Functions of a Complex Variable I
Math 561, Fall 2009

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1 The Field \( C \) of Complex Numbers; Some Simple Concepts

Summary: The set of all complex numbers \( z = x + iy \) forms a commutative field, denote by \( C \), were addition and multiplication are defined. The mapping a complex conjugation, \( z = x + iy \rightarrow \bar{z} = z - iy \), commutes with addition and multiplication. With the distance function \( d(z_1, z_2) = |z_1 - z_2| \) the set \( C \) becomes a complete metric space. Important analytical concepts are convergence of series and complex differentiability of functions \( f : U \rightarrow \mathbb{C} \) where \( U \) denotes an open subset of \( \mathbb{C} \).

1.1 The Field \( C \) of Complex Numbers and the Euclidean Plane

Let \( \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\} \) denote the Euclidean plane, consisting of all ordered pairs of real numbers \( x, y \). One defines addition in \( \mathbb{R}^2 \) by

\[
(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).
\]

It is remarkable that one can define multiplication in \( \mathbb{R}^2 \) which, together with the above addition, turns \( \mathbb{R}^2 \) into a commutative field.

To motivate the definition of multiplication, let us identify the pair \((x, 0)\) with \(x \in \mathbb{R}\) and let \((0, 1) = i\). Then

\[
(x, y) = (x, 0) + (0, y) = x + iy.
\]

If one now postulates that \( i^2 = -1 \) and also postulates distributive laws, one obtains

\[
(x_1, y_1) \cdot (x_2, y_2) = (x_1 + iy_1) \cdot (x_2 + iy_2) = x_1 x_2 - y_1 y_2 + i(y_1 x_2 + x_1 y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2).
\]

This motivates to define multiplication in \( \mathbb{R}^2 \) by

\[
(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2).
\]

It is tedious, but not difficult to prove:

Theorem 1.1 The set \( \mathbb{R}^2 \), together with addition and multiplication defined above, is a commutative field.

Partial Proof: The zero–element in \( \mathbb{R}^2 \) is \((0, 0) = 0\) and the one–element is \((1, 0) = 1\). We want to check that every element \((x, y) \neq (0, 0)\) has a multiplicative inverse. Motivation for the formula for the inverse: Let
\[ z = (x, y) = x + iy. \]

Then we have
\[
\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2}.
\]

This motivates to set
\[
(a, b) = \left( \frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).
\]

Then we calculate
\[
(x, y) \cdot (a, b) = \left( \frac{x^2}{x^2 + y^2} - \frac{y(-y)}{x^2 + y^2}, \frac{yx}{x^2 + y^2} + \frac{x(-y)}{x^2 + y^2} \right)
\]
\[
= (1, 0)
\]
\[
= 1
\]

This shows that \((a, b)\) is indeed the multiplicative inverse of \((x, y)\).

It is, of course, also important to check that
\[
i^2 = i \cdot i
\]
\[
= (0, 1) \cdot (0, 1)
\]
\[
= (-1, 0)
\]
\[
= -1
\]

As usual, we will identify \((x, 0)\) with \(x \in \mathbb{R}\) and write \(i = (0, 1)\),
\[
(x, y) = x + iy.
\]

With these notations and the above definitions of addition and multiplication, one writes \(\mathbb{C}\) for the plane \(\mathbb{R}^2\). It is convenient to think of \(\mathbb{R}\) as a subfield of \(\mathbb{C}\), i.e. \(\mathbb{R} \subset \mathbb{C}\).

**Summary:** The Euclidean plane \(\mathbb{R}^2\) and the field of complex numbers \(\mathbb{C}\) can be identified via the mapping
\[
\mathbb{R}^2 \leftrightarrow \mathbb{C}, \quad (x, y) \leftrightarrow z = x + iy.
\]

Addition and multiplication in \(\mathbb{C}\) are defined by
\[
(x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + i(y_1 + y_2)
\]
\[
(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)
\]
1.2 Some Simple Concepts

Complex Conjugation. If \( z = x + iy \) with real \( x, y \), then \( \bar{z} = x - iy \) is called the complex conjugate of \( z \). One easily checks the rules:

\[
\bar{z_1 + z_2} = \bar{z}_1 + \bar{z}_2
\]

and

\[
\bar{z_1 z_2} = \bar{z}_1 \bar{z}_2.
\]

Furthermore, \( z = \bar{z} \) if and only if \( z \) is real.

Exercise: Prove: If \( z \neq 0 \) then \( \frac{1}{\bar{z}} = \frac{1}{\bar{z}} \).

A simple consequence of the rules for taking complex conjugates is the following:

**Lemma 1.1** Let \( p(z) = a_0 + a_1 z + \ldots + a_k z^k \) be a polynomial with real coefficients, \( a_j \in \mathbb{R} \). If \( p(z_0) = 0 \) for some \( z_0 \in \mathbb{C} \), then \( p(\bar{z}_0) = 0 \). In other words, the non–real roots of a polynomial with real coefficients come in pairs of complex conjugate numbers. Further implication: The non–real eigenvalues of a real matrix \( A \in \mathbb{R}^{n \times n} \) come in complex conjugate pairs.

**Absolute Value.** If \( z = x + iy \) with real \( x, y \), then

\[
|z| = \sqrt{x^2 + y^2}
\]

is the Euclidean distance of \( z \) from 0. We have the triangle inequality,

\[
|z + w| \leq |z| + |w|,
\]

and the multiplication rule:

\[
|zw| = |z||w|.
\]

**Distance and Convergence of Sequences.** If \( z_1, z_2 \in \mathbb{C} \) are two complex numbers then their Euclidean distance is

\[
|z_1 - z_2|.
\]

This distance concept leads, as usual, to a concept of convergence for sequences: If \( z_n \) is a sequence in \( \mathbb{C} \) and \( z \in \mathbb{C} \), then \( z_n \) converges to \( z \) (for short: \( z_n \to z \)) if and only if for all \( \varepsilon > 0 \) there is \( N \in \mathbb{N} \) with

\[
|z_n - z| < \varepsilon \quad \text{for} \quad n \geq N.
\]

A sequence \( z_n \) of complex numbers is called a Cauchy sequence in \( \mathbb{C} \) if for all \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) so that \( |z_m - z_n| < \varepsilon \) for \( m, n \geq N \). An important result of analysis says that every Cauchy sequence in \( \mathbb{C} \) has a limit in \( \mathbb{C} \). In other words, the metric space \( \mathbb{C} \) with distance \( d(z_1, z_2) = |z_1 - z_2| \) is complete.

**Convergence of Series.** Similar as in real analysis, we will consider series, which are expressions of the form
\[
\sum_{j=0}^{\infty} a_j
\]
where \(a_j \in \mathbb{C}\). The sequence

\[
s_n = \sum_{j=0}^{n} a_j
\]
is the corresponding sequence of partial sums. The series \(\sum_{j=0}^{\infty} a_j\) is called convergent if the sequence \(s_n\) of partial sums converges. If \(s_n \rightarrow s\) then one writes

\[
\sum_{j=0}^{\infty} a_j = s.
\]

In other words, the symbol \(\sum_j a_j\) may denote just an expression, but it also may denote the complex number

\[
\lim_{n \to \infty} \sum_{j=0}^{n} a_j.
\]

This double meaning of \(\sum_j a_j\), though sometimes confusing, turns out to be very convenient.

The series \(\sum_j a_j\) is called absolutely convergent if the series \(\sum_j |a_j|\) converges. If the series \(\sum_j a_j\) is convergent, but not absolutely convergent, then one calls it conditionally convergent. The standard example of a conditionally convergent series is

\[
\sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{j} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots
\]

In the theory of complex variables one typically works with absolutely convergent series.

Two simple results on convergent series are the comparison and the quotient criteria.

**Theorem 1.2 (Comparison criterion)** Assume that \(|a_j| \leq |b_j|\) for all \(j\). If \(\sum_j |b_j|\) converges, then \(\sum_j |a_j|\) converges, too.

The proof uses the completeness of \(\mathbb{C}\).

**Theorem 1.3 (Quotient criterion)** Assume that there exists \(J \in \mathbb{N}\) and \(q < 1\) so that

\[
\left| \frac{a_{j+1}}{a_j} \right| \leq q < 1 \quad \text{for} \quad j \geq J.
\]

Then the series \(\sum_j a_j\) converges absolutely.
The proof uses convergence of the geometric series,
\[\sum_{j=0}^{\infty} q^j = \frac{1}{1-q}, \quad |q| < 1.\]

**Example 1.1:** The quotient criterion can be used to prove absolute convergence of the series defining the exponential function,
\[\exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}, \quad z \in \mathbb{C}.\]

In this case, \(a_j = z^j/j!\) and
\[|a_{j+1}/a_j| = |z|/(j+1) \leq \frac{1}{2} \quad \text{for} \quad j + 1 \geq 2|z|.\]

**Continuity.** Let \(U \subset \mathbb{C}\) and let \(f : U \to \mathbb{C}\) denote a function. Let \(z_0 \in U\). The function \(f\) is called continuous at \(z_0\) if for all \(\varepsilon > 0\) there is \(\delta > 0\) so that
\[|f(z_0) - f(z)| < \varepsilon\]
for all \(z \in U\) with \(|z_0 - z| < \delta\).

### 1.3 Complex Differentiability

An important concept is complex differentiability of a function. Here the field structure of \(\mathbb{C}\) is used in an essential way since in the formula (1.1) below division by the complex number \(h\) occurs.

**Definition 1.1:** Let \(U \subset \mathbb{C}\) be an open set and let \(f : U \to \mathbb{C}\) be a function. Let \(z_0 \in U\). The function \(f\) is called complex differentiable in \(z_0\) if
\[\lim_{h \to 0} \frac{1}{h} (f(z_0 + h) - f(z_0)) \quad (1.1)\]
exists. If the limit exists, we denote it by
\[f'(z_0) = \frac{df}{dz}(z_0)\]
and call the number \(f'(z_0)\) the complex derivative of \(f\) in \(z_0\). The function \(f : U \to \mathbb{C}\) is called complex differentiable in \(U\) if it is complex differentiable in every point \(z_0\) in \(U\). We then write \(f \in H(U)\) and call \(f\) a holomorphic function in \(U\).

**Example 1.2** Let \(U = \mathbb{C}\) and let \(f(z) = z^n\) where \(n\) is a positive integer. We have
\[f(z + h) - f(z) = (z + h)^n - z^n = \left(z^n + nhz^{n-1} + R(h)\right) - z^n = nhz^{n-1} + R(h)\]
where $|R(h)| \leq C|h|^2$ for $|h| \leq 1$. It follows that
\[
\lim_{h \to 0} \frac{1}{h} (f(z + h) - f(z)) = nz^{n-1},
\]
thus the function $f(z) = z^n$ is complex differentiable with derivative
\[
(z^n)' = nz^{n-1}.
\]

**Example 1.3** The function $f(z) = x$ where $z = x + iy$ with real $x, y$ is nowhere complex differentiable. To see this, take first $h = h_1, h_1 \in \mathbb{R}, h_1 \neq 0$ and obtain
\[
\frac{1}{h} (f(z + h) - f(z)) = \frac{h_1}{h_1} = 1.
\]
Second, let $h = ih_2, h_2 \in \mathbb{R}, h_2 \neq 0$. In this case
\[
\frac{1}{h} (f(z + h) - f(z)) = \frac{0}{ih_2} = 0.
\]
Therefore, the limit
\[
\lim_{h \to 0} \frac{1}{h} (f(z + h) - f(z))
\]
does not exist.

The theory of complex variables is the study of functions $f : U \to \mathbb{C}$ where $U \subset \mathbb{C}$ is an open set and where $f$ is complex differentiable in $U$.

Any complex function $f : U \to \mathbb{C}$ can be written as
\[
f(z) = u(x, y) + iv(x, y) \quad \text{with} \quad z = x + iy
\]
where $u(x, y)$ and $v(x, y)$ are real valued. It is important to understand the relation between complex differentiability of $f$ and real differentiability of the functions $u(x, y)$ and $v(x, y)$. As Example 1.3 shows, complex differentiability is more than just smoothness.

Roughly speaking, differentiation corresponds to approximation by a linear map. We can consider $\mathbb{R}^2 \simeq \mathbb{C}$ as a 2-dimensional real vector space or as a 1-dimensional complex vector space. If we have a map
\[
L : \mathbb{R}^2 \simeq \mathbb{C} \to \mathbb{R}^2 \simeq \mathbb{C}
\]
we then must distinguish between real and complex linearity of $L$. This distinction is of an algebraic nature.

Therefore, as we will explain in Chapter 3, the difference between real and complex differentiability is of an algebraic nature. The main issue is addressed in the following question: Which real-linear maps $L : \mathbb{R}^2 \to \mathbb{R}^2$ correspond to complex-linear maps from $\mathbb{C}$ to $\mathbb{C}$?
2 The Cauchy Product of Two Series and the Addition Theorem for the Exponential Function

For the exponential function,

\[ \exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}, \quad z \in \mathbb{C}, \]

the fundamental addition theorem,

\[ \exp(a + b) = \exp(a) \exp(b), \quad a, b \in \mathbb{C}, \]

will be derived from a result about the Cauchy product of two series.

2.1 The Cauchy Product of Two Series

Let

\[ \sum_{j=0}^{\infty} a_j \quad \text{and} \quad \sum_{j=0}^{\infty} b_j \]  \hspace{1cm} (2.1)

denote two series of complex numbers. Proceeding formally, we obtain for their product

\[
(a_0 + a_1 + a_2 + \ldots) \cdot (b_0 + b_1 + b_2 + \ldots) = a_0b_0 + a_0b_1 + a_0b_2 + \ldots \\
+ a_1b_0 + a_1b_1 + a_1b_2 + \ldots \\
+ a_2b_0 + a_2b_1 + a_2b_2 + \ldots \\
+ \ldots \\
= c_0 + c_1 + c_2 + \ldots
\]

with

\[ c_0 = a_0b_0, \quad c_1 = a_0b_1 + a_1b_0, \quad c_2 = a_0b_2 + a_1b_1 + a_2b_0, \quad \text{etc.} \]

In general, set

\[ c_n = a_0b_n + a_1b_{n-1} + \ldots + a_nb_0. \]  \hspace{1cm} (2.2)

Then the series

\[ \sum_{n=0}^{\infty} c_n \]

is called the Cauchy product of the series (2.1).
Theorem 2.1  a) Assume both series (2.1) converge, and at least one of them converges absolutely. Then their Cauchy product also converges, and for the values of the series we have

\[
\left( \sum_{j=0}^{\infty} a_j \right) \cdot \left( \sum_{j=0}^{\infty} b_j \right) = \sum_{n=0}^{\infty} c_n .
\]  

(2.3)

b) If both series (2.1) converge absolutely, then their Cauchy product also converges absolutely.

