 + k} \leq \frac{1}{s + k} \leq \frac{1}{k} ,
\]

thus

\[
\ln n + \left( \sum_{k=1}^{n-1} \frac{1}{1 + k} - \ln n \right) = \sum_{k=1}^{n-1} \frac{1}{1 + k} \leq \sum_{k=1}^{n-1} \frac{1}{s + k} \leq \sum_{k=1}^{n-1} \frac{1}{k} = \ln n + \left( \sum_{k=1}^{n-1} \frac{1}{k} - \ln n \right) .
\]

This yields \(^3\)

\[
\ln n - C \leq \sum_{k=1}^{n-1} \frac{1}{s + k} \leq \ln n + C .
\]

Since

\[
\int_0^1 \sin(\pi s) \, ds = \frac{2}{\pi}
\]

the claim,

\[
L_n = \frac{4}{\pi^2} \ln n + \mathcal{O}(1) ,
\]

follows. \( \diamond \)

\(^3\)Recall that \( \gamma_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n \to \gamma = 0.57721 \ldots \)
11.7.1 The Lebesgue Constant as Norm of a Functional

As above, let \( X \) denote the linear space of all continuous functions \( f : [0, 1] \to \mathbb{C} \) with \( f(0) = f(1) \).

On \( X \) we use the maximum norm, \( |\cdot|_{\infty} \). Then \((X, |\cdot|_{\infty})\) is a Banach space. Define the linear functional \( A_n : X \to \mathbb{C} \) by

\[
A_n f = S_n f(0) = \int_0^1 D_n(y) f(y) \, dy .
\]

One defines the norm of \( A_n \) by

\[
\|A_n\| = \sup \{|A_n f| : f \in X, \, |f|_{\infty} = 1\} .
\]

It is not difficult to show that \( A_n : X \to \mathbb{C} \) is a bounded linear functional with norm \( \|A_n\| = L_n \). In fact, the estimate \( \|A_n\| \leq L_n \) is easy to show. To see that it is sharp, set

\[ h(y) = sgn(D_n(y)) \]

and let \( h_\varepsilon \in X \) denote a smooth approximation \(^5\) of \( h \) with \( |h_\varepsilon|_{\infty} = 1 \). We then have for \( \varepsilon \to 0 \):

\[
A_n h_\varepsilon = \int_0^1 D_n(y) h_\varepsilon(y) \, dy \to \int_0^1 |D_n(y)| \, dy = L_n .
\]

The main point is that, by the previous lemma, we obtain that

\[
\|A_n\| \to \infty \quad \text{as} \quad n \to \infty .
\]

11.8 Divergence of the Fourier Series of a Continuous Function at a Point

In 1873, du Bois–Raymond constructed a continuous function whose Fourier series diverges at a point.

We use the Uniform–Boundedness Principle of functional analysis. (This is also called the Banach–Steinhaus Theorem or the Resonance Principle.)

First recall some simple concepts from functional analysis. If \( X, Y \) are normed spaces, then a linear operator \( T : X \to Y \) is called bounded if there is a constant \( C \geq 0 \) with

\[
\|Tf\|_Y \leq C\|f\|_X \quad \text{for all} \quad f \in X . \tag{11.5}
\]

If \( T : X \to Y \) is a bounded linear operator, then its operator norm is defined by

\[^4\text{Here} \ sgn \ \text{denotes the sign–function,} \ sgn(x) = 1 \ \text{for} \ x > 0, \ sgn(x) = -1 \ \text{for} \ x < 0, \ \text{and} \ sgn(0) = 0.\]

\[^5\text{The functions} \ h_\varepsilon(y) \ \text{must satisfy} \int_0^1 |h(y) - h_\varepsilon(y)| \, dy \to 0 \ \text{as} \ \varepsilon \to 0. \ \text{One can choose} \ h_\varepsilon(y) \ \text{as a continuous, piecewise linear function.}\]
\[ \|T\| = \min\{C \geq 0 : \text{(11.5) holds}\} \]
\[ = \sup\{\|Tx\|_Y : x \in X, \|x\|_X = 1\} \]

The linear space \( L(X,Y) \) of all bounded linear operators from \( X \) to \( Y \), together with the operator norm, is a normed space.

The Uniform-Boundedness Principle is the following remarkable result:

**Theorem 11.5** Let \( X \) be a Banach space and \( Y \) be a normed space. Let \( T \subset L(X,Y) \). Assume that for every \( f \in X \) there is a constant \( C_f \) with

\[ \|Tf\|_Y \leq C_f \quad \text{for all} \quad T \in T . \]

Then there is a constant \( C \) with

\[ \|T\| \leq C \quad \text{for all} \quad T \in T . \]

The following is a reformulation:

**Theorem 11.6** Let \( X \) be a Banach space and \( Y \) be a normed space. Let \( T \subset L(X,Y) \).

Assume that

\[ \sup\{\|T\| : T \in T\} = \infty . \quad (11.6) \]

Then there is \( f \in X \) so that

\[ \sup\{\|Tf\|_Y : T \in T\} = \infty . \quad (11.7) \]

This reformulation is called Resonance Principle. Roughly, if (11.6) holds, then there exists \( f \in X \) which resonates with the family \( T \) of operators, and (11.7) holds.

Application: Let \( (X, |\cdot|_\infty) \) denote the Banach space of all continuous functions

\[ f : [0,1] \to \mathbb{C} \]

with \( f(0) = f(1) \) equipped with the maximum norm. Let \( Y = \mathbb{C} \) with \( \|z\|_Y = |z| \). For \( n = 1, 2, \ldots \) let

\[ A_n : X \to \mathbb{C} \]

be defined by

\[ A_nf = S_nf(0) = \int_0^1 D_n(y)f(y)\,dy, \quad f \in X . \]

As stated above, the operator norm of \( A_n \) is \( L_n \) and \( L_n \to \infty \) by Lemma 11.4.

We apply the resonance principle to the family
\[ T = \{ A_n : n = 1, 2, \ldots \} . \]

Since \( \| A_n \| \to \infty \) as \( n \to \infty \) the resonance principle implies existence of a function \( f \in X \) for which

\[ A_n f = S_n f(0) \]

is unbounded.

### 11.9 Isomorphy of \( L_2(0, 1) \) and \( l_2 \)

So far, we have shown that

\[ |f - \sigma_n f|_{\infty} \to 0 \text{ as } n \to \infty \text{ for } f \in X \]

and

\[ \|f - S_n f\|_{L_2} \to 0 \text{ as } n \to \infty \text{ for } f \in L_2(0, 1) . \]

(11.8)

Also, if \( f \in L_2(0, 1) \), then the construction of \( S_n f \) shows that the approximation error

\[ f - S_n f \]

is orthogonal to \( T_n \). Therefore,

\[ \|f\|_{L_2}^2 = \|f - S_n f\|_{L_2}^2 + \|S_n f\|^2 \text{ for } f \in L_2(0, 1) . \]

Since

\[ \|S_n f\|_{L_2}^2 = \sum_{k=-n}^{n} |\hat{f}(k)|^2 \]

one obtains Parseval’s relation:

\[ \|f\|_{L_2}^2 = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 . \]

(11.9)

**Definition:** Let \( l_2 \) denote the linear space of all sequences

\[ a = (a_k)_{k \in \mathbb{Z}} , \quad a_k \in \mathbb{C} , \]

with

\[ \sum_{k} |a_k|^2 < \infty . \]

For \( a, b \in l_2 \) define the \( l_2 \) inner product by

\[ (a, b)_{l_2} = \sum_{k} \bar{a}_k b_k . \]
As usual, the corresponding norm is defined by

\[ \|a\|_{l^2}^2 = (a, a)_{l^2} . \]

It is not difficult to show:

**Theorem 11.7** The sequence space \( l^2 \) with the above inner product is complete, i.e., it is a Hilbert space.

**Theorem 11.8** The mapping \( F : L^2(0, 1) \to l^2 \) defined by

\[ F(f) = (\hat{f}(k))_{k \in \mathbb{Z}}, \quad f \in L^2(0, 1) , \]

is a Hilbert space isomorphism.

**Proof:** Parseval's relation (11.9) shows that \( F \) maps \( L^2(0, 1) \) into \( l^2 \) and is norm preserving. The linearity of \( F \) is clear. Since \( \|Ff\|_{l^2} = \|f\|_{L^2} \) it is also clear that \( F \) is one–to–one. To prove that \( F \) is onto, let \( a \in l^2 \) be given and define

\[ f_n(x) = \sum_{k=-n}^{n} a_k \phi_k(x) . \]

Then \( f_n \) is a Cauchy sequence in \( L^2(0, 1) \), thus there exists \( f \in L^2(0, 1) \) with

\[ \|f - f_n\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty . \]

We have

\[ (\phi_k, f)_{L^2} = \lim_{n \to \infty} (\phi_k, f_n)_{L^2} = a_k . \]

Therefore, \( Ff = a \), showing that \( F \) is onto.

It remains to prove that \( F \) preserves the inner product:

\[ (Ff, Fg)_{l^2} = (f, g)_{L^2} . \quad (11.10) \]

The polarization equality,

\[ 4(f, g) = \|f + g\|^2 - \|f - g\|^2 + i\|if + g\|^2 - i\|if - g\|^2 , \]

which is valid in any inner product space over \( \mathbb{C} \), shows that one can express the inner product in terms of the norm. Therefore, (11.10) follows from (11.9). ⊗
11.10 Convergence of $S_n f$ in Maximum Norm

**Lemma 11.5** Let $f : \mathbb{R} \rightarrow \mathbb{C}, f(x + 1) \equiv f(x)$. If $f \in C^r$ then

$$|\hat{f}(k)| \leq C_r |k|^{-r}, \quad k \in \mathbb{Z}, \quad k \neq 0,$$

with

$$C_r = (2\pi)^{-r} \int_0^1 |f^{(r)}(x)| \, dx.$$

Consequently,

$$|f - S_n f|_{\infty} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

if $r \geq 2$.

**Proof:** Through integration by parts one finds that

$$\hat{f}(k) = \frac{1}{2\pi ik} (f^{(r)})^{(k)}(k).$$

Applying this result $r$ times and noting that

$$(f^{(r)})^{(k)} = \int_0^1 e^{-2\pi i k x} f^{(r)}(x) \, dx$$

the estimate of $|\hat{f}(k)|$ follows. Therefore, if $r \geq 2$, we have for $n > m \geq N(\varepsilon)$:

$$|S_n f - S_m f|_{\infty} \leq \sum_{m < |k| \leq n} |\hat{f}(k)| < \varepsilon.$$

It follows that $S_n f$ is a Cauchy sequence in $(X, | \cdot |_{\infty})$, and there exists $g \in X$ with $|g - S_n f|_{\infty} \rightarrow 0$. This implies $\|g - S_n f\|_{L_2} \rightarrow 0$. We have shown that $\|f - S_n f\|_{L_2} \rightarrow 0$, and conclude that $g = f$. 

One can relax the assumption $f \in C^2$ slightly.

**Theorem 11.9** Let $f : \mathbb{R} \rightarrow \mathbb{C}, f(x + 1) \equiv f(x), f \in C^1$. Then the Fourier sums

$$S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k)e^{2\pi ikx}$$

converge uniformly to $f(x)$,

$$|f - S_n f|_{\infty} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

**Proof:** From

$$\hat{f}(k) = \frac{1}{2\pi ik} (f^{(r)})^{(k)}(k)$$

and
\[
\sum_{k=-\infty}^{\infty} |(f')^\ast(k)|^2 = \|f'\|_{L^2}^2
\]
we conclude that
\[
\sum_{k=-\infty}^{\infty} |k|^2 |f^\ast(k)|^2 =: Q < \infty .
\]
Using the Cauchy–Schwarz inequality,

\[
|S_n f - S_m f|_\infty \leq \sum_{m < |k| \leq n} |\hat{f}(k)|
\]
\[
= \sum_{m < |k| \leq n} \frac{1}{|k|} |k||\hat{f}(k)|
\]
\[
\leq \left( \sum_{m < |k| \leq n} \frac{1}{|k|^2} \right)^{1/2} Q^{1/2} .
\]
As in the proof of the previous lemma, it follows that \(S_n f\) is a Cauchy sequence in \((X, |·|_\infty)\), and the claim follows. \(\diamond\)

### 11.11 Smoothness of \(f(x)\) and Decay of \(\hat{f}(k)\)

Let \(f : \mathbb{R} \to \mathbb{C}, f(x) \equiv f(x + 1), f \in C\). Since

\[
\|f\|_{L^2}^2 = \sum_{k} |\hat{f}(k)|^2
\]
we know that

\[
\hat{f}(k) \to 0 \quad \text{as} \quad |k| \to \infty .
\]
We want to show that the smoothness of \(f\) is related to the decay rate of \(\hat{f}(k)\).

Smoothness implies decay:

**Lemma 11.6** Let \(r \in \{1, 2, \ldots\}\). Let \(f \in C^{r-1}(\mathbb{R})\) have period one, and assume that \(D^r f = f^{(r)} \in L_1\). Then we have

\[
|\hat{f}(k)| \leq C_r |k|^{-r} \quad \text{for} \quad k \neq 0
\]

with

\[
C_r = (2\pi)^{-r} \int_0^1 |D^r f(x)| \, dx .
\]

**Proof:** This follows through integration by parts,
\[
\hat{f}(k) = \int_0^1 e^{-2\pi ikx} f(x) \, dx
\]
\[
= \frac{1}{2\pi i k} \int_0^1 e^{-2\pi ikx} Df(x) \, dx
\]
\[
= \frac{1}{(2\pi i k)^r} \int_0^1 e^{-2\pi ikx} D^r f(x) \, dx
\]

\[
\hat{f}(k) = \int_0^1 e^{-2\pi ikx} f(x) \, dx
\]
\[
= \frac{1}{2\pi i k} \int_0^1 e^{-2\pi ikx} Df(x) \, dx
\]
\[
= \frac{1}{(2\pi i k)^r} \int_0^1 e^{-2\pi ikx} D^r f(x) \, dx
\]

Decay implies smoothness:

**Lemma 11.7** Let \( r \in \{0, 1, 2, \ldots \} \). Let \( f : \mathbb{R} \to \mathbb{C} \) have period one and let \( f \in L_1(0,1) \). Assume that there exists \( A > 0, \delta > 0 \) with

\[
|\hat{f}(k)| \leq A|k|^{-(r+1+\delta)} \text{ for all } k \neq 0.
\]

Then \( f \in C^r \).