Proof:  a) Let

\[
A_n := \sum_{j=0}^{n} a_j \rightarrow A
\]

\[
B_n := \sum_{j=0}^{n} b_j \rightarrow B
\]

\[
C_n := \sum_{j=0}^{n} c_j
\]

We must show that \( C_n \rightarrow AB \).

Assume that \( \sum a_j \) converges absolutely and let

\[
\alpha := \sum_{j=0}^{\infty} |a_j| .
\]

Set

\[
\beta_n = B_n - B = - \sum_{k=n+1}^{\infty} b_k .
\]

Then we have

\[
B_n = B + \beta_n , \quad \beta_n \rightarrow 0 , \quad |\beta_n| \leq \beta_{\text{max}} .
\]

We now rewrite \( C_n \):

\[
C_n = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \ldots + (a_0 b_n + \ldots + a_n b_0)
\]

\[
= a_0 B_n + a_1 B_{n-1} + \ldots + a_n B_0
\]

\[
= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \ldots + a_n (B + \beta_0)
\]

\[
= A_n B + \gamma_n
\]

with

\[
\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \ldots + a_n \beta_0 .
\]

We have to show that \( \gamma_n \rightarrow 0 .\)
Let $\varepsilon > 0$ be given. There exists $N = N_1(\varepsilon)$ with

$$|\beta_n| \leq \varepsilon \quad \text{for all} \quad n \geq N + 1. \quad (2.4)$$

In the following, $N$ is fixed with (2.4). We have for all $n \geq N$:

$$|\gamma_n| \leq |\beta_0 a_n| + \ldots + |\beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1}| + \ldots + |\beta_n a_0|$$

$$\leq |\beta_0 a_n| + \ldots + |\beta_N a_{n-N}| + \varepsilon \alpha$$

$$\leq \beta_{\text{max}} (|a_n| + \ldots + |a_{n-N}|) + \varepsilon \alpha$$

Here the bracket contains $N + 1$ terms and

$$|a_n| \leq \frac{\varepsilon}{N + 1}, \ldots, |a_{n-N}| \leq \frac{\varepsilon}{N + 1}$$

if $n \geq N_2(\varepsilon)$. It follows that

$$|\gamma_n| \leq \varepsilon (\beta_{\text{max}} + \alpha) \quad \text{for} \quad n \geq N_2(\varepsilon).$$

This proves that $\gamma_n \to 0$.

b) Assume that both series (2.1) converge absolutely. We have

$$|c_n| \leq |a_0||b_n| + \ldots + |a_n||b_0| =: d_n.$$  

Here $\sum d_n$ is the Cauchy product of the series $\sum |a_j|$ and $\sum |b_j|$. By part a), the series $\sum d_n$ converges and, therefore, $\sum |c_n|$ also converges. 

Remark: If $a_j = b_j = (-1)^j \frac{1}{\sqrt{j+1}}$ then the series (2.1) converge, but the convergence is not absolute. Here the general term of the Cauchy product is

$$c_n = (-1)^n \sum_{j=0}^{n} \frac{1}{\sqrt{j+1} \sqrt{n+1} - j}$$

and

$$\frac{1}{\sqrt{j+1} \sqrt{n+1} - j} \geq \frac{1}{\sqrt{n+1} \sqrt{n+1} - j} = \frac{1}{n+1}, \quad 0 \leq j \leq n.$$  

It follows that $|c_n| \geq 1$; the Cauchy product of $\sum a_j$ and $\sum b_j$ diverges.

2.2 The Addition Theorem for the Exponential Function

For all $z \in \mathbb{C}$ the series

$$\exp(z) := \sum_{j=0}^{\infty} \frac{z^j}{j!}$$

converges absolutely by the quotient criterion. We use the previous theorem to prove the fundamental Addition Theorem for the exponential function.
Theorem 2.2
\[ \exp(a + b) = \exp(a) \exp(b) \quad \text{for all} \quad a, b \in \mathbb{C}. \]  \hfill (2.5)

**Proof:** Note that
\[ \exp(a) = \sum_{j=0}^{\infty} a_j \quad \text{with} \quad a_j = \frac{a^j}{j!} \]
and
\[ \exp(b) = \sum_{j=0}^{\infty} b_j \quad \text{with} \quad b_j = \frac{b^j}{j!}. \]
Also,
\[ \exp(a + b) = \sum_{n=0}^{\infty} \frac{1}{n!} (a + b)^n \]
where
\[ (a + b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j} \]
and
\[ \binom{n}{j} = \frac{n!}{j!(n-j)!}. \]
It follows that
\[ \exp(a + b) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{j!} a^j \frac{1}{(n-j)!} b^{n-j}. \]
This is precisely the Cauchy product of the series for \( \exp(a) \) and \( \exp(b) \). The claim follows from Theorem 2.1. \( \diamond \)

Let us give a second proof of the Addition Theorem. It uses tools, however, which we will only justify later. The function
\[ f(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \]
is entire and \( f'(z) = f(z), f(0) = 1 \). Fix \( a \in \mathbb{C} \) and consider the function \( g(z) = f(a + z) \). Then \( g'(z) = g(z) \) and \( g(0) = f(a) \). The function \( h(z) = f(a)f(z) \) also satisfies \( h'(z) = h(z), h(0) = f(a) \). Therefore, the functions \( g(z) \) and \( h(z) \) are both solutions of the initial–value problem
\[ u'(z) = u(z), \quad u(0) = f(a). \]
Uniqueness of the solution of this initial–value problem implies that \( g(z) = h(z) \), i.e., \( f(a + z) = f(a)f(z) \).
2.3 Powers of $e$

One sets

\[ e := \exp(1) = \sum_{j=0}^{\infty} \frac{1}{j!} = 2.718281828459046 \ldots, \]

a notation due to Euler. Then, by (2.5),

\[ \exp(2) = \exp(1) \exp(1) = e \cdot e = e^2 \]
\[ \exp(3) = \exp(1) \exp(2) = e \cdot e^2 = e^3 \]

etc.

Also, since

\[ \exp(1) \exp(-1) = \exp(0) = 1, \]

we obtain

\[ \exp(-1) = \frac{1}{e} = e^{-1}. \]

More generally:

**Lemma 2.1** Let $q = m/n$ denote a positive rational number where $m,n \in \mathbb{N}$. If

\[ \alpha := e^{m/n} = e^q \]

denotes the positive $n$–th root of $e^m$, then

\[ \exp(q) = \alpha. \]

In other words,

\[ \exp(q) = e^q \]

for all positive rationals $q$.

**Proof:** Set $\beta := \exp(q)$. Then $\beta > 0$ and, using the addition theorem,

\[ \beta^n = \exp\left(\frac{m}{n}\right) \cdot \ldots \cdot \exp\left(\frac{m}{n}\right) \quad (n \text{ factors}) \]
\[ = \exp(m) \]
\[ = e^m \]

Also, $\alpha^n = e^m$ and, therefore, $\alpha^n = \beta^n$. Since $\alpha > 0$ and $\beta > 0$ we conclude that $\alpha = \beta$. $\diamond$

With similar arguments, it follows that the equation
\[
\exp(q) = e^q
\]
also holds for negative rationals \( q \). This justifies the standard notation
\[
e^z = \exp(z), \quad z \in \mathbb{C},
\]
where the exponential function is defined by the exponential series:
\[
\exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}.
\]

### 2.4 Euler’s Identity and Implications

Define
\[
\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}
\]
\[
\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}
\]
The series converge absolutely for every \( z \in \mathbb{C} \). Using the definitions by the series, it is not difficult to prove Euler’s identity,

**Lemma 2.2**
\[
e^{iz} = \cos z + i \sin z \quad \text{for all} \quad z \in \mathbb{C}.
\]

**Proof:** We have
\[
e^{iz} = \sum_{j=0}^{\infty} \frac{(iz)^j}{j!}
\]
\[
= \sum_{k=0}^{\infty} \frac{(iz)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(iz)^{2k+1}}{(2k+1)!}
\]
\[
= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}
\]
\[
= \cos z + i \sin z
\]
\[
\diamond
\]

**Lemma 2.3** For all \( z \in \mathbb{C} \):
\[
\cos^2 z + \sin^2 z = 1.
\]
Proof: We have
\[ e^{-iz} = \cos z - i \sin z \quad \text{for all} \quad z \in \mathbb{C}. \]
Therefore,
\[
\begin{align*}
\cos z &= \frac{1}{2}(e^{iz} + e^{-iz}), \\
\sin z &= \frac{1}{2i}(e^{-iz} - e^{iz}).
\end{align*}
\]
Using (2.5),
\[
\cos^2 z + \sin^2 z = \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) - \frac{1}{4}(e^{2iz} - 2 + e^{-2iz}) = 1.
\]
\[\diamondsuit\]

Lemma 2.4 For all \( \theta \in \mathbb{R} \):
\[ |e^{i\theta}| = 1. \]
Proof: By Euler’s identity:
\[ e^{i\theta} = \cos \theta + i \sin \theta. \]
For real \( \theta \), the values of \( \cos \theta \) and \( \sin \theta \) are real. Therefore,
\[ |e^{i\theta}|^2 = \cos^2 \theta + \sin^2 \theta = 1. \]
\[\diamondsuit\]

2.5 The Polar Representation of a Complex Number

Let \( z \in \mathbb{C}, z \neq 0 \). Then \( \zeta = z/|z| = x + iy \) satisfies
\[ |
\zeta|^2 = x^2 + y^2 = 1. \]
From trigonometry (or calculus) we know the following result:
Lemma 2.5 Given any two real numbers \(x, y\) with \(x^2 + y^2 = 1\) there is a unique real \(\theta\) so that \(-\pi < \theta \leq \pi\) and

\[
x = \cos \theta, \quad y = \sin \theta.
\]

**Remark:** It is not at all obvious how to prove this result using the series representations of \(\cos \theta\) and \(\sin \theta\). In particular, one has to introduce the number \(\pi\). One can define \(\pi/2\) as the smallest positive zero of the cosine–function. One can prove that the functions \(c(\theta) = \cos \theta\) and \(s(\theta) = \sin \theta\) satisfy \(c' = -s, s' = c\), thus \(c'' + c = s'' + s = 0\). A proof of the lemma can be based on properties of the solutions of the differential equation \(u'' + u = 0\).

Using the lemma we have

\[
\zeta = x + iy = \cos \theta + i \sin \theta = e^{i\theta}.
\]

The representation

\[
z = re^{i\theta} \quad \text{with} \quad r = |z| > 0, \quad \theta = \arg(z) \in (-\pi, \pi],
\]

is called the polar representation of \(z\). It is very useful if one wants to visualize complex multiplication geometrically since

\[
z_1 = r_1e^{i\theta_1} \quad \text{and} \quad z_2 = r_2e^{i\theta_2}
\]

implies

\[
z_1z_2 = r_1r_2e^{i(\theta_1 + \theta_2)}.
\]

### 2.6 Further Properties of the Exponential Function

In the following, let \(z = x + iy\) with real \(x, y\). We want to understand the map

\[
z \rightarrow e^z
\]

from \(\mathbb{C}\) into itself. We make the following observations:

1) \(e^z \neq 0\) for all \(z \in \mathbb{C}\). This follows from \(e^z e^{-z} = e^0 = 1\).

2) \(|e^z| = |e^x e^{iy}| = e^x > 0\) since \(|e^{iy}| = 1\).

3) The horizontal line

\[
H_y = \{z = x + iy : x \in \mathbb{R}\}
\]

is mapped to the half–line

\[
e^x (\cos y + i \sin y), \quad 0 < e^x < \infty.
\]

4) The vertical line

\[
V_x = \{z = x + iy : y \in \mathbb{R}\}
\]

is mapped (infinitely often) to the circle of radius \(e^x\),
\[ e^{x}(\cos y + i \sin y), \quad -\infty < y < \infty. \]

We note that the family of lines \( H_y \) is orthogonal to the family of lines \( V_x \). Orthogonality also holds for the corresponding image lines. We will see below that this is not accidental, but is generally true for a holomorphic map \( f(z) \) with \( f'(z) \neq 0 \).

Roughly, the map

\[ z \rightarrow e^{z} = e^{x}e^{iy} \]

is oscillatory in \( y \) and has real exponential behavior in \( x \). For \( x << -1 \), the complex number \( e^{z} \) is very small in absolute value; for \( x >> 1 \), the complex number \( e^{z} \) is very large in absolute value. This follows simply from

\[ |e^{z}| = e^{x}. \]

### 2.7 The Main Branch of the Complex Logarithm

Consider the open horizontal strip

\[ S_{\pi} = \{ z = x + iy : -\pi < y < \pi, \ x \in \mathbb{R} \} \]

and the slit plane

\[ \mathbb{C}^{-} = \mathbb{C} \setminus (-\infty, 0]. \]

If \( z = x + iy \in S_{\pi} \) then \(-\pi < y < \pi\), thus

\[ e^{z} = e^{x}e^{iy} \in \mathbb{C}^{-}. \]

**Lemma 2.6** The map

\[
\exp : \begin{cases} 
S_{\pi} & \mapsto \mathbb{C}^{-} \\
z & \mapsto e^{z}
\end{cases} \tag{2.6}
\]

is one-to-one and onto.

**Proof:**

a) Let \( z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in S_{\pi} \) and let \( e^{z_1} = e^{z_2} \). It follows that \( e^{x_1} = e^{x_2} \) and \( e^{iy_1} = e^{iy_2} \). Therefore, \( x_1 = x_2 \) is clear. The uniqueness statement of the previous lemma yields that \( y_1 = y_2 \).

b) Let \( w = re^{i\theta} \in \mathbb{C}^{-} \) be given. Then we have \( r > 0 \) and may assume that \(-\pi < \theta < \pi\). Let \( x = \ln r \) and set \( z = x + i\theta \). We have \( z \in S_{\pi} \) and

\[ e^{z} = e^{x}e^{i\theta} = re^{i\theta} = w. \]

By definition, the inverse function of (2.6) is the main branch of the complex logarithm:

\[
\log : \begin{cases} 
\mathbb{C}^{-} & \mapsto S_{\pi} \\
w & \mapsto \log w \tag{2.7}
\end{cases}
\]

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with

\[ \exp(\log w) = w \quad \text{for all} \quad w \in \mathbb{C}^- . \]

This log–function extends the real function

\[ \ln : \begin{cases} (0, \infty) & \mapsto (-\infty, \infty) \\ r & \mapsto \ln r \end{cases} \quad (2.8) \]

from the positive real axis into the slit plane \( \mathbb{C}^- \).

Given any \( w \in \mathbb{C}^- \), write

\[ w = re^{i\theta} \quad \text{with} \quad r > 0 \quad \text{and} \quad -\pi < \theta < \pi . \]

Then one has

\[ \log w = \ln r + i\theta . \]

If \( w = w_1 + iw_2 \in \mathbb{C}^- \) then

\[ r = (w_1^2 + w_2^2)^{1/2}, \quad \theta = \arctan(w_2/w_1) , \]

thus

\[ \log(w_1 + iw_2) = \frac{1}{2} \ln(w_1^2 + w_2^2) + i\arctan(w_2/w_1) . \]

Here one has to choose the correct branch of \( \arctan \).

**General Powers; Main Branch.** Let \( b \in \mathbb{C} \) and let \( a \in \mathbb{C}^- \). One defines the main branch of \( a^b \) by

\[ a^b = e^{b \log a} . \]

**Example:** We have

\[ e^{\pi i/2} = i , \]

thus

\[ \log i = \pi i/2 . \]

Therefore,

\[ i^i = e^{i(\pi i/2)} = e^{-\pi/2} = 0.2078... \]

Surprisingly, the result is real. Euler found this result in 1746.
3 Real and Complex Differentiability

3.1 Outline and Notations

We identify $\mathbb{R}^2$ and $\mathbb{C}$ using the relation

$$(x, y) \leftrightarrow z = x + iy.$$ 

Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a map. We then define two real functions, $u, v : U \to \mathbb{R}$, by

$$f(x + iy) = u(x, y) + iv(x, y).$$

Then the complex–valued map $f : U \to \mathbb{C}$ corresponds to the map

$$
\begin{pmatrix}
x \\
y
\end{pmatrix}
\mapsto
\begin{pmatrix}
u(x, y) \\
v(x, y)
\end{pmatrix}
= F(x, y)
$$

from $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$.

Loosely speaking, a map is differentiable at a point $P$ if it can be approximated near $P$ by a linear map. In the present context, we must distinguish clearly between $\mathbb{R}$–linearity and $\mathbb{C}$–linearity. Therefore, in the next section, we consider $\mathbb{R}$–linear maps $F : \mathbb{R}^2 \to \mathbb{R}^2$ ask when an $\mathbb{R}$–linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ corresponds to a $\mathbb{C}$–linear map $f : \mathbb{C} \to \mathbb{C}$. The condition is of algebraic nature.

In Section 3.3 we will then use this to discuss the relationship between real and complex differentiability. This leads to the Cauchy–Riemann equations.

3.2 $\mathbb{R}$–Linear and $\mathbb{C}$–Linear Maps from $\mathbb{R}^2 \simeq \mathbb{C}$ into Itself

If $V$ is a vector space over a field $\mathbb{K}$ then a map $f : V \to V$ is called $\mathbb{K}$–linear (or simply linear if the field $\mathbb{K}$ is unambiguous) if

$$f(\alpha a + \beta b) = \alpha f(a) + \beta f(b) \quad \text{for all } a, b \in V \quad \text{and} \quad \text{for all } \alpha, \beta \in \mathbb{K}. $$

The space $\mathbb{R}^2$ is a two–dimensional vector space over $\mathbb{R}$. The general $\mathbb{R}$–linear map from $\mathbb{R}^2$ into itself has the form

$$
\begin{pmatrix}
x \\
y
\end{pmatrix}
\mapsto
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
$$

(3.2)

where $a, b, c, d \in \mathbb{R}$.

The space $\mathbb{C}$ is a one–dimensional vector space over $\mathbb{C}$ and the general $\mathbb{C}$–linear map from $\mathbb{C}$ into itself has the form

$$z \mapsto wz =: f(z)$$

where $w \in \mathbb{C}$.

If $w = \alpha + i\beta$ then
\[
f(z) = (\alpha + i\beta)z = (\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\beta x + \alpha y)
\]
is a \(\mathbb{C}\)–linear map from \(\mathbb{C}\) into itself. We see that \(f\) corresponds to the \(\mathbb{R}\)–linear map (3.2) with

\[
a = d = \alpha, \quad -b = c = \beta.
\]
In other words, an \(\mathbb{R}\)–linear map (3.2) corresponds to a \(\mathbb{C}\)–linear map iff

\[
a = d \quad \text{and} \quad -b = c. \tag{3.3}
\]
If we write the \(\mathbb{R}\)–linear map (3.2) in the form (3.1), then

\[
a = u_x, \quad b = u_y, \quad c = v_x, \quad d = v_y,
\]
and the condition (3.3) becomes

\[
u_x = v_y, \quad -u_y = v_x.
\]
In a more general setting, these are the Cauchy–Riemann equations. They demand precisely that the (real) Jacobian of (3.1) corresponds to a \(\mathbb{C}\)–linear map.

To summarize:

**Theorem 3.1** The \(\mathbb{R}\)–linear map (3.2) corresponds to the \(\mathbb{C}\)–linear map

\[
z \rightarrow (\alpha + i\beta)z
\]
if and only if

\[
a = d = \alpha, \quad -b = c = \beta.
\]
In other words, the \(\mathbb{R}\)–linear map (3.2) corresponds to the \(\mathbb{C}\)–linear map \(z \rightarrow (\alpha + i\beta)z\) if and only if

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.
\]

3.2.1 The Polar Representation of a Complex Number and the Corresponding Matrix Factorization

This section can be skipped.

Let

\[
w = \alpha + i\beta = re^{i\theta}, \quad w \neq 0.
\]
The \(\mathbb{C}\)–linear map \(z \rightarrow wz\) corresponds to the \(\mathbb{R}\)–linear map
\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  \alpha & -\beta \\
  \beta & \alpha
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
=: F(x, y).
\] (3.4)

It is not difficult to show that the system matrix is
\[
\begin{pmatrix}
  \alpha & -\beta \\
  \beta & \alpha
\end{pmatrix}
= \sqrt{\alpha^2 + \beta^2}
\begin{pmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix}
\] (3.5)

We see here, in real notation, that the complex map \( z \rightarrow wz \) is the map of rotation by \( \theta \), counterclockwise, followed by stretching by the factor
\[
r = |w| = \sqrt{\alpha^2 + \beta^2}.
\]

Remark: The determinant of the matrix in (3.5) is
\[
\det F'(x, y) = \alpha^2 + \beta^2 = |w|^2.
\]

The map \( z \rightarrow wz \) stretches lengths by \( |w| \). The determinant of the Jacobian matrix \( F'(x, y) \) describes the stretching of area, which is described by the factor \( |w|^2 \).

3.2.2 Motivation for the Operators \( \partial/\partial z \) and \( \partial/\partial \bar{z} \)

Let \( \lambda, \mu \in \mathbb{C} \). Then the map
\[
z \rightarrow f(z) = \lambda z + \mu \bar{z}
\]
is \( \mathbb{R} \)-linear. Since \( \lambda = \lambda_1 + i\lambda_2 \) and \( \mu = \mu_1 + i\mu_2 \) the map \( f(z) \) depends on four real parameters, \( \lambda_1, \lambda_2, \mu_1, \mu_2 \). We also can start with formula (3.2) and see that the general \( \mathbb{R} \)-linear map from \( \mathbb{C} \) into \( \mathbb{C} \) depends on four real parameters \( a, b, c, d \).

Let us derive the relations between the parameters \( \lambda_1, \lambda_2, \mu_1, \mu_2 \) and \( a, b, c, d \).

To do this, recall that
\[
z = x + iy, \quad \bar{z} = x - iy
\]
and
\[
x = \frac{1}{2}(z + \bar{z}), \quad iy = \frac{1}{2}(z - \bar{z}).
\]

Therefore, if we start from the general form (3.2), then we have
\[
f(x + iy) = (ax + by) + i(cx + dy)
\]
\[
= (a + ic)x + (b + id)y
\]
\[
= \frac{1}{2}(a + ic)(z + \bar{z}) - \frac{i}{2}(b + id)(z - \bar{z})
\]
\[
= \lambda z + \mu \bar{z}
\]
with
\[
\lambda = \frac{1}{2}(a + ic) - \frac{i}{2}(b + id) \\
\mu = \frac{1}{2}(a + ic) + \frac{i}{2}(b + id)
\]

This shows how to obtain the representation \( z \to \lambda z + \mu \bar{z} \) from (3.2).

Conversely, if we start from the general form

\[
f(z) = \lambda z + \mu \bar{z}, \quad \lambda = \lambda_1 + i\lambda_2, \quad \mu = \mu_1 + i\mu_2,
\]

then we have

\[
f(z) = \lambda z + \mu \bar{z} \\
= (\lambda_1 + i\lambda_2)(x + iy) + (\mu_1 + i\mu_2)(x - iy) \\
= (\lambda_1 + \mu_1)x + (\mu_2 - \lambda_2)y + i(\lambda_2 + \mu_2)x + (\lambda_1 - \mu_1)y
\]

We see that

\[
a = \lambda_1 + \mu_1 \\
b = \mu_2 - \lambda_2 \\
c = \lambda_2 + \mu_2 \\
d = \lambda_1 - \mu_1
\]

We obtain that the Cauchy–Riemann equations,

\[
a = d \quad \text{and} \quad -b = c,
\]

are equivalent to the condition

\[
\mu = 0.
\]

**Lemma 3.1** The map

\[
f(z) = \lambda z + \mu \bar{z}
\]

is complex differentiable if and only if \( \mu = 0 \).

**Proof:** This is clear since \( z \to \lambda z \) is complex differentiable and \( z \to \bar{z} \) is not complex differentiable. Another proof follows from the Cauchy–Riemann equations. \( \Box \)

In the present example, we have

\[
\lambda = \frac{1}{2}f_x - \frac{i}{2}f_y
\]

and

\[
\mu = \frac{1}{2}f_x + \frac{i}{2}f_y.
\]
Introduce the operators
\[
\frac{\partial}{\partial z} = \frac{1}{2} \frac{\partial}{\partial x} - \frac{i}{2} \frac{\partial}{\partial y}, \\
\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \frac{\partial}{\partial x} + \frac{i}{2} \frac{\partial}{\partial y}
\]

Then we have shown:

**Lemma 3.2** Let \( f : \mathbb{C} \to \mathbb{C} \) be an \( \mathbb{R} \)-linear function, i.e.,
\[
f(x + iy) = (a + ic)x - i(b + id)y = \lambda z + \mu \bar{z}.
\]
Then \( f \) is complex differentiable if and only if \( \mu = f_{\bar{z}} = 0 \).

Let \( f : \mathbb{C} \to \mathbb{C} \) be complex differentiable. Then we have
\[
f(x + iy) = (a + ic)x - i(b + id)y = \lambda z
\]
with
\[
a = d = \lambda_1, \quad -b = c = \lambda_2.
\]
Therefore,
\[
f' = f_{z} = \lambda = \lambda_1 + i\lambda_2
\]
\[= a + ic
\]
\[= -i(b + id)
\]
\[= f_x
\]
\[= -if_y
\]

### 3.3 Real and Complex Differentiability

In the following, \( \psi(h) \) denotes a function with \( \psi(h) \to 0 \) as \( h \to 0 \).

Let \( a < c < b \) be real numbers and let \( f : (a, b) \to \mathbb{R} \) be a real function. The function \( f \) is real–differentiable at \( c \) if there is a number \( w \in \mathbb{R} \) so that
\[
f(c + h) = f(c) + wh + h\psi(h)
\]
where \( \psi(h) \) is a function with
\[
\lim_{h \to 0} \psi(h) = 0.
\]
(One writes \( \psi(h) = o(1) \).)
One can show that $w$, if it exists, is uniquely determined. One writes $w = f'(c)$.

Let $U \subset \mathbb{R}^m$ be an open set and let $f : U \to \mathbb{R}^n$ be a function. Let $c \in U$. The function $f$ is real–differentiable in $c$ if there is a matrix $A \in \mathbb{R}^{n \times m}$ with

$$f(c + h) = f(c) + Ah + \|h\|\psi(h)$$

where

$$\lim_{h \to 0} \psi(h) = 0.$$  

One can show that the matrix $A$, if it exists, is uniquely determined. The entries of $A$ then agree with the partial derivatives of the components of $f$,

$$a_{jk} = \frac{\partial f_j}{\partial x_k}(c).$$

Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a function. Let $z_0 \in U$. Then $f$ is complex differentiable at $z_0$ if there is $w \in \mathbb{C}$ such that

$$f(z_0 + h) = f(z_0) + wh + h\psi(h)$$

where

$$\lim_{h \to 0} \psi(h) = 0.$$  

One can show that the number $w$, if it exists, is unique. One writes $f'(z_0) = w$.

**Examples:** Show that $f(z) = z^n$ is c.d. with $f'(z) = nz^{n-1}$. Show that $f(z) = \bar{z}$ is not c.d. at any point.

In the following, let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a function. We write

$$f(x + iy) = u(x, y) + iv(x, y)$$

and identify $f$ with the function

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

from $U$ into $\mathbb{R}^2$.

**Theorem 3.2** Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a function. Let $z_0 \in U$. Then the following two conditions are equivalent:

1) $f$ is complex differentiable at $z_0$.