**Proof:** We assume that the above estimate holds for \( r = 1 \) and must show that \( f \in C^1 \). We have

\[
S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k)e^{2\pi ikx}
\]
\[
(S_n f)'(x) = \sum_{k=-n}^{n} 2\pi ik \hat{f}(k)e^{2\pi ikx}
\]

For \( n > m \):

\[
|(S_n f)' - (S_m f)'|_\infty \leq 2\pi \sum_{m < |k| \leq n} |k| |\hat{f}(k)|
\]
\[
\leq 2\pi A \sum_{m < |k| \leq n} |k|^{-(1+\delta)}
\]
\[
\leq \varepsilon
\]

for \( n > m \geq N(\varepsilon) \). Therefore, there exists \( g \in X \) with

\[
|g - (S_n f)'|_\infty \to 0 \quad \text{as } n \to \infty.
\]

Also,

\[
|f - S_n f|_\infty \to 0 \quad \text{as } n \to \infty.
\]

Let \( n \to \infty \) in the equation

\[
(S_n f)(x) - (S_n f)(0) = \int_0^x (S_n f)'(y) \, dy
\]
to obtain that

\[ f(x) - f(0) = \int_0^x g(y) \, dy . \]

It follows that \( f \in C^1 \) and \( f' = g \). The proof for other values of \( r \) is similar. \( \diamond \)

11.12 Exponential Decay of \( \hat{f}(k) \) and Analyticity of \( f \)

For \( \alpha > 0 \) let \( S_\alpha \) denote the horizontal strip of width \( 2\alpha \) along the real axis:

\[ S_\alpha = \{ z = x + iy : x \in \mathbb{R}, |y| < \alpha \} . \]

Lemma 11.8 Let \( f \in H(S_\alpha) \), \( f(z) \equiv f(z+1) \). Then, for any \( 0 \le \beta < \alpha \), there is a constant \( C_\beta = C_\beta(f) \) with

\[ |\hat{f}(k)| \le C_\beta e^{-2\pi|k|\beta} \text{ for all } k \in \mathbb{Z} . \]

Proof: For definiteness, let \( k < 0 \). Consider the rectangle

\[ \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \]

where

\[
\begin{align*}
\Gamma_1 & : \quad z(x) = x, \quad 0 \le x \le 1 , \\
\Gamma_2 & : \quad z(y) = 1 + iy, \quad 0 \le y \le \beta , \\
-\Gamma_3 & : \quad z(x) = x + i\beta, \quad 0 \le x \le 1 , \\
-\Gamma_4 & : \quad z(y) = iy, \quad 0 \le y \le \beta .
\end{align*}
\]

Let

\[ g(z) = f(z)e^{-2\pi ikz} . \]

We then have

\[ \hat{f}(k) = \int_{\Gamma_1} g(z) \, dz \]

and, by Cauchy’s theorem,

\[ \sum_{j=1}^4 \int_{\Gamma_j} g(z) \, dz = 0 . \]

The periodicity of \( f \) also implies that

\[ \int_{\Gamma_2} g(z) \, dz + \int_{\Gamma_4} g(z) \, dz = 0 . \]

Consequently,
\[ \hat{f}(k) = \int_{-\Gamma_3} g(z) \, dz . \]

We have justified to move the integration path \( \Gamma_1 \) upwards by \( \beta \). If \( z \in \Gamma_3 \) then \( z = x + i\beta \) and

\[ |e^{-2\pi ikz}| = e^{2\pi k\beta} = e^{-2\pi|k|\beta} \quad \text{for} \quad k < 0 . \]

It follows that

\[ |\hat{f}(k)| \leq C_{\beta} e^{-2\pi|k|\beta} \]

with

\[ C_{\beta} = \max_{0 \leq x \leq 1} |f(x + i\beta)| . \]

If \( k > 0 \) we can argue similarly, moving the path of integration down by \( \beta \).

The previous lemma says that the Fourier coefficients \( \hat{f}(k) \) of a periodic function \( f(x) \) decay exponentially as \( |k| \to \infty \) if \( f(x) \) can be continued analytically into a strip along the real axis. The following lemma shows a converse.

**Lemma 11.9** Let \( \gamma > 0 \) and let \( a = (a_k) \in l_2 \) denote a sequence with

\[ |a_k| \leq Ce^{-2\pi|k|\gamma} \quad \text{for all} \quad k \in \mathbb{Z} . \]

Then there is a unique function \( f \in H(S_{\gamma}) \) with \( f(z) \equiv f(z + 1) \) and

\[ \hat{f}(k) = a_k, \quad k \in \mathbb{Z} . \]

**Proof:** Set

\[ f_n(z) = \sum_{k=-n}^{n} a_k e^{2\pi ikz} . \]

Fix any \( \beta \) with \( 0 < \beta < \gamma \). Let \( z = x + iy, |y| \leq \beta < \gamma \) and obtain that

\[ |e^{2\pi ikz}| \leq e^{2\pi|k|\beta} . \]

Therefore,

\[ |a_k||e^{2\pi ikz}| \leq Ce^{-2\pi|k|\gamma} \quad \text{where} \quad \gamma - \beta > 0 . \]

If \( K \subset S_{\gamma} \) is any compact set, then \( K \subset \bar{S}_{\beta} \) for some \( \beta \) with \( 0 < \beta < \gamma \), and the above estimate implies that the series

\[ \sum_{k} a_k e^{2\pi ikz} \]

converges normally in \( S_{\gamma} \) to an analytic function \( f \in H(S_{\gamma}) \). It is clear that \( \hat{f}(k) = a_k \) and that \( f \) is unique.
11.13 Divergence of $S_n f(0)$: Explicit Construction of $f$

The following is close to [Koerner].

We show existence of a function $f \in X$ for which

$$S_n f(0)$$

is unbounded. Recall that

$$S_n f(y) = \int_{-1/2}^{1/2} D_n(y-x) f(x) \, dx ,$$

thus

$$S_n f(0) = \int_{-1/2}^{1/2} D_n(-x) f(x) \, dx = \int_{-1/2}^{1/2} D_n(x) f(x) \, dx .$$

Here

$$D_n(x) = \frac{\sin(\pi(2n+1)x)}{\sin(\pi x)}$$

is the Dirichlet kernel. We have shown that the Lebesgue constants diverge,

$$L_n = \int_{-1/2}^{1/2} |D_n(x)| \, dx \to \infty \quad \text{as} \quad n \to \infty .$$

Let

$$h_n(x) = \text{sgn} \, D_n(x) .$$

We then have

$$S_n h_n(0) = L_n \to \infty \quad \text{as} \quad n \to \infty .$$

The functions $h_n$ are not continuous, but we can approximate $h_n$ by $g_{n,\varepsilon} \in X$ so that

$$\int_{-1/2}^{1/2} |h_n(x) - g_{n,\varepsilon}| \, dx < \varepsilon \quad \text{and} \quad |g_{n,\varepsilon}|_{\infty} = 1 .$$

Choosing $\varepsilon(n) > 0$ small enough, one finds that

$$S_n g_{n,\varepsilon}(0) \geq L_n - 1 , \quad |g_{n,\varepsilon}|_{\infty} = 1 .$$

Since the trigonometric polynomials are dense in $(X, | \cdot |_{\infty})$, there exists $f_n \in T$ with $|g_{n,\varepsilon} - f_n|_{\infty} < \varepsilon'$. One then finds that

$$S_n f_n(0) \geq L_n - 2 , \quad |f_n|_{\infty} \leq 2 .$$

Dividing by 2, we have obtained a trigonometric polynomial $\frac{1}{2} f_n$ with

$$S_n \left( \frac{1}{2} f_n \right)(0) \geq \frac{1}{2} (L_n - 2) , \quad \left| \frac{1}{2} f_n \right|_{\infty} \leq 1 .$$

Since $L_n \to \infty$ as $n \to \infty$, the following lemma holds:
Lemma 11.10 Given any constant $A > 0$ there exists $H \in \mathcal{T}$ and $N \in \mathbb{N}$ with

$$|S_N H(0)| \geq A, \quad |H|_{\infty} \leq 1.$$  

The remaining part of the proof may be called *piling up of bad functions*. Take $A = A_k = 2^{2^k}$ for $k = 1, 2, \ldots$ By the previous lemma we obtain sequences $H_k \in \mathcal{T}$ and $n(k) \in \mathbb{N}$ with

$$|S_{n(k)} H_k(0)| \geq 2^{2^k}, \quad |H_k|_{\infty} \leq 1.$$  

We write

$$H_k(x) = \sum_{|j| \leq q(k)} \hat{H}_k(j)e^{2\pi ijx} \quad (11.11)$$  

and may assume that

$$q(k) \geq n(k).$$  

Set

$$p(k) = \sum_{j=1}^{k} (2q(j) + 1)$$  

and define the trigonometric polynomials

$$f_n(x) = \sum_{k=1}^{n} 2^{-k} H_k(x) e^{2\pi ip(k)x}, \quad n = 1, 2, \ldots \quad (11.12)$$  

If $n > m$ then

$$|f_n - f_m|_{\infty} \leq \sum_{k=m+1}^{n} 2^{-k} \leq 2^{-m}.$$  

This shows that $f_n$ is a Cauchy sequence in $X$. Since $(X, \cdot_{\infty})$ is complete there exists $f \in X$ with

$$|f - f_n|_{\infty} \to 0.$$  

We will show that the sequence $S_n f(0)$ is unbounded.

Convergence $|f - f_n|_{\infty} \to 0$ implies that

$$\hat{f}_n(r) \to \hat{f}(r) \quad \text{as} \quad n \to \infty \quad \text{for all} \quad r \in \mathbb{Z}.$$  

The main point of the construction is the following: The integers $p(k)$ are so large that the terms in the sum (11.12) have no common frequencies. This makes the following analysis possible.

Substituting the right side of (11.11) for $H_k(x)$ in (11.12) yields

$$f_n(x) = \sum_{k=1}^{n} 2^{-k} \sum_{|j| \leq q(k)} \hat{H}_k(j) \exp(2\pi i(p(k) + j)x)$$  

115
and we obtain
\[ \hat{f}_n(p(k) + j) = 2^{-k}\hat{H}_k(j) \quad \text{if} \quad n \geq k \quad \text{and} \quad |j| \leq q(k). \]
Letting \( n \to \infty \) we see that
\[ \hat{f}(p(k) + j) = 2^{-k}\hat{H}_k(j) \quad \text{if} \quad |j| \leq q(k). \]
Now consider
\[
S_{p(k)+n(k)}f(0) - S_{p(k)-n(k)-1}f(0) = \sum_{|j| \leq n(k)} \hat{f}(p(k) + j) \\
= 2^{-k} \sum_{|j| \leq n(k)} \hat{H}_k(j) \\
= 2^{-k}S_{n(k)}H_k(0)
\]
(In the second equation we have used that \( n(k) \leq q(k) \).) We now take absolute values in the above equation and recall the estimate
\[ |S_{n(k)}H_k(0)| \geq 2^{2k} \]
to obtain
\[ |S_{p(k)+n(k)}f(0) - S_{p(k)-n(k)-1}f(0)| \geq 2^k, \quad k = 1, 2, \ldots \]
This implies that the sequence \( S_n f(0) \) is unbounded.

11.14 Fourier Series and the Dirichlet Problem for Laplace’s Equation on the Unit Disk

The Poisson kernel \( P_r(\alpha) \) is the Abel sum of
\[
\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ik\alpha}.
\]
In other words,
\[
P_r(\alpha) = \frac{1}{2\pi} \sum_{k=0}^{\infty} (re^{i\alpha})^k + \sum_{k=1}^{\infty} (re^{-i\alpha})^k \quad \text{for} \quad 0 \leq r < 1.
\]
12 Fourier Transformation

12.1 Motivation: Application to PDEs

Example 1: Consider the PDE

\[ u_t + au_x = 0 \]

for a function \( u(x,t) \) where \( a \in \mathbb{R} \). Give an initial condition

\[ u(x,0) = e^{ikx}, \quad x \in \mathbb{R}. \]

The initial function

\[ e^{ikx} = \cos(kx) + i\sin(kx) \]

is a wave with wavenumber \( k \) (assumed to be real). Roughly, the wave \( e^{ikx} \) has \( |k| \) waves in the interval \( 0 \leq x \leq 2\pi \) and the wavelength is \( 2\pi/|k| \). To obtain a solution of the PDE \( u_t + au_x = 0 \) with initial condition \( u(x,0) = e^{ikx} \) try the ansatz

\[ u(x,t) = \alpha(t)e^{ikx}. \]

One obtains that

\[ \alpha'(t) + iak\alpha(t) = 0, \quad \alpha(0) = 1, \]

thus

\[ \alpha(t) = e^{-iakt}, \quad u(x,t) = e^{i(k(x-at))}. \]

The solution \( u(x,t) \) describes that the initial wave \( e^{ikx} \) moves undistorted at speed \( a \). If \( a > 0 \) the wave moves to the right, if \( a < 0 \) it moves to the left.

Example 2: Consider the heat equation

\[ u_t = u_{xx} \quad \text{for} \quad x \in \mathbb{R}, \quad t \geq 0, \]

with initial condition

\[ u(x,0) = e^{ikx}, \quad x \in \mathbb{R}. \]

The ansatz

\[ u(x,t) = \alpha(t)e^{ikx} \]

leads to

\[ \alpha'(t) = -k^2\alpha(t), \quad \alpha(0) = 1, \]

with solution

\[ \alpha(t) = e^{-k^2t}. \]
The solution of the PDE $u_t = u_{xx}$ with initial condition $u(x, 0) = e^{ikx}$ is

$$u(x, t) = e^{-k^2 t} e^{ikx}.$$  

We see that the amplitude decays exponentially as $t$ increases if $k \neq 0$. The smaller the wavelength of the initial function $u(x, 0) = e^{ikx}$, the more rapid is the decay in time. We expect, therefore, that the solution of the heat equation will become smooth if we start with rough initial data $u(x, 0) = f(x)$.

**Example 3:** Consider the linearized Korteweg–de Vries equation

$$u_t = u_{xxx}$$

for $x \in \mathbb{R}$, $t \geq 0$,

with initial condition

$$u(x, 0) = e^{ikx}, \quad x \in \mathbb{R}.$$  

The ansatz

$$u(x, t) = \alpha(t)e^{ikx}$$

leads to

$$\alpha'(t) = -ik^3 \alpha(t), \quad \alpha(0) = 1,$$

with solution

$$\alpha(t) = e^{-ik^3 t}.$$  

The solution of the PDE $u_t = u_{xxx}$ with initial condition $u(x, 0) = e^{ikx}$ is

$$u(x, t) = e^{ik(x-k^2 t)}.$$  

We see that the amplitude $\alpha(t)$ satisfies $|\alpha(t)| = 1$, thus there is neither decay nor growth. Here the wave speed is $k^2$, i.e., the wave speed depends on the wave length of the initial function $e^{ikx}$. We have the effect of dispersion.

**Example 4:** Consider the free–space Schrödinger equation

$$u_t = iu_{xx}$$

for $x \in \mathbb{R}$, $t \geq 0$,

with initial condition

$$u(x, 0) = e^{ikx}, \quad x \in \mathbb{R}.$$  

The ansatz

$$u(x, t) = \alpha(t)e^{ikx}$$

leads to

$$\alpha'(t) = -ik^2 \alpha(t), \quad \alpha(0) = 1,$$

with solution
\[ \alpha(t) = e^{-ik^2t}. \]

The solution of the PDE \( u_t = iu_{xx} \) with initial condition \( u(x,0) = e^{ikx} \) is
\[ u(x,t) = e^{ik(x-kt)}. \]

We see that the amplitude \( \alpha(t) \) satisfies \( |\alpha(t)| = 1 \), thus there is neither decay nor growth. Here the wave speed is \( k \), i.e., the wave speed depends on the wave length of the initial function \( e^{ikx} \). We have the effect of dispersion.