2) $f$ is real differentiable at $(x_0, y_0)$ and the real matrix

$$A = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} (x_0, y_0)$$

determines a $\mathbb{C}$–linear map, i.e.,
\[ u_x = v_y, \quad -u_y = v_x \quad \text{at} \quad (x_0, y_0). \]

**Proof:** First let \( f \) be c.d. and let

\[ f(z_0 + h) = f(z_0) + wh + o(h). \]

It is clear that \( f \) is real differentiable with Jacobian determined by \( w \). The converse is also clear. ◦

### 3.4 The Operators \( \partial/\partial x, \partial/\partial y, \partial/\partial z, \partial/\partial \bar{z}, d/dz \)

Let \( \lambda, \mu \in \mathbb{C} \). Consider the \( \mathbb{R} \)-linear map

\[ z \rightarrow f(z) = \lambda z + \mu \bar{z}. \]

We have seen that that \( f \) is \( \mathbb{C} \)-linear if and only if \( \mu = 0 \).

### 3.5 The Complex Logarithm as an Example

We have for \( z = x + iy \in \mathbb{C}^- \):

\[ f(z) = \log z = \frac{1}{2} \ln(x^2 + y^2) + i \arctan(y/x), \]

thus

\[ u = \frac{1}{2} \ln(x^2 + y^2), \]

\[ v = \arctan(y/x). \]

The partial derivatives are

\[
\begin{align*}
  u_x &= \frac{x}{x^2 + y^2} \\
  u_y &= \frac{y}{x^2 + y^2} \\
  v_x &= -\frac{y}{x^2 + y^2} \cdot \frac{1}{1 + (y/x)^2} \\
      &= -\frac{y}{x^2 + y^2} \\
  v_y &= \frac{1}{x} \cdot \frac{1}{1 + (y/x)^2} \\
      &= \frac{x}{x^2 + y^2}
\end{align*}
\]

We see that

\[ u_x = v_y, \quad u_y = -v_x. \]
Since the Cauchy–Riemann equations are satisfied, the function \( f(z) = \log z \) is complex–differentiable in \( \mathbb{C}^- \). We compute its complex derivative:

\[
f'(z) = f_x = u_x + iv_x = \frac{x - iy}{x^2 + y^2} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{z}
\]

This, of course, is not unexpected since the derivative of \( \ln x \) is \( \frac{1}{x} \). We will see later that the functions

\[
f(z) = \log z
\]

and

\[
f'(z) = \frac{1}{z}
\]

are the only holomorphic extensions of the functions \( \ln x \) and \( 1/x \), defined for \( x > 0 \), into the set \( \mathbb{C}^- \).
4 Complex Line Integrals and Cauchy’s Theorem

4.1 Curves

Let $\gamma : [a, b] \rightarrow \mathbb{C}$ denote a $C^1$–map. This means the following: If we write

$$\gamma(t) = \gamma_1(t) + i\gamma_2(t), \quad a \leq t \leq b,$$

then the two functions $\gamma_1, \gamma_2 : [a, b] \rightarrow \mathbb{R}$ are differentiable and their derivatives are continuous. Intuitively, we think of the image set

$$\{\gamma(t) : a \leq t \leq b\}$$

as a curve in $\mathbb{C}$ parametrized by $\gamma$. A curve will have different parametrization. For example, the mappings

$$\gamma(t) = e^{it}, \quad 0 \leq t \leq 2\pi,$$

and

$$\delta(s) = e^{2is}, \quad 0 \leq s \leq \pi,$$

both parametrize the circle $C_1$ of radius one, centered at the origin. The map

$$\varepsilon(t) = e^{-it}, \quad 0 \leq t \leq 2\pi,$$

has the same image set as $\gamma$ but parametrizes $C_1$ in opposite direction. We say that $\gamma$ and $\delta$ both parametrize $C_1$ whereas the map $\varepsilon$ parametrizes $-C_1$.

It is not trivial to define the notion of a curve precisely. One can proceed as follows: Let

$$\gamma : [a, b] \rightarrow \mathbb{C} \quad \text{and} \quad \delta : [c, d] \rightarrow \mathbb{C}$$

denote two $C^1$–maps. Call these two maps equivalent if there exists a $C^1$–map

$$\phi : [a, b] \rightarrow [c, d]$$

(a parameter transformation) which is one–to–one and onto and satisfies

$$\delta(\phi(t)) = \gamma(t) \quad \text{and} \quad \phi'(t) > 0 \quad \text{for all} \quad t \in [a, b].$$

Then an equivalence class of such maps $\gamma$ is called a $C^1$–curve or simply a curve.

Often it is convenient to identify a curve with any of its parametrizations. Somewhat imprecisely, we also refer to the image set

$$\Gamma = \{\gamma(t) : a \leq t \leq b\} \subset \mathbb{C}$$

as a $C^1$–curve if $\gamma : [a, b] \rightarrow \mathbb{C}$ is a $C^1$–map. Furthermore, it is convenient to work with curves that are only piecewise $C^1$ and with parametrizations $\gamma(t)$ where $t$ varies in an unbounded interval. A curve which is piecewise $C^1$ has a continuous parametrization that is piecewise $C^1$. 
**Length of a $C^1$–curve:** Let $\Gamma$ denote a $C^1$–curve with parametrization $\gamma(t), a \leq t \leq b$. Using real analysis, one obtains that

$$\text{length}(\Gamma) = \int_a^b \sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2} \, dt$$

$$= \int_a^b |\gamma'(t)| \, dt .$$

**Example 4.1:** Let

$$\gamma(t) = r e^{it}, \quad 0 \leq t \leq 2\pi ,$$
denote a parametrization of the circle $C_r$ of radius $r$ centered at the origin. One obtains that $|\gamma'(t)| = r$ and $\text{length}(C_r) = 2\pi r$.

### 4.2 Definition and Simple Properties of Line Integrals

Let $\gamma : [a, b] \to \mathbb{C}$ denote a $C^1$–map parametrizing the curve

$$\Gamma = \{ \gamma(t) : a \leq t \leq b \}$$

and let

$$f : \Gamma \to \mathbb{C}$$

denote a continuous function. We want to define the line integral of $f$ along $\Gamma$, which we denote by

$$\int_{\Gamma} f(z) \, dz \quad \text{or} \quad \int_{\gamma} f(z) \, dz .$$

This line integral can be defined as a limit of Riemann sums as follows: Let $a = t_0 < t_1 < \ldots < t_n = b$ denote a partition of the parametrization interval $[a, b]$ and let $t_{j-1} \leq s_j \leq t_j$. The points $z_j = \gamma(t_j)$ and $w_j = \gamma(s_j)$ line up along $\Gamma$. We have:

$$\int_{\Gamma} f(z) \, dz \approx \sum_{j=1}^{n} f(w_j)(z_j - z_{j-1})$$

$$= \sum_{j=1}^{n} f(\gamma(s_j))(\gamma(t_j) - \gamma(t_{j-1}))$$

$$\approx \sum_{j=1}^{n} f(\gamma(s_j))\gamma'(s_j)(t_j - t_{j-1})$$

$$\approx \int_a^b f(\gamma(t))\gamma'(t) \, dt$$

As the partition is refined, the sums converge. One obtains:
\[ \int_{\Gamma} f(z) \, dz = \int_{a}^{b} f(\gamma(t)) \gamma'(t) \, dt \quad (4.1) \]

We may also use the latter equation as the definition of \( \int_{\Gamma} f(z) \, dz \). This is justified since the right-hand side is independent of the parametrization \( \gamma \) of the curve \( \Gamma \). To see this, use substitution.

**Note on Computation:** To compute the integral on the right-hand side of (4.1), note the following: If \( \psi : [a, b] \to \mathbb{C} \) is a continuous complex-valued function,

\[ \psi(t) = \psi_1(t) + i\psi_2(t) , \]

then

\[ \int_{a}^{b} \psi(t) \, dt = \int_{a}^{b} \psi_1(t) \, dt + i \int_{a}^{b} \psi_2(t) \, dt \quad (4.2) \]

Using (4.1) and (4.2) with \( \psi(t) = f(\gamma(t))\gamma'(t) \) we see that, in principle, the evaluation of line integrals is standard calculus.

**Example 4.2:** Let \( \Gamma \) denote a curve in \( \mathbb{C} \) from \( P \) to \( Q \) and let \( f(z) = c = \text{const.} \). Applying the Riemann sum definition one obtains:

\[ \int_{\Gamma} c \, dz = c(Q - P) . \]

**Example 4.3:** Using the parametrization

\[ \gamma(t) = e^{it}, \quad 0 \leq t \leq 2\pi , \]

of the unit circle \( C_1 \), obtain for any integer \( n \):

\[ \int_{C_1} z^n \, dz = 0 \quad \text{for} \quad n \neq -1 \quad \text{and} \quad \int_{C_1} \frac{dz}{z} = 2\pi i . \]

If one integrates over \( C_r \) with parametrization

\[ \gamma(t) = re^{it}, \quad 0 \leq t \leq 2\pi , \]

one obtains the same result. It is also easy to check that

\[ \int_{C_1} \bar{z} \, dz = 2\pi i . \]

Note that \( \bar{z} = 1/z \) for all \( z \in C_1 \).

**A Simple Estimate:** The estimate

\[ | \int_{\Gamma} f(z) \, dz | \leq \max_{z \in \gamma} | f(z) | \cdot \text{length}(\Gamma) \]

can be obtained using Riemann sums or (4.1).

To practically evaluate line integrals, the following result, which is analogous to the fundamental theorem of calculus, is very useful:
Theorem 4.1 Let $U \subset \mathbb{C}$ denote an open set and let $f : U \to \mathbb{C}$ be a continuous function. Suppose that $g : U \to \mathbb{C}$ is complex differentiable and $g' = f$ in $U$. If $\gamma : [a, b] \to U$ parametrizes a $C^1$-curve $\Gamma$, then
\[ \int_{\Gamma} f(z) \, dz = g(\gamma(b)) - g(\gamma(a)) = g(Q) - g(P). \]
Here $\Gamma$ goes from $P = \gamma(a)$ to $Q = \gamma(b)$. (If $\Gamma$ is piecewise $C^1$, the same result holds.)

Proof: We have
\[
\int_{\Gamma} f(z) \, dz = \int_{a}^{b} f(\gamma(t))\gamma'(t) \, dt \\
= \int_{a}^{b} g'(\gamma(t))\gamma'(t) \, dt \\
= \int_{a}^{b} \frac{d}{dt}(g(\gamma(t))) \, dt \\
= g(\gamma(b)) - g(\gamma(a))
\]

Definition: If $g \in H(U)$ and $g' = f$ in $U$, then $g$ is called an antiderivative or a primitive of $f$ in $U$.

The previous theorem says that we can evaluate line integrals of $f$ easily if we have an antiderivative $g$ of $f$. We will also see below that, conversely, line integrals can be used to construct an antiderivative of $f$ if $f$ is complex differentiable.

Example 4.4: Let $f(z) = z^n$ where $n$ is an integer, $n \neq -1$. If $n \geq 0$ then we can take $U = \mathbb{C}$ and $g(z) = \frac{1}{n+1} z^{n+1}$. If $n \leq -2$ we can take $U = \mathbb{C} \setminus \{0\}$ and again $g(z) = \frac{1}{n+1} z^{n+1}$. In both cases we have $g'(z) = f(z) = z^n$ in $U$. It follows that
\[ \int_{\Gamma} z^n \, dz = 0 \]
for any closed curve $\Gamma$ in $U$.

Example 4.5: Consider $f(z) = 1/z$ in $U = \mathbb{C} \setminus \{0\}$. Since
\[ \int_{C_1} \frac{dz}{z} = 2\pi i \neq 0 \] (4.3)
one obtains the following: There is no complex differentiable function $g : U \to \mathbb{C}$ with $g'(z) = 1/z$ in $U$. We have seen that we can extend the real function $g(x) = \ln x$ into the open slit plane
\[ \mathbb{C}^- = \mathbb{C} \setminus (-\infty, 0]. \]
The extended function is the main branch of the complex logarithm, $g(z) = \log z$. One can show that $g(z) = \log z$ is holomorphic in $\mathbb{C}^-$ and $g'(z) = \frac{1}{z}$ in
However, because of (4.3), one cannot extend $g(z) = \log z$ holomorphically into $U = \mathbb{C} \setminus \{0\}$.

**Example 4.6:** Let $\Gamma$ be a curve from $\gamma(a) = z_0$ to $\gamma(b) = z_1$. Then

$$
\int_{\gamma} z^3 \, dz = \frac{1}{4} z_1^4 - \frac{1}{4} z_0^4 .
$$

### 4.3 Goursat’s Lemma

Cauchy’s Integral Theorem can be stated, somewhat loosely, as follows:

**Theorem 4.2** Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be complex differentiable. Let $\Gamma$ be a closed $C^1$ curve in $U$ whose interior lies in $U$, i.e., $\Gamma$ does not surround any holes of $U$. Then

$$
\int_{\Gamma} f(z) \, dz = 0 .
$$

It is not easy to make precise what the interior of a closed curve is. (A possibility is to use the Jordan curve theorem, a result of topology that is notoriously difficult to prove.)

We prove Cauchy’s theorem first for the case that the curve $\Gamma$ is the boundary of a triangle $\Delta$ in $U$. The corresponding result is then known as Goursat’s Lemma.

Note that $f \in H(U)$ implies that $f$ is continuous in $U$. Therefore, $\int_{\Gamma} f(z) \, dz$ is defined for any $C^1$–curve $\Gamma$ in $U$.

**Theorem 4.3** (*Goursat’s Lemma*) Let $U \subset \mathbb{C}$ denote an open set and let $f : U \to \mathbb{C}$ be complex differentiable. Let $\Delta$ be a closed triangle, $\Delta \subset U$, with boundary curve $\partial \Delta$. Then we have

$$
\int_{\partial \Delta} f(z) \, dz = 0 .
$$

**Proof:** All the triangles below are assumed to be closed. Also, if $P \in \mathbb{C}$ and $\delta > 0$, then

$$
D(P, \delta) = \{ z \in \mathbb{C} : |z - P| < \delta \}
$$
denotes the open disk of radius $\delta$ centered at $P$.