**More General Equation:** Consider the constant–coefficient equation
\[ u_t = Lu \equiv \sum_{j=0}^{m} a_j D^j u \quad \text{for} \quad x \in \mathbb{R}, \quad t \geq 0, \]
where
\[ a_j \in \mathbb{C}, \quad D = \frac{\partial}{\partial x}. \]

Given an initial condition
\[ u(x,0) = e^{ikx}, \quad x \in \mathbb{R}, \]
the ansatz
\[ u(x,t) = \alpha(t)e^{ikx} \]
leads to
\[ \alpha'(t) = \hat{L}(ik)\alpha(t), \quad \alpha(0) = 1, \]
where
\[ \hat{L}(ik) = \sum_{j=0}^{m} a_j (ik)^j \]
is the so–called symbol of \( L \). The solution of the amplitude equation is
\[ \alpha(t) = e^{\hat{L}(ik)t} \]
and the solution of the PDE \( u_t = Lu \) with initial condition \( u(x,0) = e^{ikx} \) is
\[ u(x,t) = e^{\hat{L}(ik)t}e^{ikx}. \]

If one wants to solve the initial value problem
\[ u_t = Lu, \quad u(x,0) = f(x), \]
with more general functions \( f(x) \) then one might try to write \( f(x) \) as a superposition of the simple waves \( e^{ikx} \). The tool is the Fourier transform.
12.2 The Fourier Transform of an $L_1$ Function

For $f \in L_1 = L_1(\mathbb{R}, \mathbb{C})$ one defines its Fourier transform by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \xi} \, dx, \quad \xi \in \mathbb{R}.$$  \hfill (12.1)

(We follow the conventions in [Stein, Shakarchi]. Other forms, like

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix \xi} \, dx, \quad \xi \in \mathbb{R},$$

are also used.)

**Remark:** The condition $f \in L_1$ is necessary if one wants to use the standard meaning of the integral in (12.1). However, using the notions of distribution theory, one can define a Fourier transform for functions and distributions that are not in $L_1$.

**Lemma 12.1** If $f \in L_1$ then $\hat{f}$ is uniformly continuous.

**Proof:** Let $\varepsilon > 0$ be given. We must show existence of $\delta > 0$ so that $|\xi_1 - \xi_2| < \delta$ implies $|\hat{f}(\xi_1) - \hat{f}(\xi_2)| < \varepsilon$.

Since $f \in L_1$ there exists $R > 0$ with

$$\int_{|x| > R} |f(x)| \, dx < \frac{\varepsilon}{4}.$$  

This can be shown by a cut–off argument and Lebesgue’s dominated convergence theorem. We have

$$|\hat{f}(\xi_1) - \hat{f}(\xi_2)| \leq \int_{|x| > R} |f(x)||e^{-2\pi i x \xi_1} - e^{-2\pi i x \xi_2}| \, dx$$

$$+ \int_{|x| \leq R} |f(x)||e^{-2\pi i x \xi_1} - e^{-2\pi i x \xi_2}| \, dx$$

$$\leq \frac{\varepsilon}{4} \cdot 2 + \|f\|_{L_1} \max_{|x| \leq R} |e^{-2\pi i x \xi_1} - e^{-2\pi i x \xi_2}|.$$

Here

$$\|f\|_{L_1} = \int_{-\infty}^{\infty} |f(x)| \, dx.$$  

The maximum in the above formula equals

$$M = \max_{|x| \leq R} |1 - e^{-2\pi x \xi}| \quad \text{where} \quad \xi = \xi_2 - \xi_1.$$  

Since the function

$$g(x, \xi) = |1 - e^{-2\pi x \xi}|$$

is uniformly continuous on the set
and since
\[ g(x,0) = 0 , \]
there is \( \delta > 0 \) so that
\[ g(x,\xi) < \varepsilon' \quad \text{for} \quad |\xi| < \delta, \quad |x| \leq R . \]
Choosing
\[ \varepsilon' = \frac{\varepsilon}{2 \| f \|_{L^1}} \]
we obtain \( |\hat{f}(\xi_1) - \hat{f}(\xi_2)| < \varepsilon \) for \( |\xi_1 - \xi_2| < \delta. \)

What can be said about the decay of \( |\hat{f}(\xi)| \) as \( |\xi| \to \infty? \) Through integration by parts one obtains:

**Lemma 12.2** Let \( f \in C^1_0 \), i.e., \( f \) is a \( C^1 \) function with compact support. Then
\[ |\hat{f}(\xi)| \leq \frac{1}{2\pi|\xi|} \| f' \|_{L^1}, \quad \xi \neq 0 . \]

**Proof:** We have
\[
\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx \\
= f(x) \frac{1}{-2\pi i \xi} e^{-2\pi i x \xi} |_{x=-\infty} + \frac{1}{2\pi i \xi} \int_{-\infty}^{\infty} f'(x) e^{-2\pi i x \xi} \, dx
\]
The estimate follows. \( \diamond \)

### 12.3 The Riemann–Lebesgue Lemma

The following result is known as the Riemann–Lebesgue Lemma:

**Lemma 12.3** If \( f \in L^1 \) then \( |\hat{f}(\xi)| \to 0 \) as \( |\xi| \to \infty. \)

To prove the Riemann–Lebesgue Lemma, we need the following auxiliary result:

**Lemma 12.4** The set \( C^\infty_0 \) of all \( C^\infty \) functions with compact support is dense in \( (L^1, \| \cdot \|_{L^1}) \).

**Proof:** Let \( f \in L^1 \) and let \( \varepsilon > 0 \) be given. Choose \( R > 0 \) so large that
\[ \int_{|x|>R} |f(x)| \, dx < \frac{\varepsilon}{2} . \]
We apply a cut–off to the function $f$ and set

$$g(x) = \begin{cases} f(x) & |x| \leq R \\ 0 & |x| > R \end{cases}.$$  

We then have $\|f - g\|_{L_1} \leq \frac{\varepsilon}{2}$ and $g$ has compact support.

We now mollify $g$. This process is as follows: Choose a $C^\infty$ function $\Phi : \mathbb{R} \to \mathbb{R}$ with

$$\Phi(y) > 0 \quad \text{for} \quad -1 < y < 1, \quad \Phi(y) = 0 \quad \text{for} \quad |y| \geq 1$$

and

$$\int_{-\infty}^{\infty} \Phi(y) \, dy = 1.$$ 

Note that

$$\int_{-\infty}^{\infty} \frac{1}{\varepsilon} \Phi(y/\varepsilon) \, dy = 1 \quad \text{for} \quad \varepsilon > 0$$

and the function $y \to \Phi(y/\varepsilon)$ is supported in $|y| \leq \varepsilon$. Define

$$g_\varepsilon(x) = \int_{-\infty}^{\infty} \frac{1}{\varepsilon} \Phi((x-y)/\varepsilon) g(y) \, dy.$$ 

It is not difficult to prove that $g_\varepsilon \in C^\infty_0$ and $\|g - g_\varepsilon\|_{L_1} \to 0$ as $\varepsilon \to 0$.  

\textbf{Proof of Riemann–Lebesgue Lemma:} Let $f \in L_1$ and let $\varepsilon > 0$ be given. There exists $f_\varepsilon \in C^\infty_0$ so that

$$\|f - f_\varepsilon\|_{L_1} < \frac{\varepsilon}{2}.$$  

This implies that

$$|\hat{f}(\xi) - \hat{f}_\varepsilon(\xi)| < \frac{\varepsilon}{2} \quad \text{for} \quad \xi \in \mathbb{R}.$$  

Since

$$|\hat{f}_\varepsilon(\xi)| \leq \frac{C_\varepsilon}{|\xi|} \quad \text{for} \quad \xi \neq 0$$

we obtain the estimate

$$|\hat{f}(\xi)| \leq \frac{\varepsilon}{2} + \frac{C_\varepsilon}{|\xi|} < \varepsilon \quad \text{for} \quad |\xi| \geq R_\varepsilon.$$  

\textbf{Example 1:} Let

$$f(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}.$$ 

We have
\[
\hat{f}(\xi) = \int_{-1}^{1} e^{-2\pi i x \xi} \, dx
\]
\[
= \frac{1}{-2\pi i \xi} e^{-2\pi i x |x=1} |_{x=-1}
\]
\[
= \frac{\sin(2\pi \xi)}{\pi \xi}.
\]

Since
\[
\int_{0}^{R} |\hat{f}(\xi)| \, d\xi \to \infty \quad \text{as} \quad R \to \infty
\]
we conclude that \( \hat{f} \notin L_1 \). The example shows that, in general, the assumption \( f \in L_1 \) does not imply \( \hat{f} \in L_1 \).

### 12.4 The Fourier Transform on the Schwartz Space \( S \)

A function \( f : \mathbb{R} \to \mathbb{C} \) is called rapidly decreasing if
\[
|x^n f(x)|, \quad x \in \mathbb{R},
\]
is bounded for all \( n \in \mathbb{N} \). The Schwartz space \( S = S(\mathbb{R}, \mathbb{C}) \) consists of all \( f \in C^\infty(\mathbb{R}, \mathbb{C}) \) for which all derivatives \( f^{(k)}(x) \), \( k = 0, 1, 2, \ldots \) are rapidly decreasing.

For \( f \in S \) the Fourier transform
\[
\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx, \quad \xi \in \mathbb{R},
\]
is well-defined.

**Theorem 12.1**

\( a) \) If \( f \in S \) then \( \hat{f} \in S \).

\( b) \) If \( f \in S \) then
\[
(f')^\wedge(\xi) = 2\pi i \xi \hat{f}(\xi).
\]

\( c) \) If \( f \in S \) then \( -2\pi i x f(x) \in S \) and
\[
(f')^\wedge(\xi) = \left( -2\pi i x f(x) \right)^\wedge(\xi).
\]

**Proof:** We know that \( \hat{f} \in C \) and \( \hat{f} \) is bounded. Further, \( f' \in S \) and, through integration by parts,
\[
(f')^\wedge(\xi) = \int f'(x) e^{-2\pi i x \xi} \, dx
\]
\[
= \int f(x) (2\pi i \xi) e^{-2\pi i x \xi} \, dx
\]
\[
= 2\pi i \xi \hat{f}(\xi).
\]
This shows that \( \xi \hat{f}(\xi) \) is bounded and b) holds. By further differentiations, we see that \( |\xi^n \hat{f}(\xi)| \) is bounded for every \( n \).

We now show that \( \hat{f} \in C^1 \) and

\[
(\hat{f})'(\xi) = \hat{g}(\xi) \quad \text{with} \quad g(x) = -2\pi ixf(x).
\]

To this end, let \( \xi \in \mathbb{R} \) be fixed and let \( h \) be a real number, \( h \neq 0 \). Consider

\[
Q_h := \frac{1}{h}(\hat{f}(\xi + h) - \hat{f}(\xi)) - \hat{g}(\xi)
\]

\[
= \int f(x)e^{-2\pi ix\xi}\left(\frac{1}{h}(e^{-2\pi ixh} - 1) + 2\pi ix\right)dx.
\]

We must show that for any \( \varepsilon > 0 \) there is \( h_0 > 0 \) so that \( |Q_h| < \varepsilon \) for \( 0 < |h| \leq h_0 \).

It is clear that for real \( \alpha \):

\[
\left|\frac{1}{\alpha}(e^{-2\pi i\alpha} - 1)\right| \leq \begin{cases} 
2/\alpha & \text{for } |\alpha| \geq 1 \\
C & \text{for } 0 < |\alpha| \leq 1
\end{cases}
\]

Therefore, the function \((e^{-2\pi i\alpha} - 1)/\alpha\) is bounded for real \( \alpha \neq 0 \). It follows that

\[
\left|\frac{1}{h}(e^{-2\pi ixh} - 1) + 2\pi ix\right| \leq C|x|
\]

for \( xh \neq 0 \). For any \( R > 0 \) we have

\[
|Q_h| \leq C \int_{|x| \geq R} |f(x)||x| dx + \int_{|x| \leq R} |f(x)||\frac{1}{h}(e^{-2\pi ixh} - 1) + 2\pi ix| dx.
\]

We choose \( R > 0 \) so that the first term is bounded by \( \varepsilon/2 \). Then, for \( |x| \leq R \), we have

\[
\left|\frac{1}{h}(e^{-2\pi ixh} - 1) + 2\pi ix\right| \leq C_R|h|.
\]

It is then clear that \( |Q_h| < \varepsilon \) if \( |h| \) is small enough. This proves c). With the same arguments as above, it follows that \( |\xi^n (\hat{f})'(\xi)| \) is bounded for every \( n \).

The argument can be repeated for every derivative of \( \hat{f}(\xi) \), and the theorem is proved. \( \diamond \)

12.5 The Fourier Inversion Formula on the Schwartz Space: Preparations

The following is an important definite integral:

**Lemma 12.5**

\[
J := \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.
\]
Proof: Using Fubini’s theorem and transformation to polar coordinates, we have

\[
J^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} \, dx \, dy
= 2\pi \int_{0}^{\infty} r e^{-\pi r^2} \, dr
= \int_{0}^{\infty} e^{-\rho} \, d\rho
= 1.
\]

\[\blacksquare\]

We next show that the Fourier transform of a Gaussian is a Gaussian:

**Lemma 12.6** The Fourier transform of

\[f(x) = e^{-\pi x^2}\]

is

\[\hat{f}(\xi) = e^{-\pi \xi^2} .\]

Proof: Set

\[F(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi ix \xi} \, dx .\]

We then have \(F(0) = 1\) and, using integration by parts,

\[
F'(\xi) = \int e^{-\pi x^2} (-2\pi ix) e^{-2\pi i x \xi} \, dx
= i \int \frac{d}{dx} \left( e^{-\pi x^2} \right) e^{-2\pi i x \xi} \, dx
= -2\pi \xi F(\xi)
\]

Thus we have shown that the function \(\hat{f}(\xi) = F(\xi)\) satisfies

\[F'(\xi) = -2\pi \xi F(\xi), \quad F(0) = 1 .\]

Unique solubility of this initial value problem proves that

\[\hat{f}(\xi) = F(\xi) = e^{-\pi \xi^2} .\]

\[\blacksquare\]

By linear scalings \((x/\sqrt{\delta} = y)\) one obtains:
Lemma 12.7 For $\delta > 0$ let

$$K_\delta(x) = \frac{1}{\sqrt{\delta}} e^{-\pi x^2/\delta}, \quad G_\delta(\xi) = e^{-\pi \xi^2}.$$ 

Then we have

$$\hat{K}_\delta = G_\delta \quad \text{and} \quad \hat{G}_\delta = K_\delta.$$ 

Further properties of the family of functions $K_\delta(x)$ are:

Lemma 12.8 a)

$$\int_{-\infty}^{\infty} K_\delta(x) \, dx = 1, \quad \delta > 0.$$ 

b) If $\eta > 0$ is fixed, then

$$\int_{|x| \geq \eta} K_\delta(x) \, dx \to 0 \quad \text{as} \quad \delta \to 0.$$ 

c)

$$\int_{-\infty}^{\infty} |x| K_\delta(x) \, dx \to 0 \quad \text{as} \quad \delta \to 0.$$ 

d) If $f \in S$ then

$$\int_{-\infty}^{\infty} f(x) K_\delta(x) \, dx \to f(0) \quad \text{as} \quad \delta \to 0.$$ 

Proof: a) follows from

$$\int_{-\infty}^{\infty} K_\delta(x) \, dx = G_\delta(0) = 1.$$ 

Using the substitution $x/\sqrt{\delta} = y$ we have

$$\int_{\eta}^{\infty} K_\delta(x) \, dx = \frac{1}{\sqrt{\delta}} \int_{\eta}^{\infty} e^{-\pi (x/\sqrt{\delta})^2} \, dx$$

$$= \int_{\eta/\sqrt{\delta}}^{\infty} e^{-\pi y^2} \, dy$$

and b) follows. To show c) we compute

$$\int_{0}^{\infty} x K_\delta(x) \, dx = \frac{1}{\sqrt{\delta}} \int_{0}^{\infty} xe^{-\pi (x/\sqrt{\delta})^2} \, dx$$

$$= \sqrt{\delta} \int_{0}^{\infty} e^{-\pi y^2} \, dy.$$ 

Finally,
\[
\int_{-\infty}^{\infty} f(x)K_\delta(x) \, dx = f(0) + \int_{-\infty}^{\infty} (f(x) - f(0))K_\delta(x) \, dx .
\]

Here \(|f(x) - f(0)| \leq C|x|\), and the claim follows from c). \(\diamondsuit\)

We next prove the following multiplication rule:

**Theorem 12.2**

a) If \(f, g \in S\) then \(fg \in S\). Therefore, \(f\hat{g} \in S\).

b) If \(f, g \in S\) then

\[
\int_{-\infty}^{\infty} f(x)\hat{g}(x) \, dx = \int_{-\infty}^{\infty} \hat{f}(y)g(y) \, dy .
\]

**Proof:** If \(f, g \in S\) then Leibniz’ rule yields \(fg \in S\). Furthermore, using Fubini’s theorem,

\[
\int_{-\infty}^{\infty} f(x)\hat{g}(x) \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y)e^{-2\pi ixy} \, dy \, dx
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y)e^{-2\pi ixy} \, dx \, dy
\]

\[
= \int_{-\infty}^{\infty} \hat{f}(y)g(y) \, dy
\]

This proves the multiplication rule. \(\diamondsuit\)

### 12.6 The Fourier Inversion Formula on the Schwartz Space

Any function \(f \in S\) can be written as a Fourier integral. The formula (12.2) is called the Fourier representation of \(f(x)\) or the Fourier inversion formula.