Some simple observations:

1) If $\Delta$ is any triangle then

$$
w, z \in \Delta \quad \text{implies} \quad |w - z| \leq \text{length}(\partial \Delta) . \tag{4.4}
$$

2) If $\Delta$ is a triangle we subdivide it into four similar triangles by connecting the midpoints of the sides of $\Delta$. Then, if $\Delta'$ is any of the four subtriangles, we have

$$
\text{length}(\partial \Delta') = \frac{1}{2} \text{length}(\partial \Delta) . \tag{4.5}
$$
3) We use the abbreviation

\[ a(\Delta) = \int_{\partial \Delta} f(z) \, dz . \]

If \( \Delta_1, \Delta_2, \Delta_3, \Delta_4 \) are the four subtriangles of \( \Delta \) obtained by the subdivision, then

\[ a(\Delta) = \sum_{j=1}^{4} a(\Delta_j) . \]

4) Choose \( \Delta' \in \{ \Delta_1, \Delta_2, \Delta_3, \Delta_4 \} \) with

\[ |a(\Delta')| = \max_j |a(\Delta_j)| . \]

Then we have

\[ |a(\Delta)| \leq \sum_{j=1}^{4} |a(\Delta_j)| \leq 4|a(\Delta')| . \quad (4.6) \]

By subdividing \( \Delta' \) etc. we obtain a sequence of triangles \( \Delta^n \) with

\[ \Delta^{n+1} \subset \Delta^n \subset \ldots \subset \Delta \]

and

\[ \text{length}(\partial \Delta^n) = \frac{1}{2^n} \text{length}(\partial \Delta) \]

and

\[ |a(\Delta)| \leq 4|a(\Delta^1)| \leq 4^2|a(\Delta^2)| \leq 4^n|a(\Delta^n)| \]

Using a compactness argument, it is easy to show that there is a unique point \( P \in \Delta \subset U \) with

\[ \bigcap_{n=1}^{\infty} \Delta^n = \{ P \} . \]

We now use complex differentiability of \( f \) in \( P \) and write

\[ f(z) = f(P) + f'(P)(z - P) + R(z), \quad z \in U , \]

where

\[ R(z) = (z - P)\phi(z), \quad \phi \in C(U), \quad \phi(P) = 0 . \]

It is easy to see that
\[ \int_{\partial \Delta^n} f(z)dz = \int_{\partial \Delta^n} R(z)dz. \]

One obtains

\[
|a(\Delta)| \leq 4^n |a(\Delta^n)| \\
= 4^n \left| \int_{\partial \Delta^n} f(z)dz \right| \\
= 4^n \left| \int_{\partial \Delta^n} R(z)dz \right| \\
\leq 4^n \text{length}(\partial \Delta^n) \cdot \max\{|R(z)| : z \in \partial \Delta^n\} \\
\leq 4^n \text{length}(\partial \Delta^n) \cdot \text{length}(\partial \Delta^n) \cdot \max\{|\phi(z)| : z \in \partial \Delta^n\} \\
= \text{length}(\partial \Delta) \cdot \text{length}(\partial \Delta) \cdot \max\{|\phi(z)| : z \in \partial \Delta^n\}
\]

Thus, we have shown that

\[ |\int_{\partial \Delta} f(z)dz| = |a(\Delta)| \leq (\text{length}(\partial \Delta))^2 \cdot \max\{|\phi(z)| : z \in \partial \Delta^n\}. \]

Given \( \varepsilon > 0 \) there is \( \delta > 0 \) so that

\[ |\phi(z)| \leq \varepsilon \quad \text{if} \quad |z - P| < \delta. \]

Also, if \( n \) is large enough, then

\[ \Delta^n \subset D(P, \delta). \]

Therefore, given \( \varepsilon > 0 \) there is \( n \) with

\[ |\int_{\partial \Delta} f(z)dz| \leq (\text{length}(\partial \Delta))^2 \cdot \varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, we obtain that the integral is zero. \( \diamond \)

### 4.4 Construction of a Primitive in a Disk

We now use Goursat’s Lemma to construct an antiderivative of a given function \( f \in H(U) \) where \( U \) is an open disk.

**Theorem 4.4** Let \( U = D(P, r) = \{z \in \mathbb{C} : |z - P| < r\} \) denote an open disk. If \( f \in H(U) \) then there is \( g \in H(U) \) with \( g' = f \).

**Proof:** For any \( z_0 \in U \) let \( \Gamma_{z_0} \) denote the straight line from \( P \) to \( z_0 \) and define

\[ g(z_0) = \int_{\Gamma_{z_0}} f(z)dz. \]

We claim that \( g \in H(U) \) and \( g'(z_0) = f(z_0) \) for every \( z_0 \in U \).
Fix $z_0 \in U$ and let $\varepsilon = r - \vert P - z_0 \vert$, thus $\varepsilon > 0$. If $\vert h \vert < \varepsilon$ then

$$
\vert P - (z_0 + h) \vert < \vert P - z_0 \vert + \varepsilon = r,
$$

thus $z_0 + h \in U$. Also,

$$
g(z_0 + h) = \int_{\Gamma_{z_0 + h}} f(z) \, dz.
$$

Let $C_h$ denote the straight line from $z_0$ to $z_0 + h$. We have, by Goursat’s Lemma:

$$
\int_{\Gamma_{z_0 + h}} f(z) \, dz = \int_{\Gamma_{z_0}} f(z) \, dz + \int_{C_h} f(z) \, dz,
$$

thus

$$
g(z_0 + h) = g(z_0) + \int_{C_h} f(z) \, dz.
$$

Since $C_h$ has the parametrization

$$
\gamma(t) = z_0 + th, \quad 0 \leq t \leq 1,
$$

one obtains

$$
g(z_0 + h) - g(z_0) = \int_0^1 f(z_0 + th) h \, dt.
$$

Therefore,

$$
\frac{1}{h}(g(z_0 + h) - g(z_0)) = \int_0^1 f(z_0 + th) \, dt =: \text{Int}(h).
$$

We write

$$
f(z_0 + th) = f(z_0) + \left( f(z_0 + th) - f(z_0) \right).
$$

Therefore,

$$
\text{Int}(h) = f(z_0) + R(h)
$$

with

$$
\vert R(h) \vert \leq \max_{0 \leq t \leq 1} \vert f(z_0 + th) - f(z_0) \vert.
$$

Continuity of $f$ in $z_0$ implies that $\vert R(h) \vert \to 0$ as $h \to 0$. This shows that $g'(z_0) = f(z_0)$. ∘

**Remark:** Let $U \subset \mathbb{C}$ be an open set that is star–shaped with respect to $P \in U$. Let $f \in H(U)$. The same proof as above can be used to construct a primitive $g$ of $f$ in $U$.  

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4.5 Cauchy’s Theorem in a Disk

**Theorem 4.5** Let \( U = D(P, r) \) and let \( f \in H(U) \). If \( \Gamma \) is a closed \( C^1 \) curve in \( U \), then

\[
\int_{\Gamma} f(z) \, dz = 0.
\]

**Proof:** Using the previous theorem, there is \( g \in H(U) \) with \( g' = f \). Then, if \( \Gamma \) is a curve in \( U \) from \( P \) to \( Q \), we have seen that

\[
\int_{\Gamma} f(z) \, dz = g(Q) - g(P).
\]

If \( \Gamma \) is closed then \( Q = P \), and the integral is zero. \( \diamond \)

4.6 Extensions

If \( U \subset \mathbb{C} \) is any open set and \( f \in H(U) \), will it hold that

\[
\int_{\Gamma} f(z) \, dz = 0 \tag{4.7}
\]

whenever \( \Gamma \) is a closed curve in \( U \)? The example

\[
U = \mathbb{C} \setminus \{0\}, \quad f(z) = \frac{1}{z}, \quad \Gamma = C_1,
\]

shows that the answer is no, in general, since \( \int_{C_1} dz/z = 2\pi i \). However, if \( U \) is simply connected, then (4.7) does hold whenever \( f \in H(U) \) and \( \Gamma \) is a closed curve in \( U \). We will explain this below.

**Definition:** An open set \( U \subset \mathbb{C} \) is called connected if for any two points \( P, Q \in U \) there is a curve in \( U \) from \( P \) to \( Q \). An open connected set is called a region.

Let \( \Gamma_0 \) and \( \Gamma_1 \) be two \( C^1 \)–curves in \( U \) from \( P \) to \( Q \) parametrized by \( \gamma_0(t) \) and \( \gamma_1(t), a \leq t \leq b \). The curve \( \Gamma_0 \) is called homotopic to \( \Gamma_1 \) in \( U \) with fixed endpoints if there exists a continuous function

\[
\gamma : [0, 1] \times [a, b] \to U
\]

with:

\[
\begin{align*}
\gamma(0, t) &= \gamma_0(t), \quad a \leq t \leq b \\
\gamma(1, t) &= \gamma_1(t), \quad a \leq t \leq b \\
\gamma(s, a) &= P, \quad 0 \leq s \leq 1 \\
\gamma(s, b) &= Q, \quad 0 \leq s \leq 1 \\
\gamma(s, \cdot) &\in C^1[a, b] \quad \text{for} \quad 0 \leq s \leq 1.
\end{align*}
\]

One can show:
Theorem 4.6  Let $U$ be a region in $\mathbb{C}$ and let $\Gamma_0$ and $\Gamma_1$ be two $C^1$ curves in $U$ which are homotopic in $U$ with fixed endpoints. If $f \in H(U)$ then

$$\int_{\Gamma_0} f(z) \, dz = \int_{\Gamma_1} f(z) \, dz .$$

Definition: A region $U$ in $\mathbb{C}$ is called simply connected if every closed curve $\Gamma$ in $U$, which goes from a point $P \in U$ to itself, is homotopic in $U$ with fixed endpoints to the constant curve $P$.

Theorem 4.7  Let $U$ be a simply connected region in $\mathbb{C}$ and let $f \in H(U)$. If $\Gamma$ is any closed $C^1$ curve in $U$, then

$$\int_{\Gamma} f(z) \, dz = 0 .$$

Notations: Let $P \in \mathbb{C}$ and let $r > 0$. We set

\[
D = D(P, r) = \{z : |z - P| < r\} \\
\bar{D} = \bar{D}(P, r) = \{z : |z - P| \leq r\} \\
\partial D = \partial D(P, r) = \{z : |z - P| = r\}
\]

With

$$\gamma(t) = \gamma(t, P, r) = P + re^{it}, \quad 0 \leq t \leq 2\pi ,$$

we denote the standard parametrization of the boundary curve of $D(P, r)$.

4.7 Cauchy’s Integral Formula in a Disk

Theorem 4.8  Let $U \subset \mathbb{C}$ be open and let $f \in H(U)$. Let $\bar{D} = \bar{D}(P, r) \subset U$ and let $\partial D$ denote the boundary curve of $D$. Then we have for all $z_0 \in \bar{D}$:

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} \, dz . \quad (4.8)$$

Proof: Deform $\partial D$ to a small curve $\Gamma_\varepsilon$ about $z_0$ with parametrization

$$\gamma_\varepsilon(t) = z_0 + \varepsilon e^{it}, \quad 0 \leq t \leq 2\pi .$$

Write

$$f(z) = f(z) - f(z_0) + f(z_0)$$

and

$$\frac{f(z)}{z - z_0} = \frac{f(z) - f(z_0)}{z - z_0} + \frac{f(z_0)}{z - z_0}, \quad z \neq z_0 .$$

Integrate over $\Gamma_\varepsilon$ to obtain
\[
\int_{\partial D} \frac{f(z)}{z - z_0} \, dz = \int_{\Gamma_\varepsilon} \frac{f(z) - f(z_0)}{z - z_0} \, dz + 2\pi i f(z_0) .
\]

Use that

\[
\left| \frac{f(z) - f(z_0)}{z - z_0} \right| \leq C \quad \text{for} \quad 0 < |z - z_0| \leq \varepsilon_0 .
\]

Here \( C \) is a constant depending on \( z_0 \), but not on \( z \). Then obtain for \( \varepsilon \to 0 \):

\[
\int_{\partial D} \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0) .
\]

The formula (4.8) follows. \( \diamond \)

With a change of notation, the formula (4.8) is also written as

\[
f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta , \quad z \in D .
\] (4.9)
5 Holomorphic Functions Written As Power Series

In this section we will prove the following important result.

**Theorem 5.1** Let \( U \) denote an open subset of \( \mathbb{C} \) and let \( f : U \to \mathbb{C} \) be a holomorphic function. Let \( z_0 \in U \) be arbitrary and assume

\[ D(z_0, \rho) \subset U, \quad \rho > 0. \]

Then there exist complex numbers \( a_0, a_1, \ldots \) so that

\[ f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{for} \quad |z - z_0| < \rho. \quad \tag{5.1} \]

The series converges absolutely for every \( z \in D(z_0, \rho) \) and the convergence is uniform for \( |z - z_0| \leq r \) if \( 0 < r < \rho \) is fixed.

We will see that the theorem follows rather easily from Cauchy’s Integral Theorem and convergence of the geometric series. Also, we will see below that the coefficients \( a_j \) of the power series (5.1) are uniquely determined.

The theorem says that any holomorphic function \( f \) can locally be written as a power series. Furthermore, the power series expansion is valid in any open disk \( D(z_0, \rho) \) which lies completely in the region \( U \) where \( f \) is holomorphic.

5.1 The Geometric Series

For \( w \in \mathbb{C} \) with \( |w| < 1 \) the geometric series converges,

\[ \sum_{j=0}^{\infty} w^j = \frac{1}{1 - w}. \quad \tag{5.2} \]

The convergence is absolute since \( \sum_{j=0}^{\infty} |w|^j \) converges if \( |w| < 1 \). Also, if \( 0 < r < 1 \) is fixed, then the convergence is uniform for all \( w \) with \( |w| \leq r \).

Let us recall what uniform convergence means here. We use the following notation: If \( V \subset \mathbb{C} \) and if \( g : V \to \mathbb{C} \) is any bounded function, then let

\[ |g|_V := \sup \{|g(w)| : w \in V\}. \]

In case of the geometric series, let

\[ s_n(w) = \sum_{j=0}^{n} w^j = \frac{1 - w^{n+1}}{1 - w} \]

denote the \( n \)-th partial sum. We have, for \( |w| \leq r < 1 \):

\[ |s_n(w) - \frac{1}{1 - w}| = \frac{|w|^{n+1}}{|1 - w|} \leq \frac{r^{n+1}}{|1 - r|} \]
The estimate is attained for \( w = r \). Therefore,

\[
|s_n(w) - \frac{1}{1 - w}|_{V_r} = \frac{r^{n+1}}{1 - r}, \quad V_r = D(0, r).
\]

Thus, if \( 0 < r < 1 \) is fixed, then

\[
\max_{|w| \leq r} |s_n(w) - \frac{1}{1 - w}| \to 0 \quad \text{as} \quad n \to \infty.
\]

This means that the convergence in formula (5.2) is uniform for \(|w| \leq r\) if \( 0 < r < 1 \) is fixed.

### 5.2 Expansion Using the Geometric Series

Let \( U \subset \mathbb{C} \) be an open set and let \( f : U \to \mathbb{C} \) be holomorphic. Let \( D = D(z_0, r) \) and assume that \( \overline{D} \subset U \). Let \( \partial D \) denote the boundary curve of \( D \).

We have, for all \( z \in D \),

\[
f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta.
\]

Let us first assume that \( z_0 = 0 \). Then we have

\[
|z| < |\zeta| = r
\]

and can write

\[
\zeta - z = \zeta \left(1 - \frac{z}{\zeta}\right) \quad \text{with} \quad \left|\frac{z}{\zeta}\right| = \frac{|z|}{r} < 1,
\]

thus

\[
\frac{1}{\zeta - z} = \frac{1}{\zeta} \cdot \frac{1}{1 - \frac{z}{\zeta}} = \frac{1}{\zeta} \sum_{j=0}^{\infty} \left(\frac{z}{\zeta}\right)^j.
\]

For fixed \( z \in D \) the convergence of the series is uniform for \( \zeta \in \partial D \). Therefore, we may exchange the order of integration and summation to obtain

\[
f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]

\[
= \frac{1}{2\pi i} \sum_{j=0}^{\infty} z^j \int_{\partial D} \frac{f(\zeta)}{\zeta^{j+1}} \, d\zeta
\]

\[
= \sum_{j=0}^{\infty} a_j z^j
\]

with
\[ a_j = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta^{j+1}} \, d\zeta. \]

Clearly, the numbers \( a_j \) do not depend on \( z \in D \). We have written \( f(z) \) as a convergent power series in \( z \) for \( z \in D \).

In the general case, where \( z_0 \) is not assumed to be \( z_0 = 0 \), we write

\[ \zeta - z = (\zeta - z_0) - (z - z_0) = (\zeta - z_0) \left(1 - \frac{z - z_0}{\zeta - z_0}\right) \]

and find that

\[ \frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \sum_{j=0}^{\infty} \frac{(z - z_0)^j}{(\zeta - z_0)^j}. \]

In the same ways as for \( z_0 = 0 \) one obtains that

\[ f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \]

with

\[ a_j = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta. \]

The numbers \( a_j \) do not depend on \( z \in D \).

**Theorem 5.2** Let \( U \subset \mathbb{C} \) be an open set and let \( f : U \to \mathbb{C} \) be holomorphic.

Let \( D = D(z_0, r) \), and assume that \( \bar{D} \subset U \). With \( \partial D \) we denote the boundary curve of \( D \). We have, for all \( z \in D \),

\[ f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \]

with

\[ a_j = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta. \]

This shows that any function \( f \in H(U) \) can locally be written as a power series. If \( D(z_0, r) \subset U \) then the power series with expansion point \( z_0 \) converges to \( f(z) \) at least in \( D(z_0, r) \).

We now make a further fine point. Let \( f \in H(U) \) and consider an open disk \( D(z_0, \rho) \). Assume

\[ D(z_0, \rho) \subset U, \quad \rho > 0. \]

Fix \( 0 < r < \rho \). Set

\[ a_j = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta. \]
Our previous considerations show that

\[ f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{for} \quad |z - z_0| < r . \]

It is clear, by Cauchy's integral theorem, that the coefficients \( a_j \) are independent of \( r \). Therefore, since the number \( r \) with \( 0 < r < \rho \) was arbitrary, one obtains that

\[ f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{for} \quad |z - z_0| < \rho \]

if \( D(z_0, \rho) \subset U \).
6 Functions Defined by Power Series

By the main result of the previous section, Theorem 5.1, any holomorphic function \( f(z) \) can locally be written as a convergent power series. In this section, we prove the converse: A function which can be written as a convergent power series in a disc is holomorphic in this disc.

More is true:

**Theorem 6.1** Assume that the power series \( \sum a_j z^j \) has radius of convergence \( r \) where \( 0 < r \leq \infty \). Then the function

\[
f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad |z| < r ,
\]

is holomorphic in \( D(0, r) \). Furthermore,

\[
f'(z) = \sum_{j=1}^{\infty} j a_j z^{j-1}, \quad |z| < r ,
\]

and the series \( \sum ja_j z^{j-1} \) also has radius of convergence equal to \( r \).

The theorem says, among others, that the power series representation (6.1) of \( f(z) \) can be differentiated term by term to give the power series representation (6.2) of \( f'(z) \). In other words, two limit processes, differentiation and summation, can be exchanged for power series.

We review some simple properties of \( \limsup s_j \) of a sequence of real numbers \( s_j \) and prove Hadamard’s formula for the radius \( r \) of convergence of a power series,

\[
r = \frac{1}{\limsup_{j \to \infty} |a_j|^{1/j}}.
\]

6.1 Remarks on the Exchange of Limits

Let us recall the basic concept of a uniformly convergent sequence of functions \( s_n : \Omega \to \mathbb{C} \), where \( \Omega \subset \mathbb{R}^m \) is a nonempty set. Let \( s : \Omega \to \mathbb{C} \) be a function. The sequence \( s_n \) converges uniformly on \( \Omega \) to \( s \) if for any \( \varepsilon > 0 \) there is \( N \in \mathbb{N} \) with

\[
|s_n(x) - s(x)| < \varepsilon \quad \text{for} \quad n \geq N \quad \text{and} \quad \text{for all} \quad x \in \Omega .
\]

We know from real analysis that the uniform limit of a sequence of continuous functions is continuous:

**Theorem 6.2** If \( s_n \in C(\Omega) \) for all \( n \) and if \( s_n \) converges to \( s \) uniformly on \( \Omega \), then \( s \in C(\Omega) \).
Proof: Fix any $z_0 \in \Omega$, and let $\varepsilon > 0$ be given. There is $N \in \mathbb{N}$ so that

$$\sup_{z \in \Omega} |s_N(z) - s(z)| < \varepsilon/3.$$ 

Use the continuity of $s_N$: There is $\delta > 0$ so that $|s_N(z) - s_N(z_0)| < \varepsilon/3$ if $z \in \Omega$ and $|z - z_0| < \delta$. Then, using the triangle inequality,

$$|s(z) - s(z_0)| \leq |s(z) - s_N(z)| + |s_N(z) - s_N(z_0)| + |s_N(z_0) - s(z_0)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for $z \in \Omega$ with $|z - z_0| < \delta$. ♦

Under the assumptions of the above theorem, let $x_k, x_0 \in \Omega$ and let $x_k \to x_0$ as $k \to \infty$. Consider the values $s_n(x_k)$. The above theorem, stating the continuity of $s$, expresses that the limits $k \to \infty$ and $n \to \infty$ can be exchanged. Precisely, first take the limit $k \to \infty$, then $n \to \infty$:

$$s_n(x_k) \to s_n(x_0) \quad \text{as} \quad k \to \infty; \quad s_n(x_0) \to s(x_0) \quad \text{as} \quad n \to \infty.$$ 

Now exchange the limit processes and first take the limit $n \to \infty$, then $k \to \infty$:

$$s_n(x_k) \to s(x_k) \quad \text{as} \quad n \to \infty; \quad s(x_k) \to s(x_0) \quad \text{as} \quad k \to \infty.$$ 

Validity of the last limit, $s(x_k) \to s(x_0)$, holds precisely because of continuity of the function $s$ at the point $x_0$.

A natural question is: Suppose that, under the above assumptions, $s_n$ and $s$ are smooth function, for example infinitely often differentiable functions. Is it allowed to exchange differentiation with taking the limit $n \to \infty$? In real analysis, the answer is No, in general. The sequence $s_n(x, y) = \frac{1}{n} \cos(n^2(x + y))$ gives a simple example. Clearly, $s_n$ converges uniformly to $s(x, y) \equiv 0$, but the derivatives of $s_n$ do not converge to the derivatives of $s$ as $n \to \infty$.

It is, therefore, remarkable and important that for functions defined by power series, $f(z) = \sum_j a_j(z - P)^j$, one can differentiate term by term within the open disk of convergence. We will prove this in Section 6.6.

6.2 The Disk of Convergence of a Power Series

An expression

$$\sum_{j=0}^{\infty} a_j(z - z_0)^j$$

is called a power series centered at $z_0$. We often take $z_0 = 0$ for convenience.

The following simple result is very important.
Lemma 6.1 (Abel) Suppose that
\[ \sum_{j=0}^{\infty} a_j z^j \]
converges for some \( z \neq 0 \). If \( |w| < |z| \), then the series
\[ \sum_{j=0}^{\infty} a_j w^j \]
converges absolutely. If a number \( r \) with \( 0 < r < |z| \) is fixed, then the convergence is uniform for all \( w \) with \( |w| \leq r \).

Proof: Since \( |a_j||z|^j \to 0 \) as \( j \to \infty \) there exists \( M > 0 \) so that
\[ |a_j||z|^j \leq M \quad \text{for all } \quad j = 0, 1, \ldots \]
Also,
\[ q := \frac{|w|}{|z|} < 1 \]
Therefore,
\[ |a_j||w|^j = |a_j||z|^j\left(\frac{|w|}{|z|}\right)^j \leq Mq^j. \]
Since \( \sum q^j \) converges, the claim follows by the Comparison Theorem. \( \diamond \)

Definition 6.1 For any given power series,
\[ \sum_{j=0}^{\infty} a_j z^j \]
define the radius \( r \) of convergence as follows:
\[ r := \sup\{|z| : \sum_{j=0}^{\infty} a_j z^j \text{ converges}\}. \]
Clearly, we have \( 0 \leq r \leq \infty \).

There are three cases:

a) \( r = \infty \): In this case, by the previous lemma, the series converges for every \( z \). (It defines an entire function.)

b) \( r = 0 \): In this case the series converges only for \( z = 0 \).

c) \( 0 < r < \infty \): In this case, the series converges absolutely for \( |z| < r \) and diverges for \( |z| > r \).

In many simple cases, one can obtain \( r \) as follows:
Theorem 6.3  Let \( \sum_{j=0}^{\infty} a_j z^j \) denote a power series and assume \( a_j \neq 0 \) for all large \( j \). If

\[
\left| \frac{a_{j+1}}{a_j} \right| \to q \quad \text{as} \quad j \to \infty
\]

with \( 0 \leq q \leq \infty \), then the radius of convergence is

\[
 r = \frac{1}{q}.
\]

Here one uses the conventions \( 1/\infty = 0 \) and \( 1/0 = \infty \).

Proof:  Let \( \alpha_j = a_j z^j, z \neq 0 \). We have

\[
\left| \frac{\alpha_{j+1}}{\alpha_j} \right| \to q|z| \quad \text{as} \quad j \to \infty.
\]

By the quotient criterion, the power series \( \sum a_j z^j \) converges if \( q|z| < 1 \) and diverges if \( q|z| > 1 \). This implies that \( r = 1/q \).

Example 6.1:  For \( \sum_{j=0}^{\infty} j! z^j \) the radius of convergence is \( r = 0 \) by Theorem 6.3.

Example 6.2:  For \( \sum_{j=0}^{\infty} \frac{1}{j!} z^j \) the radius of convergence is \( r = \infty \) by Theorem 6.3. We have

\[
\sum_{j=0}^{\infty} \frac{1}{j!} z^j = e^z, \quad z \in \mathbb{C}.
\]

Example 6.3:  For \( \sum_{j=0}^{\infty} z^j \) the radius of convergence is \( r = 1 \) by Theorem 6.3. We have

\[
\sum_{j=0}^{\infty} z^j = \frac{1}{1 - z}, \quad |z| < 1.
\]

Example 6.4:  For \( \sum_{j=1}^{\infty} j z^j \) the radius of convergence is \( r = 1 \) by Theorem 6.3. We have

\[
\sum_{j=1}^{\infty} j z^j = z \sum_{j=0}^{\infty} \frac{d}{dz} z^j
\]

\[
= z \frac{d}{dz} \sum_{j=0}^{\infty} z^j
\]

\[
= \frac{z}{(1 - z)^2}
\]

The fact that we can take \( d/dz \) out of the infinite sum will be justified below.

Example 6.5:  Taylor expansion of the real function

\[
f(x) = \ln(1 + x), \quad x > -1,
\]
about $x = 0$ leads to the series

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} x^j.$$  

The radius of convergence of the corresponding complex series is $r = 1$ by Theorem 6.3. This suggest that

$$\log(1 + z) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} z^j, \quad |z| < 1,$$

where $\log$ denotes the main branch of the complex logarithm. In other words, if $w = 1 + z$,

$$\log w = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} (w - 1)^j, \quad |w - 1| < 1.$$  

We will see that this expansion is indeed valid.

**Example 6.6:** The Taylor expansion of the real function

$$f(x) = \frac{1}{1 + x^2}$$

about $x = 0$ can best be obtained using the geometric series: With $\varepsilon = -x^2$ we have for $|x| < 1$:

$$\frac{1}{1 + x^2} = \frac{1}{1 - \varepsilon} = \sum_{j=0}^{\infty} \varepsilon^j = \sum_{j=0}^{\infty} (-1)^j x^{2j} = 1 - x^2 + x^4 \ldots$$

The corresponding complex series

$$\sum_{j=0}^{\infty} (-1)^j z^{2j} = \frac{1}{1 + z^2}$$

has the radius of convergence equal to 1.

### 6.3 Remarks on Lim Sup

Let $s_j$ denote a sequence of real numbers. One defines

$$L := \limsup_{j \to \infty} s_j := \lim_{n \to \infty} \left( \sup_{j \geq n} s_j \right).$$

Let us first show that the limit always exists. Set
\[ L_n := \sup_{j \geq n} s_j . \]

**Case 1:** The sequence \( s_j \) is not bounded from above. In this case \( L_n = \infty \) for all \( n \) and, therefore, \( L = \infty \).