**Theorem 12.3** If \(f \in S\) then we have

\[
f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x \xi} \, d\xi, \quad x \in \mathbb{R} .
\]  

(12.2)

**Proof:** First, since the Fourier transform of \(G_\delta(\xi)\) is \(K_\delta(x)\), we have for any \(\delta > 0\):

\[
\int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{\delta}} e^{-\pi x^2/\delta} \, dx = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{-\pi \xi^2 \delta} \, d\xi .
\]  

(12.3)

In this equation we take the limit \(\delta \to 0\) and obtain

\[
f(0) = \int_{-\infty}^{\infty} \hat{f}(\xi) \, d\xi .
\]  

(12.4)

This proves the Fourier inversion formula for \(x = 0\). Next, fix \(x\) and set

\[
F(y) = f(x + y), \quad y \in \mathbb{R} .
\]

We have
\[ \hat{F}(\xi) = \int_{-\infty}^{\infty} f(x+y) e^{-2\pi i y \xi} \, dy \]
\[ = \int_{-\infty}^{\infty} f(z) e^{-2\pi i z \xi} e^{2\pi i x \xi} \, dz \]
\[ = \hat{f}(\xi) e^{2\pi i x \xi} . \]

Using (12.4) with \( f \) replaced by \( F \) yields

\[ f(x) = F(0) \]
\[ = \int_{-\infty}^{\infty} \hat{F}(\xi) \, d\xi \]
\[ = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi . \]

This proves the inversion formula. \( \diamond \)

### 12.7 Operators

We can summarize the results by defining the following operators from \( S \) into itself:

\[ (\mathcal{F}f)(\xi) = \int f(x) e^{-2\pi i x \xi} \, dx \]
\[ (\mathcal{G}g)(x) = \int g(\xi) e^{2\pi i x \xi} \, d\xi \]
\[ (Sf)(x) = f(-x) \]
\[ (Bf)(x) = \bar{f}(x) \]

The Fourier Inversion Theorem says that

\[ \mathcal{G}\mathcal{F} = \text{id}_S . \]

We also see that

\[ \mathcal{G} = SF = BFB . \]

Since \( SS = BB = \text{id}_S \) one obtains that

\[ \mathcal{F}\mathcal{F} = S, \quad \mathcal{F}^4 = \text{id}_S . \]

In particular, \( \mathcal{F} : S \to S \) is linear, one-to-one, and onto.

The following result is Plancherel’s formula.

**Theorem 12.4** For \( f, g \in S \) define the \( L_2 \) inner product by

\[ (f, g)_{L_2} = \int \bar{f}(x)g(x) \, dx . \]
Then we have

$$(f, g)_{L^2} = (\hat{f}, \hat{g})_{L^2}.$$

**Proof:** Define

$$\phi = GBf = BF f.$$  

Then, using the multiplication formula,

$$(f, g)_{L^2} = \int \hat{f} \hat{g} d\xi = \int \phi \hat{g} d\xi = \int (B\hat{f})\hat{g} d\xi = (\hat{f}, \hat{g})_{L^2}$$

**Corollary** For all $f \in S$ we have

$$\|f\| = \|\hat{f}\|.$$  \hspace{1cm} (12.5)

### 12.8 Elementary Theory of Tempered Distributions

On the Schwartz space $S$ one defines a convergence concept as follows: If $g_n \in S$ then $g_n \to 0$ in $S$ means that for all $j, k \in \mathbb{N}$ we have

$$\sup_x |x^j g_n^{(k)}(x)| \to 0 \quad \text{as} \quad n \to \infty.$$  

If $g_n, g \in S$ then $g_n \to g$ in $S$ means that $g - g_n \to 0$ in $S$. This convergence concept is very restrictive.

A linear functional $\phi : S \to \mathbb{C}$ is called continuous if $g_n \to 0$ in $S$ implies that $\phi(g_n) \to 0$ in $\mathbb{C}$. Since the convergence concept in $S$ is very restrictive, the continuity requirement of a linear functional is very mild.

The linear space of all continuous linear functionals on $S$ is called the dual space of $S$, denote by $S'$. Every element of $S'$ is called a tempered distribution.

**Notation:** If $\phi \in S'$ and $g \in S$ we often write

$$\phi(g) = (\phi, g)_{S' \cdot S}$$

to denote the value of $\phi$ at $g$. This notation emphasizes the linearity of the expression $\phi(g)$ in $\phi$ and in $g$.  

129
12.8.1 Ordinary Functions as Tempered Distributions

If $f \in L_{1,\text{loc}}$ satisfies an estimate

$$|f(x)| \leq K|x|^k$$

for some constants $K, K_1$ and $k$, then $f$ determines the tempered distribution $\phi_f$ defined by

$$\phi_f(g) = \int_{-\infty}^{\infty} f(x)g(x) \, dx, \quad g \in \mathcal{S}.$$  

We also denote the expression by

$$(f, g)_{\mathcal{S}' \mathcal{S}}$$

and identify $\phi_f$ with $f$. In particular, since every $f \in \mathcal{S}$ satisfies an estimate (12.6), we obtain the inclusion $\mathcal{S} \subset \mathcal{S}'$.

12.9 The Fourier Transform on $\mathcal{S}'$: An Example Using Complex Variables

For $f, g \in \mathcal{S}$ we have

$$\int_{-\infty}^{\infty} (\mathcal{F}f)(x)g(x) \, dx = \int_{-\infty}^{\infty} f(x)(\mathcal{F}g)(x) \, dx.$$ 

Therefore, if $\phi \in \mathcal{S}'$ we define $\mathcal{F}\phi \in \mathcal{S}'$ by

$$(\mathcal{F}\phi, g)_{\mathcal{S}' \mathcal{S}} = (\phi, \mathcal{F}g)_{\mathcal{S}' \mathcal{S}} \quad \text{for all} \quad g \in \mathcal{S}.$$ 

We want to show by an example how the theory of complex variables is used in determining the Fourier transform of the distribution (determined by)

$$f(x) = e^{-\pi x^2 is}, \quad x \in \mathbb{R},$$

where $s \in \mathbb{R}, s \neq 0$, is fixed. Note that $f \in L_{\infty}$, so $f \in \mathcal{S}'$. However, $f \notin L_1$, so the Fourier transform of $f$ cannot be obtained directly by evaluating the Fourier integral formula.

Recall the definitions

$$G_\delta(x) = e^{-\pi x^2 \delta}, \quad K_\delta(\xi) = \frac{1}{\sqrt{\delta}} e^{-\pi \xi^2 / \delta}$$

where $x$ and $\xi$ are real and where $\delta > 0$. The functions $G_\delta$ and $K_\delta$ belong to $\mathcal{S}$ and we have

$$\mathcal{G}_\delta = K_\delta \quad \text{in} \quad \mathcal{S}$$

for every $\delta > 0$. Formally setting $\delta = is$ suggests that the Fourier transform of $f(x)$ is

$$F(\xi) = \frac{1}{\sqrt{i s}} e^{-\pi \xi^2 / is}$$
in the sense of distributions. Here \( \sqrt{\delta} \) is obtained by analytically continuing the function \( \sqrt{\delta}, \delta > 0 \), to \( \mathbb{C} \setminus (-\infty, 0] \).

Note that \( f \) and \( F \) both belong to \( L_\infty \), thus \( f \) and \( F \) both belong to \( S' \). We claim that \( \mathcal{F}(f) = F \) in \( S' \). This means that we have for all \( g \in S \):

\[
\int_{-\infty}^{\infty} e^{-\pi x^2 is} \hat{g}(x) \ dx = \frac{1}{\sqrt{i s}} \int_{-\infty}^{\infty} e^{-\pi \xi^2 / is} g(\xi) \ d\xi.
\]

To show this, fix \( g \in S \) and introduce the functions

\[
L(z) = \int_{-\infty}^{\infty} e^{-\pi x^2 z} \hat{g}(x) \ dx
\]

\[
R(z) = \frac{1}{\sqrt{z}} \int_{-\infty}^{\infty} e^{-\pi \xi^2 / z} g(\xi) \ d\xi
\]

for \( \text{Re} \ z \geq 0, z \neq 0 \). Then we know that

\[
L(\delta) = R(\delta) \quad \text{for} \quad \delta > 0.
\]

Furthermore, \( L(z) \) and \( R(z) \) are analytic functions in the open right half–plane

\[
H_r = \{ z : \text{Re} \ z > 0 \}.
\]

Therefore, \( L(z) = R(z) \) in \( H_r \). Now fix \( s \in \mathbb{R}, s \neq 0 \). The point \( z_0 = is \) is a boundary point of the half–plane \( H_r \). We claim that

\[
\lim_{z \rightarrow is, z \in H_r} L(z) = L(is),
\]

and a similar relation hold for \( R(z) \). To see this, let \( z \in H_r \) and let \( \varepsilon > 0 \) be given. For sufficiently large \( x_0 \) we have

\[
|L(is) - L(z)| \leq \int_{-\infty}^{\infty} |e^{-\pi x^2 is} - e^{-\pi x^2 z}||\hat{g}(x)| \ dx
\]
\[
\leq 2 \int_{|x| \geq x_0} |\hat{g}(x)| \ dx + \int_{|x| \leq x_0} |e^{-\pi x^2 is} - e^{-\pi x^2 z}||\hat{g}(x)| \ dx
\]
\[
\leq \varepsilon + \|\hat{g}\|_{L^1} \max_{|x| \leq x_0} |e^{-\pi x^2 is} - e^{-\pi x^2 z}|.
\]

This estimate shows that \( |L(is) - L(z)| \leq 2\varepsilon \) if \( z \in H_r \) is sufficiently close to \( is \). Since a similar result holds for \( R(z) \) and since \( L(z) = R(z) \) for \( z \in H_r \), the claim \( \hat{f} = F \) follows.

12.10 Decay of the Fourier Transform of \( f \) and Analyticity of \( f \)

12.11 The Paley–Wiener Theorem

Application to the finite speed of propagation for the wave equation and the Klein–Gordon equation.
12.12 The Laplace Transform and Its Inversion

Relation between the Laplace and the Fourier transform:

Let \( f : [0, \infty) \rightarrow \mathbb{C} \) denote a continuous function satisfying the estimate

\[
|f(t)| \leq Ce^{\alpha t}, \quad t \geq 0,
\]

for some \( C > 0 \) and real \( \alpha \). For complex \( s \) with \( \text{Re}\, s > \alpha \) the Laplace transform of \( f \) is

\[
(Lf)(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt.
\]

Using certain conventions, the Fourier transform of a function \( g : \mathbb{R} \rightarrow \mathbb{C} \) is

\[
(Fg)(y) = \int_{-\infty}^{\infty} e^{-iyt} g(t) \, dt, \quad y \in \mathbb{R}.
\]

Denote the Heaviside function by \( H(t) \) and let \( s = x + iy \). We have

\[
(Lf(t))(x + iy) = \int_{-\infty}^{\infty} H(t)f(t)e^{-xt}e^{-iyt} \, dt = \left( F(H(t)f(t)e^{-xt}) \right)(y)
\]

Assuming the Fourier inversion formula to be valid for the function \( t \rightarrow H(t)f(t)e^{-xt} \) one obtains that

\[
H(t)f(t)e^{-xt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyt} L(f(t))(x + iy) \, dy.
\]

This yields

\[
H(t)f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(x+iy)t} L(f(t))(x + iy) \, dy.
\]

In terms of a line integral one obtains that

\[
H(t)f(t) = \frac{1}{2\pi i} \int_{\Gamma_x} e^{st} (Lf)(s) \, ds.
\]

Here \( \Gamma_x \) is the straight line parameterized by

\[
s = x + iy, \quad -\infty < y < \infty.
\]

For applications, it is important that the function \( (Lf)(s) \) is often analytic in a region larger than the half–plane \( \text{Re}\, s > \alpha \) and that the curve \( \Gamma_x \) may be deformed in the region of analyticity.
13 Growth and Zeros of Entire Functions

13.1 Jensen’s Formula

**Theorem 13.1** Let $D = D(0, R + \varepsilon)$ and let $f \in H(D)$. Assume that $f(0) \neq 0$ and $f(z) \neq 0$ for $|z| = R$.

Let $a_1, \ldots, a_N$ denote the zeros of $f$ with $|a_j| < R$, repeated according to their multiplicity. Then the following formula holds:

$$
\ln |f(0)| = \sum_{j=1}^{N} \ln \left( \frac{|a_j|}{R} \right) + \frac{1}{2\pi} \int_{0}^{2\pi} \ln |f(Re^{it})| \, dt . \tag{13.1}
$$

Before proving the formula, we recall Cauchy’s integral formula: If $f \in D(z_0, R + \varepsilon)$ then

$$
f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - z_0} \, dz .
$$

Here $\Gamma$ has the parameterization

$$
z(t) = z_0 + Re^{it}, \quad 0 \leq t \leq 2\pi .
$$

We can also write the formula in mean–value form:

$$
f(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} f(z_0 + Re^{it}) \, dt .
$$

Here we have used that

$$
dz = Re^{it} \, dt = (z - z_0)i \, dt ,
$$

thus

$$
\frac{dz}{i(z - z_0)} = dt .
$$

If $u : D \to \mathbb{R}$ is harmonic, then $u$ is the real–part of a function $f \in H(D)$. Therefore,

$$
u(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} u(z_0 + Re^{it}) \, dt . \tag{13.2}
$$

**Lemma 13.1** Let $a \in \mathbb{C}, |a| < 1$. Then we have

$$
\int_{0}^{2\pi} \ln |e^{it} - a| \, dt = 0 .
$$

**Proof:** Set $f(z) = 1 - za$. Then $f$ has no zero in $D(0, 1 + \varepsilon) =: D$. We have $f(z) = e^{\phi(z)}$ for some $\phi \in H(D)$. Therefore, $|f(z)| = e^{\text{Re}\phi(z)}$ and

$$
u(z) := \ln |f(z)| = \text{Re}\phi(z)
$$
is harmonic in $D$. Also, $u(0) = \ln 1 = 0$. Formula (13.2) with $R = 1$ and $u_0 = 0$ yields

$$0 = u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \, dt = \frac{1}{2\pi} \int_0^{2\pi} \ln |1 - ae^{it}| \, dt ,$$

thus

$$0 = \int_0^{2\pi} \ln |1 - ae^{it}| \, dt . \quad (13.3)$$

(Note that (13.3) is the special case of (13.1) obtained for $R = 1$, $f(z) = 1 - az$.) In (13.3) substitute $t = -s$ to obtain

$$0 = -\int_{-2\pi}^{0} \ln |1 - ae^{-is}| \, ds$$

$$= \int_{0}^{2\pi} \ln |1 - ae^{-is}| \, ds$$

$$= \int_{-2\pi}^{0} \ln |1 - ae^{-is}| \, ds .$$

In the last equation we have used periodicity of the function which is integrated. Note that

$$|1 - ae^{-is}| = |e^{-is}(e^{is} - a)| = |e^{is} - a| .$$

This proves the lemma. ◦

**Proof of Theorem 13.1:** a) Assume that $f$ has no zero $a_j$ with $|a_j| < R$. Then we can write

$$f(z) = e^{\phi(z)} , \quad \phi \in H(D) .$$

The function

$$u(z) = \ln |f(z)| = \text{Re} \phi(z)$$

is harmonic in $D$. Formula (13.2) yields

$$\ln |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(Re^{it})| \, dt .$$

This shows formula (13.1) if $f$ has no zero.

b) Fix $w \in \mathbb{C}$ with $0 < |w| < R$ and consider the function $f(z) = z - w$. Then formula (13.1) claims that

$$\ln |w| = \ln \left( \frac{|w|}{R} \right) + \frac{1}{2\pi} \int_0^{2\pi} \ln |Re^{it} - w| \, dt . \quad (13.4)$$

Set $a = w/R$. Then we have

$$\ln |Re^{it} - w| = \ln R + \ln |e^{it} - a| .$$

Lemma 13.1 yields that
\[
\int_0^{2\pi} \ln |e^{it} - a| \, dt = 0 .
\]

Formula (13.4) follows. This shows that (13.1) holds for a function of the form \( f(z) = z - w \).

c) Assume two functions \( f_1(z) \) and \( f_2(z) \) satisfy the assumptions of the theorem and assume formula (13.1) holds for \( f_1 \) and \( f_2 \). Then the product \( f(z) = f_1(z)f_2(z) \) also satisfies the assumptions of the theorem and formula (13.1) holds for \( f \).

d) Let \( f \in H(D) \) denote any function satisfying the assumptions of the theorem. We can write

\[
f(z) = g(z)(z - a_1)(z - a_2)\ldots(z - a_N)
\]

where \( g \in H(D) \) has no zeros in \( D \). Using a), b), and c), the formula (13.1) follows for \( f \).

13.2 The Order of Growth

Definition: Let \( f \) denote an entire function. One says that \( f \) has finite growth order if there exist positive constants \( A, B, \rho \) so that

\[
|f(z)| \leq Ae^{B|z|^\rho} \quad \text{for all} \quad z \in \mathbb{C} . 
\]

(13.5)

If \( f \) is an entire function of finite growth order one defines its order of growth, denote by \( \rho_0 = \rho_0(f) \), as the infimum of all positive \( \rho \) for which the estimate (13.5) holds for some constants \( A = A_\rho \) and \( B = B_\rho \).

Examples:
1. Every polynomial has order of growth \( \rho_0 = 0 \).
2. The function \( f(z) = e^z \) has order of growth \( \rho_0 = 1 \).
3. The function \( f(z) = \cos z \) has order of growth \( \rho_0 = 1 \).
4. Recall that

\[
\cos w = \sum_{j=0}^{\infty} (-1)^j \frac{w^{2j}}{(2j)!}
\]

and define

\[
f(z) = \cos \sqrt{z} = \sum_{j=0}^{\infty} (-1)^j \frac{z^j}{(2j)!} .
\]

It is then clear that

\[
f(z^2) = \cos z .
\]

The order of growth of \( f \) is \( \rho_0 = \frac{1}{2} \).
5. One can show that

\[
\frac{1}{|\Gamma(z)|} \leq Ae^{B|z|\ln |z|} \quad \text{for} \quad |z| \geq 1 .
\]
The function $1/\Gamma(z)$ has order of growth $\rho_0 = 1$. [Stein, Shakarchi], p. 165
(The estimate should also follow from the infinite product representation.)

6. It can be shown that the function $(s - 1)\zeta(s)$ has order of growth $\rho_0 = 1$. [Stein, Shakarchi], p. 202

13.3 Zeros and Growth Estimates

Let $f$ denote an entire functions which has a sequence of zeros $a_j$. We order the zeros so that

$$|a_1| \leq |a_2| \leq \ldots$$

Each zero is listed according to its multiplicity. For $0 \leq r < \infty$ let $n(r)$ denote the number of zeros $a_j$ with $|a_j| \leq r$. Clearly, $n(r)$ is a piecewise constant function which increases. We will show that a growth estimate for $|f(z)|$ implies a growth estimate for $n(r)$ as $r \to \infty$.

**Theorem 13.2** Assume that the entire function $f(z)$ satisfies a growth estimate

$$|f(z)| \leq Ae^{B|z|^\rho} \quad \text{for all} \quad z \in \mathbb{C} \quad (13.6)$$

with positive constants $A, B, \rho$. Then there are positive constants $C$ and $r_0$ so that

$$n(r) \leq Cr^\rho \quad \text{for} \quad r \geq r_0 \quad (13.7)$$

**Proof:** a) Using the Heaviside function

$$H(x) = \begin{cases} 
1, & x \geq 0 \\
0, & x < 0
\end{cases}$$

we have

$$H(r - |a_j|) = \begin{cases} 
1, & |a_j| \leq r \\
0, & |a_j| > r
\end{cases}$$

and can write

$$n(r) = \sum_{j=1}^{\infty} H(r - |a_j|).$$

For each fixed $r$ the sum is finite.

b) Assume $f(0) \neq 0$. Then we can write
\[ \int_0^R n(r) \frac{dr}{r} = \sum_{j=1}^N \int_0^R H(r-|a_j|) \frac{dr}{r} \]
\[ = \sum_{j=1}^N \int_{|a_j|}^R \frac{dr}{r} \]
\[ = \sum_{j=1}^N \ln \left( \frac{R}{|a_j|} \right). \]

Here the zeros \( a_1, \ldots, a_N \) are the zeros of \( f \) with \( |a_j| \leq R \). Using Theorem 13.1 we obtain the following result:

**Lemma 13.2** As above, let \( f \) denote an entire function and assume

\[ f(0) \neq 0 \quad \text{and} \quad f(z) \neq 0 \quad \text{for} \quad |z| = R. \tag{13.8} \]

Then we have

\[ \int_0^R n(r) \frac{dr}{r} = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(Re^{i\theta})| \, dt - \ln |f(0)|. \]

\( c) \) We continue the proof of Theorem 13.2 and let \( f \) denote an entire function with \( f(0) \neq 0 \) and \( f(z) \neq 0 \) for \( |z| = R \). We let \( R = 2r \) and have

\[ \ln |f(Re^{i\theta})| \leq \ln A + BR^\rho \leq C_1 r^\rho \quad \text{for} \quad r \geq r_0. \]

Now the estimate of the previous lemma yields

\[ \int_0^R n(s) \frac{ds}{s} \leq C_2 r^\rho \quad \text{for} \quad r \geq r_0. \]

Furthermore,

\[ n(r) \ln 2 = n(r) \int_r^{2r} \frac{ds}{s} \leq \int_r^{2r} n(s) \frac{ds}{s} \leq C_2 r^\rho \]

for \( r \geq r_0 \).

This proves (13.7) as long as \( f \) satisfies (13.8) for \( R = 2r \). If \( f \) has a zero \( a_j \) with \( |a_j| = R = 2r \), we simply choose \( 0 < \varepsilon \leq r_0 \) so that \( f \) has no zero on the circle with radius \( 2r + 2\varepsilon \). We then have
\[ n(r) \leq n(r + \varepsilon) \leq C_2(r + \varepsilon)^\rho \leq C_3r^\rho \quad \text{for} \quad r \geq r_0. \]

d) Assume that \( f \) has a zero of order \( m \) at \( z = 0 \). We set \( F(z) = f(z)/z^m \). The entire function \( F(z) \) satisfies the same growth estimate as \( f \) does. Since \( n_f(r) = n_F(r) + m \) the claim \( n_f(r) \leq C r^\rho \) follows.

**Theorem 13.3** Let \( f \in H(\mathbb{C}) \) satisfy the growth estimate of the previous theorem and let \( a_j \) denote the non–zero zeros of \( f \). If \( s > \rho \) then we have

\[
\sum_j |a_j|^{-s} < \infty. 
\]

**Proof:** For any positive integer \( l \) the number of zeros \( a_j \) of \( f \) with

\[ 2^l < |a_j| \leq 2^{l+1} \]

is less than or equal to \( n(2^{l+1}) \). Therefore, for any large \( l \),

\[
\sum_{2^l < |a_j| \leq 2^{l+1}} |a_j|^{-s} \leq 2^{-ls} n(2^{l+1}) 
\leq C 2^{-ls} 2^{(l+1)\rho} 
\leq C_1 2^{-(s-\rho)}
\]

Since \( 2^{-(s-\rho)} < 1 \) the geometric series

\[
\sum_l 2^{-l(s-\rho)}, \quad s > \rho,
\]

converges, and the claim follows.

**13.4 Hadamard’s Factorization Theorem**

Let \( f \) denote an entire function with growth order \( \rho_0 \) and let \( k = [\rho_0] \) denote the integer part of \( \rho_0 \). We choose \( s \) and \( \rho \) with

\[ k = [\rho_0] \leq \rho_0 < \rho < s \leq k + 1. \]

Then \( f \) satisfies a growth estimate

\[ |f(z)| \leq Ae^{B|z|^\rho} \quad \text{for all} \quad z \in \mathbb{C}. \]

We assume that \( f \) has a sequence of zeros \( a_j \) which we order so that

\[ a_1 = \ldots = a_m = 0 < |a_{m+1}| \leq |a_{m+2y}| \leq \ldots \]

As above, \( n(r) \) denotes the number of zeros \( a_j \) with \( |a_j| \leq r \). We know that \( n(r) \leq C r^\rho \) for large \( r \) and

\[
\sum_{j=m+1}^{\infty} |a_j|^{-s} < \infty. \quad (13.9)
\]
Recall the Weierstrass’ canonical factors,

\[ E_k(w) = (1 - w) \exp \left( w + \frac{w^2}{2} + \ldots + \frac{w^k}{k} \right) . \]

Convergence of the series

\[ \sum_{j=m+1}^{\infty} |a_j|^{-k-1} < \infty \] (13.10)

implies that

\[ F(z) = z^m \prod_{j=m+1}^{\infty} E_k \left( \frac{z}{a_j} \right) \] (13.11)

is an entire function which has precisely the same zeros as \( f(z) \). Consequently, one can write the given function \( f(z) \) in the form

\[ f(z) = e^{g(z)} F_z \]

where \( g(z) \) is an entire function.

Hadamard’s Factorization Theorem is the following remarkable result. It gives more information about the function \( g(z) \).

**Theorem 13.4** Let \( f(z) \) denote an entire function with growth order \( \rho_0 \) and let \( k = \lfloor \rho_0 \rfloor \) denote the integer part of \( f \). Assume that \( f(z) \) has a sequence of zeros \( a_j \) with

\[ a_1 = \ldots = a_m = 0 < |a_{m+1}| \leq |a_{m+2}\nu| \leq \ldots \]

Then \( f(z) \) has the form

\[ f(z) = e^{g(z)} z^m \prod_{j=m+1}^{\infty} E_k \left( \frac{z}{a_j} \right) \] (13.12)

where \( g(z) \) is a polynomial of degree \( \leq k \). The function \( g(z) \) is unique, except that one can add a constant \( 2\pi i m, m \in \mathbb{Z} \).

An idea of the proof is to turn the growth estimate for \( f(z) \) into an estimate for \( \text{Re} \, g(z) \). This requires to show lower bounds for the product \( |F(z)| \) and is rather technical. Once these lower bounds are derived, the following lemma will complete the proof of Hadamard’s Factorization Theorem.

**Lemma 13.3** Let \( g(z) \) denote an entire function. Let \( k \in \{0, 1, \ldots\} \) and let \( k \leq s < k + 1 \). Assume that there is a constant \( C \) and a sequence of positive numbers \( r_\nu \) with \( r_\nu \to \infty \) so that \( \text{Re} \, g(z) \) satisfies the following upper bound:

\[ \text{Re} \, g(z) \leq Cr_\nu^s \quad \text{if} \quad |z| = r_\nu, \quad \nu = 1, 2, \ldots \] (13.13)

Under these assumptions, the function \( g(z) \) is a polynomial of degree \( \leq k \).
**Proof:** First fix any $r > 0$ and any integer $n \geq 0$. With $\Gamma_r$ we denote the circle with parameterization $z = re^{it}, 0 \leq t \leq 2\pi$. Let

$$g(z) = \sum_{n=0}^{\infty} g_n z^n.$$  

We obtain

$$\frac{g(z)}{z^{n+1}} = \ldots + \frac{g_n}{z} + \ldots$$

thus

$$2\pi i g_n = \int_{\Gamma_r} \frac{g(z)}{z^{n+1}} \, dz = i \int_{0}^{2\pi} g(re^{it}) r^{-n} e^{-int} \, dt.$$  

This shows that

$$2\pi g_n r^n = \int_{0}^{2\pi} g(re^{it}) e^{-int} \, dt \quad \text{for} \quad n \geq 0. \quad (13.14)$$

Also, by Cauchy’s theorem,

$$\int_{0}^{2\pi} g(re^{it}) e^{int} \, dt = 0 \quad \text{for} \quad n \geq 1,$$

thus

$$\int_{0}^{2\pi} \bar{g}(re^{it}) e^{-int} \, dt = 0 \quad \text{for} \quad n \geq 1.$$  

Therefore, if we set $u(z) = \text{Re} \, g(z) = \frac{1}{2} (g(z) + \bar{g}(z))$, then we find that

$$\pi g_n r^n = \int_{0}^{2\pi} u(re^{it}) e^{-int} \, dt \quad \text{for} \quad n \geq 1.$$  

Furthermore, setting $n = 0$ in (13.14) and taking real-parts yields

$$2\pi \text{Re} \, g_0 = \int_{0}^{2\pi} u(re^{it}) \, dt.$$  

We then have, for $n \geq 1,$

$$\pi g_n = \frac{1}{r^n} \int_{0}^{2\pi} u(re^{it}) e^{-int} \, dt = \frac{1}{r^n} \int_{0}^{2\pi} \left( u(re^{it}) - C r^s \right) e^{-int} \, dt.$$  

Here the term in (\ldots) is $\leq 0i$ for $r = r_\nu$, by assumption. Taking absolute values we obtain
\[ \pi |g_n| \leq \frac{1}{r^n} \int_0^{2\pi} \left( C r^s - u(r e^{it}) \right) dt \]
\[ = 2\pi C r^{s-n} - 2\pi (\text{Re} g_0) r^{-n} \]

Now recall the assumption \( s < k+1 \) and assume \( n \geq k+1 \). Letting \( r = r_0 \to \infty \) in the above estimate yields that \( g_n = 0 \) for \( n > k \). This proves the lemma. \( \diamond \)

13.5 Entire Functions of Non–Integer Order of Growth

The previous lemma has the following important implication.