**Case 2:** Assume \( s_j \to -\infty \). In this case \( L_n \to -\infty \), thus \( L = -\infty \).

**Case 3:** In all other cases, the numbers \( L_n \) form a monotonically decreasing sequence and

\[ L \leq \ldots \leq L_{n+1} \leq L_n, \quad n = 1, 2, \ldots \]

with finite \( L \).

Therefore, \( L = \lim \sup_{j \to \infty} s_j \) always exists if we allow the limits \( \pm \infty \):

\[ L \in \mathbb{R} \cup \{\pm \infty\} . \]

**Lemma 6.2** If

\[ L = \lim \sup_{j \to \infty} s_j \]

then the sequence \( s_j \) has a subsequence \( s_{j_n} \) converging to \( L \).

**Proof:** Exercise.

**Lemma 6.3** Set

\[ L = \lim \sup_{j \to \infty} s_j . \]

Let \( s_{j_n} \) be a convergent subsequence of \( s_j \) with limit \( K \). Then \( K \leq L \).

**Proof:** Exercise.

The last two lemmas say that \( L = \lim \sup_{j \to \infty} s_j \) is the largest limit of all the convergent subsequences of \( s_j \).

**Lemma 6.4** Let \( a_j > 0 \) for all \( j \) and set

\[ \lim \sup_{j \to \infty} \frac{a_{j+1}}{a_j} = Q_1 , \]

\[ \lim \inf_{j \to \infty} \frac{a_{j+1}}{a_j} = Q_2 , \]

\[ \lim \sup_{j \to \infty} a_j^{1/j} = L_1 , \]

\[ \lim \inf_{j \to \infty} a_j^{1/j} = L_2 . \]

Then we have

\[ Q_2 \leq L_2 \leq L_1 \leq Q_1 . \]
Proof: We show that $L_1 \leq Q_1 = Q$. Let $q_n = a_{n+1}/a_n$. Let $\varepsilon > 0$. There is $N = N \varepsilon$ so that $q_n \leq Q + \varepsilon$ for all $n \geq N$. Thus,

$$a_{n+1} \leq (Q + \varepsilon)a_n, \quad n \geq N.$$ 

It follows that

$$a_{N+j} \leq (Q + \varepsilon)^ja_N = (Q + \varepsilon)^{N+j}a_N, \quad j \geq 0.$$ 

Therefore,

$$a_{N+j}^{1/(N+j)} \leq (Q + \varepsilon)^{1/(N+j)}M^{1/(N+j)}, \quad j \geq 0.$$ 

Since $M^{1/(N+j)} \to 1$ as $j \to \infty$ it follows that

$$a_k^{1/k} \leq Q + 2\varepsilon, \quad k \geq K(\varepsilon).$$ 

This implies that $L_1 \leq Q + 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, one obtains that $L_1 \leq Q$.

A simple implication of the previous lemma is:

Lemma 6.5 Let $a_j > 0$ for all $j$. If

$$\lim_{j \to \infty} \frac{a_{j+1}}{a_j} = Q$$

then the sequence

$$a_j^{1/j}$$

also converges to $Q$.

6.4 The Radius of Convergence: Hadamard’s Formula

Hadamard gave a formula for the radius of convergence $r$ of a power series $\sum a_jz^j$. The formula has more theoretical than practical value. In other words, one often uses it in proofs, but it is less useful for computing $r$.

Theorem 6.4 (Hadamard) Let $\sum a_jz^j$ have radius of convergence equal to $r$ where $0 \leq r \leq \infty$. Then we have:

$$\frac{1}{r} = \limsup_{j \to \infty} |a_j|^{1/j}$$

with the conventions

$$\frac{1}{0} = \infty, \quad \frac{1}{\infty} = 0.$$
Remarks on $\limsup_{j \to \infty} s_j$: Let $s_j$ be a sequence of nonnegative real numbers. a) If $s_j$ is unbounded, then

$$\limsup_{j \to \infty} s_j = \infty.$$  

b) Let $s_j$ be bounded, $s_j \leq M$. Let $S$ be the set of all limits of subsequences of $s_j$. Then $S$ is not empty and $S \subset [0, M]$. One defines

$$\limsup_{j \to \infty} s_j = \sup S.$$  

One can show that $\sup S$ is an element of $S$. Thus, $\limsup_{j \to \infty} s_j$ is the largest limit of any subsequence of $s_j$.

Proof of Hadamard’s formula: Let $r$ denote the radius of convergence of the power series $\sum a_j z^j$. Set

$$q = \limsup_{j \to \infty} |a_j|^{1/j}.$$  

Assume $0 < q < \infty$.

a) Let $|z| > 1/q$. Then we have $q|z| > 1$, thus

$$|a_j|^{1/j}|z| > 1$$

for infinitely many $j$. It follows that

$$|a_j z^j| > 1$$

for infinitely many $j$. The series

$$\sum a_j z^j$$

diverges. This yields that $r \leq 1/q$.

b) Let $|z| < 1/q$. Define $d$ by

$$|z| = \frac{d}{q},$$

thus $0 \leq d < 1$. Choose $c$ with

$$d < c < 1.$$  

We have

$$\frac{q}{c} > q = \limsup_{j \to \infty} |a_j|^{1/j},$$

thus

$$|a_j|^{1/j} < \frac{q}{c}$$

for all $j \geq J$.

Therefore, for all $j \geq J$,  

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\[
|a_jz^j| = |a_j\left(\frac{d}{q}\right)^j \\
\leq \left(\frac{q}{c}\right)^j \left(\frac{d}{q}\right)^j \\
= \left(\frac{d}{c}\right)^j
\]

Since \(d/c < 1\), the series \(\sum a_jz^j\) converges. Since \(z\) with \(|z| < 1/q\) was arbitrary, this proves that \(r \geq 1/q\). ⋆

6.5 Matrix–Valued Analytic Functions and Hadamard’s Formula for the Spectral Radius

In this section we assume that \(\| \cdot \|\) denotes a norm on \(\mathbb{C}^m\). The corresponding matrix norm for matrices \(A \in \mathbb{C}^{m \times m}\) is

\[
\|A\| = \max\{\|Au\| : u \in \mathbb{C}^m, \|u\| = 1\}.
\]

Let \(A_j \in \mathbb{C}^{m \times m}\) denote a sequence a square matrices. We consider the series

\[
\sum_{j=0}^{\infty} z^j A_j \tag{6.4}
\]

with variable \(z \in \mathbb{C}\). The partial sums are the matrices

\[
S_n(z) = \sum_{j=0}^{n} z^j A_j . \tag{6.5}
\]

As \(n \to \infty\), we can consider convergence of \(S_n(z)\) in the space of matrices \(\mathbb{C}^{m \times m}\) or, alternatively, we can consider convergence of the \(m^2\) scalar series

\[
\sum_{j=0}^{\infty} z^j (A_j)_{\mu\nu}, \quad 1 \leq \mu, \nu \leq m , \tag{6.6}
\]

where \((A_j)_{\mu\nu}\) denotes the matrix entries of \(A_j\).

With arguments as in the proof of Theorem 6.4, the following result can be shown:

**Theorem 6.5** Set

\[
q = \limsup_{j \to \infty} \|A_j\|^{1/j} .
\]

a) If \(|z| < \frac{1}{q}\) then the series (6.4) converges in \(\mathbb{C}^{m \times m}\). If \(|z| > \frac{1}{q}\) then the series (6.4) diverges in \(\mathbb{C}^{m \times m}\).

b) If \(|z| < \frac{1}{q}\) then the \(m^2\) scalar series (6.6) converges in \(\mathbb{C}\). If \(|z| > \frac{1}{q}\) then at least one of the \(m^2\) scalar series (6.6) diverges.
Of particular interest if the case where \( A \in \mathbb{C}^{m \times m} \) is a fixed matrix and \( A_j = A^j \), i.e., \( A_0 = I, A_1 = A, A_2 = A^2 \), etc.

We denote the set of eigenvalues of \( A \) by
\[
\sigma(A) = \{\lambda_1, \ldots, \lambda_k\}
\]
and denote the spectral radius of \( A \) by
\[
\rho(A) = \max_j |\lambda_j|.
\]

**Theorem 6.6**  
*For any matrix \( A \in \mathbb{C}^{m \times m} \) we have*
\[
\rho(A) = \lim_{j \to \infty} \|A^j\|^{1/j} = \inf_j \|A^j\|^{1/j}.
\]  
(6.7)

**Proof:**  
First note that \( \rho(A) \leq \|A\| \) and  
\[
(\rho(A))^j = \rho(A^j) \leq \|A^j\|,
\]
thus  
\[
\rho(A) \leq \|A^j\|^{1/j}, \quad j = 1, 2, \ldots.
\]

Let \( \varepsilon > 0 \) be arbitrary and set
\[
B = B_\varepsilon = \frac{1}{\rho(A) + \varepsilon} A.
\]

Then \( \rho(B) < 1 \) and, by a theorem of linear algebra, \( B^j \to 0 \) as \( j \to \infty \). In particular, there exists \( J = J_\varepsilon \in \mathbb{N} \) with
\[
\|B^j\| \leq 1 \quad \text{for} \quad j \geq J.
\]

This says that
\[
\frac{1}{(\rho(A) + \varepsilon)^j} \|A^j\| \leq 1 \quad \text{for} \quad j \geq J.
\]

Therefore,
\[
\rho(A) \leq \|A^j\|^{1/j} \leq \rho(A) + \varepsilon \quad \text{for} \quad j \geq J_\varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, the formula (6.7) is shown. \( \diamond \)

**The Resolvent.**  
Let \( A \in \mathbb{C}^{m \times m} \). The analytic function
\[
R(\lambda) = (A - \lambda I)^{-1}, \quad \lambda \in \mathbb{C} \setminus \sigma(A),
\]
with values in \( \mathbb{C}^{m \times m} \) is called the resolvent of \( A \). (By Cramer’s rule, we know that the matrix entries \( R_{\mu\nu}(z) \) are rational functions of \( \lambda \). The eigenvalues of \( A \) are the only possible poles.)

We now use the power series
\[
\sum_{j=0}^{\infty} z^j A^j = (I - zA)^{-1} \quad \text{for} \quad |z| < \frac{1}{\rho(A)},
\]
to expand the resolvent \( R(\lambda) \) about \( \lambda = \infty \). To this end, let \(|\lambda| > \rho(A)\). Then we have

\[
R(\lambda) = \left( -\lambda I - \frac{1}{\lambda} A \right)^{-1} = -\frac{1}{\lambda} \sum_{j=0}^{\infty} \frac{1}{\lambda^j} A^j = -\sum_{j=0}^{\infty} \frac{1}{\lambda^{j+1}} A^j.
\]

According to terminology introduced below, the formula

\[
R(\lambda) = -\sum_{j=0}^{\infty} \frac{1}{\lambda^{j+1}} A^j, \quad |\lambda| > \rho(A),
\]

gives the Laurent expansion of the resolvent \( R(\lambda) \) in the region

\[
\rho(A) < |\lambda| < \infty.
\]

### 6.6 Differentiation of Power Series

Let \( \sum_{j=0}^{\infty} a_j z^j \) have radius of convergence equal to \( r > 0 \). Then the function

\[
f(z) = \sum_{j=0}^{\infty} a_j z^j = a_0 + a_1 z + a_2 z^2 + \ldots
\]
is defined for \( z \in D = D(0, r) \). Also, the convergence is uniform on any compact subset of \( D \). Therefore, \( f(z) \) is continuous in \( D \). More is true as we will show below: The formally differentiated power series has the same radius of convergence as the power series for \( f(z) \), and the formally differentiated series converges to the complex derivative of \( f(z) \).

Let

\[
g(z) = \sum_{j=1}^{\infty} j a_j z^{j-1} = a_1 + 2a_2 z + 3a_3 z^2 + \ldots = \frac{1}{z} \sum_{j=1}^{\infty} a_j z^j, \quad z \neq 0,
\]
be obtained by differentiating the series for \( f(z) \) term by term. We claim that the radius of convergence for \( g(z) \) equals \( r \) and that \( f(z) \) has the complex derivative \( g(z) \).

**Lemma 6.6**

\[
\lim_{j \to \infty} j^{1/j} = 1
\]

**Proof:** For \( t \geq 0 \) we have

\[
e^t \geq 1 + \frac{t^2}{2},
\]

thus

\[
\lim_{t \to \infty} e^{-t}t = 0.
\]

With

\[
t = \ln j, \quad e^{-t} = \frac{1}{j}
\]

obtain that

\[
\ln(j^{1/j}) = \frac{1}{j} \ln j \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty.
\]

This implies that

\[
j^{1/j} \rightarrow 1 \quad \text{as} \quad j \rightarrow \infty.
\]

\(\diamondsuit\)

**Remark:** The result \( j^{1/j} \rightarrow 1 \) also follows from Lemma 6.5.

The lemma together with Hadamard’s formula imply that the series for \( f(z) \) and \( g(z) \) have the same radius of convergence.