**Theorem 13.5** Let \( f(z) \) denote an entire function with order of growth \( \rho_0 \). If \( \rho_0 \) is not an integer, then \( f \) has infinitely many zeros.

**Proof:** Suppose \( f \) has only finitely many zeros \( a_1, \ldots, a_N \). We may assume that \( N \geq 1 \) and let
\[ p(z) = (z - a_1) \ldots (z - a_N) \]
and write
\[ f(z) = e^{g(z)} p(z) \]
where \( g(z) \) is an entire function. (If \( f(z) \) has no zeros, then take \( p(z) \equiv 1 \).) If \( k < \rho_0 < s < k + 1 \) with integer \( k \) then \( f(z) \) satisfies an estimate
\[ |f(z)| \leq A e^{B|z|^s} \]
Since \( |p(z)| \geq 1 \) for all large \(|z|\) we obtain
\[ e^{\text{Re} g(z)} = |e^{g(z)}| \leq A e^{B|z|^s} \]
if \(|z| = r\) is large. This yields
\[ \text{Re} g(z) \leq C r^s \quad \text{for} \quad |z| = r \]
if \( r \) is large. The previous lemma then yields that \( g(z) \) is a polynomial and the representation \( f(z) = e^{g(z)} p(z) \) implies the order of growth of \( f(z) \) to be an integer. This contradiction proves the theorem. \( \diamond \)

13.6 Proof of Hadamard’s Factorization Theorem

The main technical lemma is the following lower bound for the product
\[ E(z) = \prod_{j=1}^{\infty} E_k \left( \frac{z}{a_j} \right) . \]  
(13.15)
Lemma 13.4 Let $0 \leq k \leq \rho < s < k + 1$ where $k$ is an integer. Let $a_j$ denote a sequence of complex numbers with

$$0 < |a_1| \leq |a_2| \ldots$$

and assume $n(r) \leq Cr^\rho$ for $r \geq r_0$, where $n(r)$ is the number of $a_j$ with $|a_j| \leq r$. Consider the entire function $E(z)$ defined in (13.15). There is a constant $c > 0$, independent of $z$, and there is a sequence $r_\nu$ of positive radii with $r_\nu \to \infty$ so that

$$|E(z)| \geq e^{-c|z|^s} \quad \text{if} \quad |z| = r_\nu, \quad \nu = 1, 2, \ldots$$

For later reference, we recall that the assumption $n(r) \leq Cr^\rho$ implies that

$$\sum_{j=1}^{\infty} |a_j|^{-s} < \infty \quad \text{for} \quad s > \rho .$$

Before we prove the main lemma, Lemma 13.4, we show two simple auxiliary results for the Weierstrass function $E_k(z)$.

Lemma 13.5 For small $|z|$ we have

$$|E_k(z)| \geq e^{-|z|^{k+1}} \quad \text{if} \quad |z| \leq \frac{1}{2} .$$

Proof: We have $E_k(z) = e^w$ with

$$w = \log(1 - z) + h_k(z)$$

$$= -\left( \frac{z^{k+1}}{k+1} + \frac{z^{k+2}}{k+2} + \ldots \right)$$

This yields the bound $|w| \leq 2|z|^{k+1}$ if $|z| \leq \frac{1}{2}$. Therefore,

$$|E_k(z)| = |e^w| \geq e^{-|w|} \geq e^{-2|z|^{k+1}} .$$

If $|z|$ is bounded away from zero, we have the following lower bound for $|E_k(z)|$:

Lemma 13.6 There is a constant $c = c_k > 0$ with

$$|E_k(z)| \geq |1 - z| e^{-c|z|^k} \quad \text{if} \quad |z| \geq \frac{1}{2} .$$

Proof: The definition of $E_k(z)$ yields

$$|E_k(z)| = |1 - z| |e^{h_k(z)}|$$

$$\geq |1 - z| e^{-|h_k(z)|}$$
Here

\[ |h_k(z)| \leq \left| z + \frac{z^2}{2} + \ldots + \frac{z^k}{k} \right| \leq c_k |z|^k \text{ if } |z| \geq \frac{1}{2}. \]

\[ \bigcirc \]

The estimate of the next lemma is central for the proof of the main lemma, Lemma 13.4.

**Lemma 13.7** Under the assumptions of Lemma 13.4, define the following union of open discs:

\[ \Omega = \bigcup_{j=1}^{\infty} D(a_j, |a_j|^{-k-1}) . \]

There are positive constants c and R so that

\[ |E(z)| \geq e^{-c|z|^s} \text{ if } |z| \geq R \text{ and } z \in \mathbb{C} \setminus \Omega. \]

**Proof:**

a) Write \( E(z) = P_1 P_2 \) with

\[ P_1 = \prod_{|a_j| \leq 2|z|} E_k(z/a_j), \quad P_2 = \prod_{|a_j| > 2|z|} E_k(z/a_j). \]

We first estimate \( P_2 \) from below: If \( |a_j| > 2|z| \) then, by Lemma 13.5,

\[ |E_k(z/a_j)| \geq e^{-2|z/a_j|^{k+1}}. \]

Therefore,

\[ P_2 \geq e^{-2|z|^{k+1}Q} \text{ with } Q = \sum_{|a_j| > 2|z|} |a_j|^{-k-1}. \]

Since \( |z|^{-1} > 2|a_j|^{-1} \) we have

\[ |a_j|^{-k-1} = |a_j|^{-s} |a_j|^{s-k-1} \leq C|a_j|^{-s} |z|^{s-k-1}. \]

This yields the upper bound

\[ Q \leq C|z|^{s-k-1} \sum_j |a_j|^{-s} \leq C_1 |z|^{s-k-1}. \]

The lower bound

\[ P_2 \geq e^{-c|z|^s} \]

follows.

b) Next consider the finite product

\[ P_1 = \prod_{|a_j| \leq 2|z|} E_k(z/a_j). \]

Since \( |z/a_j| \geq \frac{1}{2} \) we can apply Lemma 13.6 and obtain

143
\[ |E_k(z/a_j)| \geq \left| 1 - \frac{z}{a_j} \right| e^{-c|z/a_k|^k}. \]

Therefore,
\[ |P_1| \geq P_3 P_4 \]

with
\[ P_3 = \Pi_{|a_j| \leq 2|z|} \left| 1 - \frac{z}{a_j} \right| \]

and
\[ P_4 = \Pi_{|a_j| \leq 2|z|} e^{-c|z/a_j|^k}. \]

c) To estimate \( P_4 \) from below we write
\[ P_4 = e^{-c|z|^k} Q_1 \]

with
\[ Q_1 = \sum_{|a_j| \leq 2|z|} |a_j|^{-k}. \]

Since
\[ |a_j|^{-k} = |a_j|^{-s} |a_j|^{s-k} \leq C |a_j|^{-s} |z|^{s-k} \]

we have
\[ Q_1 \leq C |z|^{s-k} \sum_j |a_j|^{-s} = C_1 |z|^{s-k}. \]

This shows that
\[ P_4 \geq e^{-c_1 |z|^s}, \quad c_1 = cC_1. \]

d) It remains to estimate
\[ P_3 = \Pi_{|a_j| \leq 2|z|} \left| 1 - \frac{z}{a_j} \right| \]

from below. Here it is important that \( z \) does not lie in any of the discs \( D(a_j, |a_j|^{-k-1}) \), i.e.,
\[ |a_j - z| \geq |a_j|^{-k-1}. \]

One obtains that
\[ \left| 1 - \frac{z}{a_j} \right| = \frac{|a_j - z|}{|a_j|} \geq |a_j|^{-k-2} \]

and
\[ P_3 \geq \Pi_{|a_j| \leq 2|z|} |a_j|^{-k-2}. \]

Therefore,

\[ \ln P_3 \geq -(k + 2) \sum_{|a_j| \leq 2|z|} \ln |a_j|. \]

Finally,

\[
\sum_{|a_j| \leq 2|z|} \ln |a_j| \leq n(2|z|) \ln(2|z|)
\leq C|z|^\rho \ln(2|z|)
\leq C_1 |z|^s
\]

for all sufficiently large \(|z|\). The lower bound

\[ \ln P_3 \geq -c|z|^s, \quad c > 0, \]

follows and the inequality

\[ P_3 \geq e^{-c|z|^s} \]

is shown.

e) To summarize, we have for \(|z| \geq R, z \in \mathbb{C} \setminus \Omega:\n
\[
E(z) = P_1 P_2
P_2 \geq e^{-d|z|^s}
|P_1| \geq P_3 P_4
P_3 \geq e^{-d|z|^s}
P_4 \geq e^{-d|z|^s}
\]

where \(c > 0\) is a constant. This implies that

\[ |E(z)| \geq e^{-3c|z|^s}. \]

The lemma is proved. ♦

Recall that

\[ 0 < |a_1| \leq |a_2| \leq \ldots \]

and \(k \leq \rho < k + 1\). Consequently,

\[ \sum_{j=1}^{\infty} |a_j|^{-k-1} < \infty. \]

Also, recall the definition

\[ \Omega = \bigcup_{j=1}^{\infty} D(a_j, |a_j|^{-k-1}). \]
**Lemma 13.8** Under the above assumptions, there is a sequence of positive radii \( r_1, r_2, \ldots \) with \( r_\nu \to \infty \) so that none of the circles 

\[ |z| = r_\nu, \quad \nu = 1, 2, \ldots \]

intersects the union of discs \( \Omega \).

**Proof:** Choose \( J \) so large that

\[ \sum_{j=J}^{\infty} |a_j|^{-k-1} \leq \frac{1}{10}. \]

The set 

\[ \Omega_J = \bigcup_{j=1}^{J-1} D(a_j, |a_j|^{-k-1}) \]

is bounded and we choose an integer \( n \) so large that \( \Omega_J \subset D(0, n) \).

We claim that the interval 

\[ n \leq r \leq n + 1 \]

contains a number \( r \) so that the circle \( |z| = r \) does not intersect the set \( \Omega \). In fact, for any \( r \geq n \), the circle \( |z| = r \) does not intersect \( \Omega_J \) since \( \Omega_J \subset D(0, r) \) and it remains to show that we can choose \( r \) in the interval \( n \leq r \leq n + 1 \) so that the circle \( |z| = r \) does not intersect the union

\[ \bigcup_{j=J}^{\infty} D(a_j, |a_j|^{-k-1}) . \]

To see this, consider the closed interval

\[ I_j = \left[ |a_j| - |a_j|^{-k-1}, |a_j| + |a_j|^{-k-1} \right], \quad j \geq J, \]

of length

\[ \text{length}(I_j) = 2|a_j|^{-k-1} . \]

We have

\[ \sum_{j \geq J} \text{length}(I_j) \leq \frac{1}{5} . \]

Therefore, there exists a number \( r \) in the interval \( n \leq r \leq n + 1 \) which does not lie in the union \( \bigcup_{j \geq J} I_j \).

It is then clear that the circle \( |z| = r \) does not intersect \( \Omega \).

This proves Lemma 13.8. The main technical lemma, Lemma 13.4, follows as a consequence of Lemma 13.7 and Lemma 13.8.

We summarize:
Theorem 13.6 (Hadamard) Let $f(z)$ denote an entire function of finite order of growth $\rho_0$ and let $k = \lfloor \rho_0 \rfloor$ denote the integer part of $\rho_0$.

**Case 1:** $f(z)$ has only finitely many zeros. Then $f(z)$ has the form

$$f(z) = e^{g(z)} p(z)$$

where $p(z)$ and $g(z)$ are polynomials. The polynomial $g(z)$ has degree $k$. The order of growth of $f(z)$ equals $k = \rho_0$.

**Case 2:** $f(z)$ has infinitely many zeros $a_j$ with

$$a_1 = \ldots = a_m = 0 < |a_{m+1}| \leq |a_{m+2}| \leq \ldots$$

Then $f(z)$ has the form

$$f(z) = e^{g(z)} z^m \prod_{j=m+1}^{\infty} E_k(z/a_j)$$

where $g(z)$ is a polynomial of degree $\leq k$.

In particular, Case 1 cannot occur unless the order of growth of $f(z)$ is an integer. If the order of growth of $f(z)$ is not an integer, then $f(z)$ has infinitely many zeros.
14 The Prime Number Theorem

The Prime Number Theorem states that

\[
\lim_{x \to \infty} \frac{\pi(x) \ln(x)}{x} = 1.
\]

If one defines a remainder \( R(x) \) by

\[
\pi(x) = \frac{x}{\ln x} \left(1 + R(x)\right)
\]

then \( R(x) \to 0 \) as \( x \to \infty \). Sharp estimates of the remainder are related to the zeros of the Riemann \( \zeta \)-function in the critical strip \( 0 < \Re s < 1 \).

14.1 Functions

The following functions will be used:

If \( n \leq x < n \) where \( n \) is an integer, then

\[ x = n + \{x\} = [x] + \{x\} \]

where \([x]\) is the integer part and \(\{x\}\) is the fractional part of the real number \(x\).

Mangoldt function:

\[
\Lambda(n) = \ln p \text{ if } n = p^m, \quad \Lambda(n) = 0 \text{ otherwise}
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \Lambda(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>\ln 2</td>
</tr>
<tr>
<td>3</td>
<td>\ln 3</td>
</tr>
<tr>
<td>4</td>
<td>\ln 2</td>
</tr>
<tr>
<td>5</td>
<td>\ln 5</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>\ln 7</td>
</tr>
<tr>
<td>8</td>
<td>\ln 2</td>
</tr>
<tr>
<td>9</td>
<td>\ln 3</td>
</tr>
</tbody>
</table>

\[
\pi(x) = \text{number of primes } \leq x
\]

\[
\psi(y) = \sum_{1 \leq n \leq y} \Lambda(n)
\]

\[
= \sum_{p^m \leq y} \ln p \quad (\text{Chebyshev’s } \psi \text{ – function})
\]

\[
= \sum_{p \leq y} \left[\frac{\ln y}{\ln p}\right] \ln p
\]

\[
\psi(y) = \sum_{1 \leq n \leq y} \Lambda(n)H(y - n) \quad (\text{with Heaviside function})
\]

\[
= \sum_{n \geq 1} \Lambda(n)H(y - n)
\]
Here we use the Heaviside function

\[ H(\xi) = \begin{cases} 
1 & \text{for } \xi \geq 0 \\
0 & \text{for } \xi < 0 
\end{cases} \]

An auxiliary integral:

**Lemma 14.1** Let \( \Gamma_c \) denote the straight line with parametrization

\[ \Gamma_c : s(t) = c + it, \quad -\infty < t < \infty . \]

For \( c > 0 \) we have

\[ \frac{1}{2\pi i} \int_{\Gamma_c} \frac{a^s}{s(s+1)} \, ds = \begin{cases} 
0 & \text{for } 0 < a \leq 1 \\
1 - \frac{1}{a} & \text{for } a > 1 
\end{cases} \]

We will show below that

\[ -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)n^{-s} \quad \text{for } \Re s > 1 . \]

Therefore,

\[ \frac{1}{2\pi i} \int_{\Gamma_c} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) = \frac{x}{2\pi i} \sum_{n=1}^{\infty} \Lambda(n) \frac{(x/n)^s}{s(s+1)} . \] (14.1)

Now let \( c > 1 \) and integrate the equation along \( \Gamma_c \). We will justify below that we may interchange summation and integration. We have, by the previous lemma:

\[ \frac{1}{2\pi i} \int_{\Gamma_c} \frac{(x/n)^s}{s(s+1)} = \begin{cases} 
0 & \text{for } 1 \leq x < n \\
1 - \frac{n}{x} & \text{for } 1 \leq n \leq x 
\end{cases} \]

Therefore, equation (14.1) yields that

\[ \frac{1}{2\pi i} \int_{\Gamma_c} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) = x \sum_{1 \leq n \leq x} \Lambda(n) \left(1 - \frac{n}{x}\right) . \]

The integrated Chebyshev function and its integral representation:

\[ \psi_1(x) = \int_{1}^{x} \psi(y) \, dy 
\]

\[ = \sum_{n \geq 1} \Lambda(n) \int_{1}^{x} H(y - n) \, dy 
\]

\[ = \sum_{1 \leq n \leq x} \Lambda(n)(x - n) 
\]

\[ = x \sum_{1 \leq n \leq x} \Lambda(n) \left(1 - \frac{n}{x}\right) 
\]

\[ = \frac{x}{2\pi i} \int_{\Gamma_c} \frac{x^s}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \, ds \]
Riemann’s $\zeta$–function:

$$
\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \in P} \left(1 - p^{-s}\right)^{-1}, \quad \text{Re } s > 1
$$

$$
-\frac{d}{ds} \log \zeta(s) = \sum_{n=1}^{\infty} \Lambda(n)n^{-s}
$$

$$
\zeta(s) = \frac{1}{s - 1} + 1 - s \int_{1}^{\infty} \frac{\{x\}}{x^{s-1}} dx, \quad \text{Re } s > 0
$$

$$
\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)
$$

14.2 Reduction to Asymptotics of $\psi_1(x)$

Write

$$
\pi(x) = \frac{x}{\ln x} (1 + R(x))
$$

$$
\psi(x) = x(1 + r(x))
$$

$$
\psi_1(x) = \frac{x^2}{2} (1 + r_1(x))
$$

**Theorem 14.1**

a) If $r(x) \to 0$ as $x \to \infty$ then $R(x) \to 0$ as $x \to \infty$.

b) If $r_1(x) \to 0$ as $x \to \infty$ then $r(x) \to 0$ as $x \to \infty$.

**Remark:** One should quantify precisely how estimates for $r_1(x)$ yield estimates for $R(x)$.

**Proof:** Part a) lower bound for

$$
\frac{\pi(x) \ln x}{x}
$$

For $x \geq 1$ we have

$$
\psi(x) = \sum_{p \leq x} \left[ \frac{\ln x}{\ln p} \right] \ln p
$$

$$
\leq \sum_{p \leq x} \frac{\ln x}{\ln p} \ln p
$$

$$
= \sum_{p \leq x} \ln x
$$

$$
= \pi(x) \ln x
$$

Therefore,

$$
\frac{\psi(x)}{x} \leq \frac{\pi(x) \ln x}{x} \quad \text{for all } x \geq 1.
$$
Therefore, if 

\[
\frac{\psi(x)}{x} \to 1 \quad \text{as} \quad x \to \infty
\]

then

\[
\lim \inf_{x \to \infty} \frac{\pi(x) \ln x}{x} \geq 1.
\]

Part a2) upper bound for

\[
\frac{\pi(x) \ln x}{x}
\]

Fix 0 < \alpha < 1. Note that if \( x \geq 1 \) and \( x^\alpha < p \) then \( \ln x^\alpha < \ln p \). We have

\[
\psi(x) = \sum_{p \leq x} \left\lfloor \frac{\ln x}{\ln p} \right\rfloor \ln p \\
\geq \sum_{p \leq x} \ln p \\
\geq \sum_{x^\alpha < p \leq x} \ln p \\
\geq \sum_{x^\alpha < p \leq x} \ln(x^\alpha) \\
= \left( \pi(x) - \pi(x^\alpha) \right) \alpha \ln x
\]

Therefore,

\[
\psi(x) + \alpha \pi(x^\alpha) \ln x \geq \alpha \pi(x) \ln x
\]

and

\[
\frac{\psi(x)}{x} + \alpha \frac{\pi(x^\alpha) \ln x}{x} \geq \alpha \frac{\pi(x) \ln x}{x}.
\]

It is clear that

\[
\pi(x^\alpha) \leq x^\alpha.
\]

Therefore, if

\[
\frac{\psi(x)}{x} \to 1 \quad \text{as} \quad x \to \infty
\]

then

\[
1 \geq \alpha \lim \sup_{x \to \infty} \frac{\pi(x) \ln x}{x}.
\]

Since 0 < \alpha < 1 was arbitrary, we obtain that
\[ 1 \geq \limsup_{x \to \infty} \frac{\pi(x) \ln x}{x} . \]

The upper and lower bounds imply that
\[ \lim_{x \to \infty} \frac{\pi(x) \ln x}{x} = 1 . \]

Part b) We claim that if
\[ \frac{\psi_1(x)}{x^{2/2}} \to 1 \quad \text{as} \quad x \to \infty \]
then
\[ \frac{\psi(x)}{x} \to 1 \quad \text{as} \quad x \to \infty . \]

The proof essentially only uses that the function \( \psi(x) \) increases monotonically.

Let \( \beta > 1 \). We have
\[
\psi(x) \leq \frac{1}{(\beta - 1)x} \int_x^{\beta x} \psi(y) \, dy \\
= \frac{1}{(\beta - 1)x} \left( \psi_1(\beta x) - \psi_1(x) \right)
\]
Therefore,
\[ \frac{\psi(x)}{x} \leq \frac{1}{\beta - 1} \left( \frac{\psi_1(\beta x)}{(\beta x)^2} \beta^2 - \frac{\psi_1(x)}{x^2} \right) . \]

As \( x \to \infty \) we obtain that
\[ \limsup_{x \to \infty} \frac{\psi(x)}{x} \leq \frac{1}{\beta - 1} \left( \frac{\beta^2}{2} - \frac{1}{2} \right) = \frac{\beta + 1}{2} . \]

Since \( \beta > 1 \) was arbitrary we obtain that
\[ \limsup_{x \to \infty} \frac{\psi(x)}{x} \leq 1 . \]

Similarly, fix \( 0 < \alpha < 1 \) and obtain that
\[
\psi(x) \geq \frac{1}{(1 - \alpha)x} \int_{\alpha x}^x \psi(y) \, dy \\
= \frac{1}{(1 - \alpha)x} \left( \psi_1(x) - \psi_1(\alpha x) \right)
\]
Therefore,
\[ \frac{\psi(x)}{x} \geq \frac{1}{1 - \alpha} \left( \frac{\psi_1(x)}{x^2} - \frac{\alpha^2 \psi_1(\alpha x)}{(\alpha x)^2} \right) . \]
As \( x \to \infty \) we obtain that

\[
\liminf_{x \to \infty} \frac{\psi(x)}{x} \geq \frac{1}{1 - \alpha} \frac{1 - \alpha^2}{2} = \frac{1 + \alpha}{2}.
\]

Since \( \alpha < 1 \) was arbitrary we obtain that

\[
\liminf_{x \to \infty} \frac{\psi(x)}{x} \geq 1.
\]

\( \diamond \)

### 14.3 Integral Representation of \( \psi_1(x) \)

For \( c \in \mathbb{R} \) let \( \Gamma^c \) denote the vertical line

\[
\Gamma^c : \quad s(t) = c + it, \quad -\infty < t < \infty.
\]

**Theorem 14.2** For \( c > 1 \) and \( x \geq 1 \) the function \( \psi_1(x) \) has the integral representation

\[
\psi_1(x) = \frac{1}{2\pi i} \int_{\Gamma^c} \frac{x^{s+1}}{s(s+1)} \left( -\frac{\zeta'(s)}{\zeta(s)} \right) ds.
\]

The proof is based on auxiliary results given below.

**Remark:** Set

\[
F(x, s) = \frac{x^{s+1}}{s(s+1)} \left( -\frac{\zeta'(s)}{\zeta(s)} \right),
\]

i.e., \( F(x, s) \) is the integrand in the above representation of \( \psi_1(x) \). Since

\[
\text{Res}(\zeta, 1) = 1
\]

it follows that

\[
\text{Res}(F(x, s), s = 1) = \frac{x^2}{2}.
\]

The claim is that this term gives the asymptotic behaviour of \( \psi_1(x) \) to leading order. If \( C_\varepsilon \) denotes the circle of radius \( \varepsilon \) centered at \( s = 1 \), then we have

\[
\frac{1}{2\pi i} \int_{C_\varepsilon} F(x, s) ds = \frac{x^2}{2}.
\]

### 14.4 Auxiliary Results

**Lemma 14.2** For \( c > 0 \) and \( a > 0 \) we have

\[
\frac{1}{2\pi i} \int_{\Gamma^c} \frac{a^s}{s(s+1)} ds = \begin{cases} 0 & \text{if } 0 < a \leq 1 \\ 1 - \frac{1}{a} & \text{if } a \geq 1 \end{cases}
\]
Lemma 14.3 For $|\varepsilon| < 1$ we have

$$-\log(1 - \varepsilon) = \sum_{j=1}^{\infty} \frac{1}{j} \varepsilon^j .$$

Proof: Set

$$f(\varepsilon) = -\log(1 - \varepsilon) \quad \text{for} \quad |\varepsilon| < 1 .$$

Then $f(0) = 0$ and

$$f'(\varepsilon) = \frac{1}{1 - \varepsilon} = \sum_{j=0}^{\infty} \varepsilon^j .$$

The claim follows by integration. \boxdot

Lemma 14.4 For $\Re s > 1$ we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)n^{-s} .$$

Proof: From the previous lemma,

$$-\log(1 - \varepsilon) = \sum_{j=1}^{\infty} \frac{1}{j} \varepsilon^j \quad \text{for} \quad |\varepsilon| < 1 .$$

Also, for $\Re s > 1$:

$$\zeta(s) = \Pi_p (1 - p^{-s})^{-1} ,$$

$$\log \zeta(s) = -\sum_p \log(1 - p^{-s})$$

$$= \sum_p \sum_{j=1}^{\infty} \frac{1}{j} p^{-js}$$

$$= \sum_{n=1}^{\infty} c_n n^{-s}$$

where

$$c_n = \begin{cases} \frac{1}{j} & \text{if} \quad n = p^j \\ 0 & \text{if} \quad p \text{ is not a prime power} \end{cases} .$$

Therefore,

$$\log \zeta(s) = \sum_{n=1}^{\infty} c_n e^{-s \ln n}$$
and differentiation yields that

\[
\frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds} \log \zeta(s) = -\sum_{n=1}^{\infty} c_n n^{-s} \ln n .
\]

If \( n = p^j \) then

\[
c_n \ln n = \frac{1}{j} \ln (p^j) = \ln p = \Lambda(n) .
\]

If \( n \) is not a prime power, then

\[
c_n \ln n = 0 = \Lambda(n) .
\]

This proves the formula

\[
-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} .
\]

\[\diamond\]

14.5 The \( \zeta \)-Function has no Zero on the Line \( s = 1 + it \)

Let

\[
s = \sigma + it, \quad \sigma > 1, \quad t \in \mathbb{R} .
\]

We have

\[
\zeta(s) = \prod_p (1 - p^{-s})^{-1}
\]

\[
\log \zeta(s) = -\sum_p \log(1 - p^{-s})
\]

\[
= \sum_p \sum_{j=1}^{\infty} \frac{1}{j} (p^j)^{-s}
\]

\[
= \sum_{n=1}^{\infty} c_n n^{-s}
\]

where

\[
c_n = \begin{cases} \frac{1}{j} & \text{if } n = p^j \\ 0 & \text{if } p \text{ is not a prime power} \end{cases}
\]

We have
\[ n^{-s} = n^{-\sigma} n^{-it} \]
\[ = n^{-\sigma} e^{-it \ln n} \]
\[ = n^{-\sigma} \left( \cos(t \ln n) - i \sin(t \ln n) \right) \]

thus
\[ \Re n^{-s} = n^{-\sigma} \cos(t \ln n) . \]

Therefore,
\[ \Re \log \zeta(s) = \sum_{n=1}^{\infty} c_n n^{-\sigma} \cos(t \ln n) . \]

From
\[ \zeta(s) = e^{\log \zeta(s)} \]
we obtain that
\[ |\zeta(s)| = e^{\Re \log \zeta(s)} \]
and
\[ \ln |\zeta(s)| = \Re \log \zeta(s) = \sum_{n=1}^{\infty} c_n n^{-\sigma} \cos(t \ln n) . \]

Now consider
\[ \ln \left| \zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it) \right| = 3 \ln |\zeta(\sigma)| + 4 \ln |\zeta(\sigma + it)| + \ln |\zeta(\sigma + 2it)| \]
\[ = \sum_{n=1}^{\infty} c_n n^{-\sigma} \left( 3 + 4 \cos(t \ln n) + \cos(2t \ln n) \right) \]

We know that
\[ \cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 2 \cos^2(\alpha) - 1 \]
thus
\[ 3 + 4 \cos(\alpha) + \cos(2\alpha) = 2(1 + 2 \cos(\alpha) + \cos^2(\alpha)) \]
\[ = 2(1 + \cos(\alpha))^2 \]
and obtain that
\[ \ln \left| \zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it) \right| \geq 0 . \]
This yields the lower bound
\[ \left| \zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it) \right| \geq 1. \]  

(14.2)

for all

\[ \sigma > 1 \quad \text{and} \quad t \in \mathbb{R}. \]

Now suppose that

\[ \zeta(1 + it) = 0 \]

for some real \( t \neq 0 \). Then consider the function

\[ f(\sigma) = \left| \zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it) \right| \quad \text{for} \quad 1 < \sigma \leq 2. \]

We obtain for some constant \( C > 0 \) the estimates

\[
\begin{align*}
|\zeta(\sigma + it)| &\leq C(\sigma - 1) \\
|\zeta(\sigma + 2it)| &\leq C' \\
|\zeta(\sigma)| &\leq 2(\sigma - 1)^{-1}
\end{align*}
\]

for \( 1 < \sigma \leq 2 \). Therefore,

\[ f(\sigma) \leq K(\sigma - 1)^{-3}(\sigma - 1)^4 = K(\sigma - 1) \quad \text{for} \quad 1 < \sigma \leq 2. \]

This estimate contradicts the lower bound which was established in (14.2). The contradiction shows that the \( \zeta \)-function has no zero on the line

\[ s = 1 + it, \quad t \in \mathbb{R}. \]
15 Complex Variables and PDEs

15.1 2D Irrotational Euler Flows

The Euler equations for 2D incompressible flows read

\[ u_t + uu_x + vu_y + p_x = 0 \]
\[ v_t + uv_x + vv_y + p_y = 0 \]
\[ u_x + v_y = 0 \]

Here the vector
\[ (u(x,y,t), v(x,y,t)) \in \mathbb{R}^2 \]

is the velocity and \( p(x,y,t) \) is the pressure. The equation \( u_x + v_y = 0 \) is the incompressibility condition. The corresponding 3D velocity field
\[ (u, v, 0) \]

has the vorticity

\[ \omega = \text{curl}(u, v, 0) \]
\[ = \nabla \times (u, v, 0) \]
\[ = (0, 0, v_x - u_y) \]

A velocity field is called irrotational if its vorticity is zero. For 2D incompressible, irrotational flows one obtains the conditions

\[ u_x + v_y = 0, \quad v_x - u_y = 0 \]

which can also be written as

\[ u_x = -v_y, \quad u_y = -(v_x). \]

These are the Cauchy–Riemann equations for the pair
\[ (u, -v). \]

**Lemma 15.1** Let \( U \subset \mathbb{C} \) denote an open set and let \( f \in H(U) \). Define \( u, v : U \to \mathbb{R} \) by

\[ f(x + iy) = u(x, y) - iv(x, y) \]

and set

\[ p(x, y) = -\frac{1}{2} \left( u^2(x, y) + v^2(x, y) \right). \]

Then
is an irrotational solution of the stationary incompressible Euler equations.

**Proof:** We know that \((u, -v)\) satisfy the Cauchy–Riemann equations, thus

\[
    u_x + v_y = 0, \quad v_x - u_y = 0.
\]

Also,

\[
    \left( \frac{1}{2} (u^2 + v^2) \right)_x = uu_x + vv_x = uu_x + vu_y = -p_x
\]

and, similarly,

\[
    \left( \frac{1}{2} (u^2 + v^2) \right)_y = uu_y + vv_y = uw_x + vv_y = -p_y
\]

\[\square\]

**Remark:** For the constructed solution \((u, v, p)\) we have

\[
    \frac{1}{2} (u^2 + v^2) + p = 0.
\]

Roughly, this implies that the pressure is low (negative, large in absolute value) where flow speed is high and, conversely, the pressure is high (negative, small in absolute value) where the flow speed is low. This is a simple form of Bernoulli’s law relating the pressure to the flow speed.

**The Velocity Field has a Potential** Let \(F \in H(U)\) and let \(F' = f\). As above, define \(u\) and \(v\) by

\[
    f(x + iy) = u(x, y) - iv(x, y).
\]

Define \(\phi\) and \(\psi\) by

\[
    F(x + iy) = \phi(x, y) + i\psi(x, y).
\]

We have

\[
    u - iv = f = F' = \phi_x + i\psi_x = \phi_x - i\phi_y
\]
thus

\[ u = \phi_x \quad \text{and} \quad v = \phi_y. \]

This says that the function \( \phi(x, y) \) is a potential of the velocity field

\[ \left( u(x, y), v(x, y) \right) = \nabla \phi(x, y). \]

**Flow Lines** From the Cauchy–Riemann equations

\[ \phi_x = \psi_y, \quad \phi_y = -\psi_x \]

one obtains that

\[ \phi_x \psi_x + \phi_y \psi_y = 0. \]

If \( \Gamma \) is a line described by

\[ \psi(x, y) = \text{const} \]

and if \( P = (x, y) \in \Gamma \), then the vector

\[ \left( \psi_x(P), \psi_y(P) \right) \]

is orthogonal to the tangent to \( \Gamma \) at \( P \). At \( P = (x, y) \) we have

\[ 0 = \phi_x \psi_x + \phi_y \psi_y = u \psi_x + v \psi_y \]

which implies that the velocity vector

\[ \left( u(P), v(P) \right) \]

is tangent to \( \Gamma \) at \( P \). In other words, the lines

\[ \psi(x, y) = \text{const} \]

are flow lines.

**Example 1:** \( F(z) = z = x + iy \)

We have

\[ \phi = x, \quad \psi = y \]

and

\[ u = 1, \quad v = 0. \]

The velocity field is

\[ \left( u(x, y), v(x, y) \right) = (1, 0) \]
is a uniform flow in the direction of the $x$–axis. The pressure is constant. The value of the constant is irrelevant since only $p_x$ and $p_y$ occur in the Euler equations.

**Example 2:** $F(z) = az = (a_1 + ia_2)(x + iy)$

We have

$$f(z) = a = a_1 + ia_2 = u - iv .$$

The velocity vector

$$\left( u(x,y), v(x,y) \right) = (a_1, -a_2)$$

is constant in space. The velocity field describes a uniform flow.

**Example 3:** $F(z) = \frac{z^2}{2}$ (flow around a corner in the first quadrant)

Here

$$f(z) = z = x + iy ,$$

thus

$$u = x , \quad v = -y .$$

We have

$$F(z) = \frac{(x + iy)^2}{2} = \frac{1}{2}(x^2 - y^2) + ixy .$$

Therefore,

$$\psi(x,y) = xy .$$

The flow lines are hyperbolas

$$y = \frac{c}{x} .$$

On the $y$–axis the velocity is

$$(u(0,y), v(0,y)) = -(0,y) ,$$

i.e., the flow is parallel to the $y$–axis.

On the $x$–axis the velocity is

$$(u(x,0), v(x,0)) = (x,0)$$

i.e., the flow is parallel to the $x$–axis.

The first quadrant is

$$U = Q_1 = \{(x,y) : 0 < x, y < \infty \} .$$

On the boundary of $U$ the velocity is tangent to the boundary. This is the usual boundary condition for Euler flow.
Example 4: \( F(z) = z + \frac{1}{z} \) (flow around a cylinder)

We have

\[
F(z) = x + iy + \frac{x - iy}{x^2 + y^2}
\]

thus

\[
\psi(x, y) = y - \frac{y}{x^2 + y^2}.
\]

The stream lines are given by the equation

\[
y - \frac{y}{x^2 + y^2} = c = \text{const}.
\]

If we take \( c = 0 \) we see that the circle

\[
x^2 + y^2 = 1
\]

and the \( y \)-axis are flow lines. We can interpret the flow as a flow around the unit–circle. We have

\[
u = \phi_x = 1 + \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2}
\]

and

\[
v = \phi_y = -\frac{2xy}{(x^2 + y^2)^2}
\]

This yields that

\[
u(x, y) \to 1 \quad \text{and} \quad v(x, y) \to 0
\]

for \( |x| + |y| \to \infty \).

Remarks: The Navier–Stokes equations for 2D incompressible flows read

\[
\begin{align*}
    u_t + uu_x + vu_y + p_x &= \nu \Delta u \\
    v_t + uv_x + vv_y + p_y &= \nu \Delta v \\
    u_x + v_y &= 0
\end{align*}
\]

where \( \nu > 0 \) is the kinematic viscosity of the fluid. If \( u \) and \( v \) are constructed as above, then

\[
\Delta u = \Delta v = 0,
\]

thus

\[
(u, v, p)
\]

is a solution of the stationary Navier–Stokes equations. However, for the Navier–Stokes equations one typically requires the boundary condition.
on every wall of the domain. For the Euler equations one only requires that 
\((u, v)\) is tangential to the wall. The solutions of the Euler equations which we 
have constructed will not satisfy the boundary conditions \(u = v = 0\) at a wall 
unless the solution is trivial.

In regions away from walls the Euler equations may still be useful since 
often one has that \(0 < \nu \ll 1\). The viscosity terms are important, however, in 
boundary layers near walls.

## 15.2 Laplace Equation

We will to derive the Poisson kernel for the Dirichlet problem for Laplace equation in the unit disc.

**Dirichlet Problem:** Let \(D = D(0, 1)\) denote the unit disc with boundary 
curve \(\Gamma = \partial D\). Let \(g : \Gamma \to \mathbb{R}\) denote a continuous function. Find

\[
u : \bar{D} \to \mathbb{R} \quad \text{with} \quad \nu \in C^2(D) \cap C(\bar{D})
\]

so that \(\Delta \nu = 0\) in \(D\), \(\nu = g\) on \(\Gamma\).

### 15.2.1 Derivation of the Poisson Kernel via Complex Variables

Let \(f \in H(D(0, 1 + \varepsilon))\). By Cauchy’s integral formula:

\[
f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{for} \quad z \in D. 
\]

For \(z \in D, z \neq 0\), define the reflected (w.r.t. \(\Gamma\)) point

\[
z_1 = \frac{z}{|z|^2} = \frac{1}{z}.
\]

Since \(|z_1| > 1\) we have that

\[
0 = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z_1} d\zeta \quad \text{for} \quad z \in D. 
\]

Subtracting this equation from the equation for \(f(z)\) we obtain

\[
f(z) = \frac{1}{2\pi} \int_{\Gamma} \left( \frac{\zeta}{\zeta - z} - \frac{\zeta}{\zeta - z_1} \right)f(\zeta) \frac{d\zeta}{i\zeta}, \quad (15.1)
\]

for \(0 < |z| < 1\).

We have for \(0 < |z| < 1\) and \(\zeta \in \Gamma:\)
\[
\frac{\zeta}{\zeta - z} - \frac{\zeta}{\zeta - z_1} = \frac{\zeta}{\zeta - z} - \frac{\zeta}{\zeta - \bar{\zeta}} \\
= \frac{\zeta}{\zeta - z} - \frac{\bar{\zeta}}{\zeta - \bar{\zeta}} \\
= \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2} \\
= \frac{1 - |z|^2}{|\zeta - z|^2}
\]

The kernel

\[
K(z, \zeta) = \frac{1}{2\pi} \frac{1 - |z|^2}{|\zeta - z|^2}
\]

where \( \zeta \in \Gamma \) and \( z \in D \) \hspace{1cm} (15.2)

is called the Poisson kernel for the unit disc \( D \). If \( f \in H(D(0, 1 + \varepsilon)) \) then we have

\[
f(z) = \int_{\Gamma} K(z, \zeta) f(\zeta) \frac{d\zeta}{i\zeta} \quad \text{for} \quad z \in D .
\]

(In (15.1) we had to exclude \( z = 0 \) since \( z_1 \) is undefined for \( z = 0 \). However, both sides of (15.1) extend continuously to \( z = 0 \) and one obtains the validity of (15.3) also for \( z = 0 \).)

**Properties of \( K(z, \zeta) \):**

1. \( K(z, \zeta) > 0 \) for \( z \in D, \zeta \in \Gamma \).
2. \[
\int_{\Gamma} K(z, \zeta) \frac{d\zeta}{i\zeta} = 1 \quad \text{for} \quad z \in D
\]
   (Take \( f \equiv 1 \) in (15.3).)
3. The function \( z \to K(z, \zeta) \) is harmonic in \( D \) for every \( \zeta \in \Gamma \). This follows since the function \( z \to \frac{1}{\zeta - z} \), \( z \in D \), is holomorphic and the function \( z \to \frac{1}{\zeta - z_1} \), \( z \in D \setminus \{0\} \), is the complex conjugate of a holomorphic function. Therefore, \( z \to \text{Re} \left( \frac{1}{\zeta - z} - \frac{1}{\zeta - z_1} \right) \) is harmonic in \( D \setminus \{0\} \), and this function extends smoothly to \( z = 0 \).

Other notation: In formula (15.2) let
\[ z = re^{i\theta} \quad \text{and} \quad \zeta = e^{i\phi} . \]

One obtains that

\[
K(z, \zeta) = K(re^{i\theta}, e^{i\phi}) = \frac{1}{2\pi} \frac{1 - r^2}{|e^{i\phi} - re^{i\theta}|^2}
\]

Here

\[
|e^{i\phi} - re^{i\theta}|^2 = (\cos \phi - r \cos \theta)^2 + (\sin \phi - r \sin \theta)^2 = 1 - 2r \cos(\phi - \theta) + r^2
\]

One defines

\[
P(r, \alpha) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r \cos \alpha + r^2} .
\]

The kernel \(P(r, \alpha)\) is also called the Poisson kernel for the unit disc.

We have

\[
K(re^{i\theta}, e^{i\phi}) = P(r, \phi - \theta) .
\]

Formula (15.3) becomes

\[
f(re^{i\theta}) = \int_{0}^{2\pi} P(r, \phi - \theta) f(e^{i\phi}) d\phi \quad \text{for} \quad 0 \leq r < 1 .
\]

This result suggests the following:

**Theorem 15.1** The solution of the Dirichlet problem is

\[
u(re^{i\theta}) = \int_{0}^{2\pi} P(r, \phi - \theta) g(e^{i\phi}) d\phi \quad \text{for} \quad 0 \leq r < 1 \quad (15.4)
\]

and

\[
u(re^{i\theta}) = g(e^{i\theta}) \quad \text{for} \quad r = 1 .
\]

**Proof:** Since the Poisson kernel \(K(z, \zeta)\) is a harmonic function of \(z \in D\) for every fixed \(\zeta \in \Gamma\) it follows that \(u(z)\) is harmonic in \(D\). The difficulty is to prove that \(u\) attains its boundary values continuously.

Fix \(\zeta_0 = e^{i\theta_0} \in \Gamma\). We must prove that

\[
\lim_{z \to \zeta_0, z \in D} u(z) = g(\zeta_0) .
\]

Using the notation

\[
\zeta = e^{i\phi}, \quad \zeta_0 = e^{i\theta_0}
\]

165
we have for all $z \in D$:

$$u(z) - g(\zeta_0) = \int_{\Gamma} K(z, \zeta) \left( g(\zeta) - g(\zeta_0) \right) \frac{d\zeta}{i\zeta}$$

$$= \int_0^{2\pi} K(z.e^{i\phi}) \left( g(e^{i\phi}) - g(e^{i\theta_0}) \right) d\phi$$

Let $\varepsilon > 0$ be given. There exists $\delta_1 > 0$ so that

$$|g(e^{i\phi}) - g(e^{i\theta_0})| \leq \frac{\varepsilon}{2} \quad \text{for} \quad |\phi - \theta_0| \leq 2\delta_1.$$

Therefore,

$$|u(z) - g(\zeta_0)| \leq \frac{\varepsilon}{2} + 2|g|_{\infty} \int_{|\phi - \theta_0| \geq 2\delta_1} K(z, e^{i\phi}) d\phi.$$ 

Let $z = re^{i\theta}$ and $|\theta - \theta_0| \leq \delta_1$.

We must bound

$$K(z, e^{i\phi}) = K(re^{i\theta}, e^{i\phi})$$

for

$$|\phi - \theta_0| \geq 2\delta_1.$$

Note that

$$|\phi - \theta_0| \geq 2\delta_1 \quad \text{and} \quad |\theta - \theta_0| \leq \delta_1$$

implies that

$$|\phi - \theta| \geq \delta_1.$$

We have

$$K(z, e^{i\phi}) = K(re^{i\theta}, e^{i\phi})$$

$$= \frac{1}{2\pi} \frac{1 - r^2}{|e^{i\phi} - re^{i\theta}|^2}$$

Here

$$|e^{i\phi} - re^{i\theta}|^2 = |e^{i\alpha} - r|^2 \quad \text{with} \quad |\alpha| = |\phi - \theta| \geq \delta_1.$$

Set

$$c_1 := \sin \delta_1 > 0.$$ 

Then we have that
\[ |e^{i\alpha} - r|^2 = (\cos \alpha - r)^2 + \sin^2 \alpha \geq c_1^2. \]

This estimate yields the bound

\[ K(z, e^{i\phi}) = K(re^{i\theta}, e^{i\phi}) \leq \frac{1}{2\pi} \frac{1 - r^2}{c_1^2} \]

for

\[ |\phi - \theta_0| \geq 2\delta_1, \quad |\theta - \theta_0| \leq \delta_1, \quad 0 \leq r < 1. \]

Therefore, there exists \( \delta_2 > 0 \) so that

\[ 2|g|_{\infty} \int_{|\phi - \theta_0|} K(z, e^{i\phi}) d\phi \leq \frac{\varepsilon}{2} \]

if

\[ |\theta - \theta_0| \leq \delta_1 \quad \text{and} \quad 1 - \delta_2 \leq r < 1. \]

We have proved the limit relation

\[ \lim_{z \to \zeta_0, z \in D} u(z) = g(\zeta_0), \]

which, together with the continuity of \( g \), implies that \( u \in C(\bar{D}). \)

**Remark:** In the circle \( D(0, R) \) of radius \( R > 0 \) the solution formula (15.4) is replaced by

\[ u(re^{i\theta}) = \int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR \cos(\phi - \theta)} g(e^{i\phi}) d\phi \quad \text{for} \quad 0 \leq r < R. \quad (15.5) \]

### 15.2.2 Derivation of the Poisson Kernel via Separation of Variables