**Theorem 6.7** Under the above assumptions the function \( f(z) \) is holomorphic in \( D(0, r) \) and

\[
f'(z) = g(z) \quad \text{for} \quad |z| < r.
\]

**Proof:** Let \( z \) with \( |z| < r \) be fixed. Fix \( r_1 \) with \( |z| < r_1 < r \). In the following, we let \( h \) be so small that

\[
|z + h| \leq |z| + |h| \leq r_1 < r.
\]

Set

\[
s_n(z) = \sum_{j=0}^{n} a_j z^j \quad \text{and} \quad \eta_n(z) = \sum_{j=n+1}^{\infty} a_j z^j
\]

and let \( \varepsilon > 0 \) be given. Then we have
\[
\left| \frac{1}{h}(f(z + h) - f(z)) - g(z) \right| \leq \left| \frac{1}{h}(s_n(z + h) - s_n(z)) - s'_n(z) \right| + \left| s'_n(z) - g(z) \right|
\]
\[
+ \left| \frac{1}{h}(\eta_n(z + h) - \eta_n(z)) \right|
\]
\[
=: A + B + C
\]

To estimate the term \( C \) we use the following lemma:

**Lemma 6.7** Let \( a, b \in \mathbb{C} \) and let \( M = \max\{|a|, |b|\} \). Then we have

\[
|a^j - b^j| \leq |a - b|jM^{j-1}.
\]

**Proof of lemma:** This follows from

\[
a^j - b^j = (a - b)(a^{j-1} + a^{j-2}b + \ldots + b^{j-1}).
\]

Applying the lemma, we obtain

\[
|(z + h)^j - z^j| \leq |h||j||z| + |h|^j \leq |h|j^i r_1^{j-1}.
\]

Therefore,

\[
C \leq \sum_{j=n+1}^{\infty} j|a_j|r_1^{j-1} \leq \varepsilon
\]

for \( n \geq N_1 \). Also,

\[
B = |s'_n(z) - g(z)| \leq \varepsilon
\]

for \( n \geq N_2 \). Fix \( n = \max\{N_1, N_2\} \). Then, since \( s_n(z) \) is a polynomial, there is \( \delta > 0 \) with

\[
A = \left| \frac{1}{h}(s_n(z + h) - s_n(z)) - s'_n(z) \right| \leq \varepsilon
\]

for \( |h| \leq \delta \). This proves the theorem. \( \Diamond \)

One can apply the previous theorem repeatedly and obtain the following result: If \( \sum a_jz^j \) has radius of convergence \( r > 0 \) then the function

\[
f(z) = \sum_{j=0}^{\infty} a_jz^j, \quad |z| < r,
\]

is infinitely often complex differentiable and all derivatives can be obtained differentiating the series term by term:

\[
f'(z) = \sum_{j=1}^{\infty} ja_jz^{j-1}
\]
\[
f''(z) = \sum_{j=2}^{\infty} j(j-1)a_jz^{j-2}
\]
etc. The power series for each derivative also has radius of convergence equal to \( r \). In particular, we have that

\[
\begin{align*}
  f(0) &= a_0 \\
  f'(0) &= a_1 \\
  f''(0) &= 2a_2 \\
  f'''(0) &= 2 \cdot 3a_3 \\
  f^{(k)}(0) &= k! a_k
\end{align*}
\]

This implies that the coefficients of a power series are uniquely determined by the function represented by the series. Precisely:

**Lemma 6.8** Assume that

\[
\begin{align*}
  f(z) &= \sum_{j=0}^{\infty} a_j z^j, \quad |z| < r_f \\
  g(z) &= \sum_{j=0}^{\infty} b_j z^j, \quad |z| < r_g
\end{align*}
\]

where \( r_f > 0 \) and \( r_g > 0 \). If, for some \( r > 0 \),

\[
f(z) = g(z) \text{ for all } z \text{ with } |z| < r
\]

then \( a_j = b_j \) for all \( j \). Therefore, \( r_f = r_g \) and \( f(z) = g(z) \) for all \( z \) with \( |z| < r_f \).

**Summary:** Let \( U \subset \mathbb{C} \) be open and let \( f \in H(U) \). Let \( D(z_0, \rho) \subset U \) and let \( 0 < r < \rho \). Let \( \Gamma \) denote the boundary curve of \( D(z_0, r) \). Set

\[
a_j = \frac{1}{j!} f^{(j)}(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta.
\]

We then have

\[
f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{for} \quad z \in D(z_0, \rho). \quad (6.8)
\]

The convergence of the series is absolute for \( z \in D(z_0, \rho) \) and uniform for \( |z - z_0| \leq r < \rho \).

In particular, if \( R \) denotes the radius of convergence of the series (6.8), then \( R \geq \rho \) as long as \( D(z_0, \rho) \subset U \). One obtains that \( R = \infty \) if \( U = \mathbb{C} \). If \( U \neq \mathbb{C} \) then the complement

\[
U^c = \mathbb{C} \setminus U
\]

is a non–empty, closed set. One obtains that
\( R \geq \text{dist}(z_0, U^c) \).

**Example 6.7:** Consider the function
\[
f(z) = \frac{e^z}{z^2 + 9}, \quad z \in U,
\]
with
\[
U = \mathbb{C} \setminus \{3i, -3i\}.
\]
Expansion of the function about \( z_0 = 4 \) yields a series
\[
f(z) = \sum_{j=0}^{\infty} a_j (z - 4)^j, \quad |z - 4| < R.
\]
By the previous considerations, the radius of convergence is at least \( R = 5 \). (Here \( R = 5 \) is the distance between the expansion point \( z_0 = 4 \) and the pole-set \( \{3i, -3i\} \).) If the power series would converge in \( D(4, R') \) with \( R' > 5 \) then the function \( f(z) \) would be bounded near \( \pm 3i \), which is not true. It follows that the radius of convergence of the series is exactly \( R = 5 \).

**Example 6.8:** The geometric sum
\[
\sum_{j=0}^{\infty} z^j
\]
has radius of convergence equal to \( r = 1 \). The value of the series is
\[
f(z) = \sum_{j=0}^{\infty} z^j = \frac{1}{1 - z}, \quad |z| < 1.
\]
The function \( f(z) = \frac{1}{1 - z} \) is holomorphic in \( U = \mathbb{C} \setminus \{1\} \). If we expand the function \( f(z) \) about \( z_0 = i/2 \), we obtain a series of the form
\[
f(z) = \sum_{j=0}^{\infty} a_j (z - \frac{i}{2})^j.
\]
Since we know that \( z_1 = 1 \) is the only singularity of \( f(z) \), the radius \( r \) of convergence of the series (6.9) is the distance between \( z_0 = i/2 \) and \( z_1 = 1 \). Thus,
\[
r = \frac{1}{2} \sqrt{5}.
\]
We can determine the precise form of the expansion (6.9) as follows:
\[ f(z) = \frac{1}{1-z} = \frac{1}{1 - \frac{i}{2} - (z - \frac{i}{2})} = \frac{1}{1 - \frac{i}{2} \left(1 - \frac{z - \frac{i}{2}}{1 - \frac{i}{2}}\right)^{-1}} = \sum_{j=0}^{\infty} a_j \left(z - \frac{i}{2}\right)^j \]

with

\[ a_j = \left(1 - \frac{i}{2}\right)^{-j-1}. \]

Using the quotient criterion, it is easy to confirm that the radius of convergence is \( r = \frac{1}{2} \sqrt{5} \). To see this, note that

\[ |a_{j+1}/a_j|^2 = \left|1 - \frac{i}{2}\right|^{-2} = \left(1 + \frac{1}{4}\right)^{-1} = \frac{4}{5}. \]

**Example 6.9:** Consider the series

\[ g(z) = \sum_{j=0}^{\infty} b_j z^j \]

where

\[ b_j = \frac{2 + \sin(j)}{3 + \cos(j^2)}. \]

Since

\[ \frac{1}{4} \leq b_j \leq \frac{3}{2} \]

it follows from Hadamard’s formula that the radius of convergence is \( r = 1 \). In this case, we do not know a simple analytic expression of \( g(z) \). If we expand \( g(z) \) about \( z_0 = i/2 \) we can say that the radius of convergence is at least \( \frac{1}{2} \), but it will be difficult to determine the radius precisely.
7 The Cauchy Estimates and Implications

For a complex differentiable function \( f(z) \) one can bound derivatives \( f'(z), f''(z), \) etc. in terms of values of the function. Here the constants in the bounds do not depend on the function \( f \), but on some distance.

The Cauchy estimates express such bounds of derivatives of \( f \) in terms of function values. They have many implications. We show Liouville’s theorem and the fundamental theorem of algebra.

7.1 The Cauchy Estimates

Let \( U \subset \mathbb{C} \) be an open set and let \( f : U \to \mathbb{C} \) be a holomorphic function. Let \( \bar{D}(z_0, r) \subset U \). We have

\[
f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j
\]

with

\[
a_j = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta, \quad \Gamma = \partial D(z_0, r),
\]

and

\[
f^{(j)}(z_0) = j! a_j.
\]

Clearly, the curve \( \Gamma \) has length \( 2\pi r \). Therefore, noting that

\[
|\zeta - z_0| = r,
\]

we obtain the following bound:

\[
|f^{(j)}(z_0)| \leq \frac{j!}{r^j} \max_{|\zeta - z_0| = r} |f(\zeta)|, \quad j = 0, 1, \ldots
\]  

(7.1)

The above estimates are called Cauchy estimates:

**Theorem 7.1** Let \( f \in H(U) \) where \( U \) is an open subset of \( \mathbb{C} \). If \( \bar{D}(z_0, r) \subset U \) then the estimates (7.1) hold.

7.2 Liouville’s Theorem

**Theorem 7.2** (Liouville) Let \( f : \mathbb{C} \to \mathbb{C} \) be holomorphic and bounded. Then \( f \) is constant.

**Proof:** We have, for all \( z \in \mathbb{C} \),

\[
f(z) = \sum_{j=0}^{\infty} a_j z^j \quad \text{with} \quad a_j = \frac{1}{j!} f^{(j)}(0).
\]

By the Cauchy estimates:
\[ |f^{(j)}(0)| \leq \frac{j!}{r^j} M(r) \]

with

\[ M(r) = \max_{|\zeta| = r} |f(\zeta)|. \]

By assumption, \( M(r) \) is bounded as \( r \to \infty \). Therefore, if \( j \geq 1 \), then the term \( M(r)/r^j \) goes to zero as \( r \to \infty \) and consequently \( a_j = 0 \) for \( j \geq 1 \). It follows that \( f(z) = a_0 \). \( \diamond \)

The following generalization says that if an entire function \( f(z) \) grows at most like \( |z|^k \) for \( z \to \infty \), then \( f(z) \) is a polynomial of degree less than or equal to \( k \). For short: Entire functions with polynomial growth are polynomials.

**Theorem 7.3** Let \( f : \mathbb{C} \to \mathbb{C} \) be holomorphic. Assume that there are constants \( C, R \) and a positive integer \( k \) with

\[ |f(z)| \leq C|z|^k \quad \text{for} \quad |z| \geq R. \]

Then \( f \) is a polynomial of degree less than or equal to \( k \).

**Proof:** By an estimate as in the previous proof obtain that \( a_j = 0 \) for \( j > k \). \( \diamond \)

### 7.3 The Fundamental Theorem of Algebra

**Theorem 7.4** Let \( p(z) = a_0 + a_1 z + \ldots + a_k z^k \) with \( a_k \neq 0 \), i.e., \( p(z) \) is a polynomial of degree \( k \). If \( k \geq 1 \) then there exists \( z_1 \in \mathbb{C} \) with \( p(z_1) = 0 \).

**Proof:** It is easy to check (see below) that \( |p(z)| \to \infty \) as \( |z| \to \infty \) because \( p(z) \) has a positive degree. If a zero \( z_1 \) of \( p(z) \) would not exist, then

\[ f(z) = \frac{1}{p(z)} \]

would be a bounded entire function. By Liouville’s theorem, \( f(z) = \text{const} \), thus \( p(z) = \text{const} \), a contradiction. \( \diamond \)

For completeness, we show here that \( |p(z)| \to \infty \) as \( |z| \to \infty \): Write

\[ p(z) = a_k z^k + q(z) \]

with

\[ q(z) = a_0 + a_1 z + \ldots + a_{k-1} z^{k-1}. \]

Let

\[ M := |a_0| + |a_1| + \ldots |a_{k-1}|. \]

Then, for all \( z \) with \( |z| \geq 1 \),

\[ |q(z)| \leq M|z|^{k-1}. \]
Therefore, for $|z| \geq 1$,

$$|p(z)| \geq |a_k||z|^k - |q(z)|$$

$$\geq |a_k||z|^k - M|z|^{k-1}$$

$$= |z|^{k-1}(|a_k||z| - M)$$

It follows that

$$|p(z)| \geq \frac{1}{2} |a_k||z|^k$$

if $|z| \geq 1$ and $|z| \geq 2M/|a_k|$. 

**Extension:** We want to show that any polynomial $p(z) = \sum_{j=0}^{k} a_j z^j$ of degree $k$ can be factorized:

$$p(z) = a_k(z - z_1) \ldots (z - z_k).$$

This follows from the fundamental theorem of algebra and the following lemma.

**Lemma 7.1** Let $p(z) = \sum_{j=0}^{k} a_j z^j$ denote a polynomial of degree $k$ where $k \geq 2$. Further, let $z_1 \in \mathbb{C}$ be a zero of the polynomial $p(z)$, i.e., $p(z_1) = 0$. Then there is a polynomial $q(z)$ of degree $k - 1$ with

$$p(z) = (z - z_1)q(z).$$

**Proof:** Using the binomial formula for $(a + b)^j$, we write

$$p(z) = \sum_{j=0}^{k} a_j z^j$$

$$= \sum_{j=0}^{k} a_j (z - z_1)^j$$

$$= \sum_{j=0}^{k} b_j (z - z_1)^j$$

where $b_k = a_k$. Since $0 = p(z_1) = b_0$ we obtain

$$p(z) = (z - z_1)\left(b_1 + b_2(z - z_1) + \ldots + b_k(z - z_1)^{k-1}\right).$$

This proves the lemma. $\Diamond$

Clearly, if $k - 1 \geq 1$, we can apply the fundamental theorem of algebra to $q(z)$, etc. This proves:

**Theorem 7.5** Let $p(z) = a_0 + a_1 z + \ldots + a_k z^k$ with $a_k \neq 0$, i.e., $p(z)$ is a polynomial of degree $k$. Let $k \geq 1$. Then there are $k$ (not necessarily distinct) numbers $z_1, z_2, \ldots, z_k \in \mathbb{C}$ with

$$p(z) = a_k(z - z_1) \ldots (z - z_k).$$
7.4 The Zeros of \( p(z) \) and \( p'(z) \)

Let

\[ p(z) = a_0(z - z_1) \cdots (z - z_n) = a \prod_{j=1}^{n} (z - z_j) \]

denote a polynomial of degree \( n \geq 2 \). We claim: If \( c \) is a zero of the derivative \( p'(z) \), then \( c \) lies in the convex hull of \( z_1, \ldots, z_n \), i.e., \( c \) can be written in the form

\[ c = \sum_j \alpha_j z_j \quad \text{with} \quad \alpha_j \geq 0 \quad \text{and} \quad \sum_j \alpha_j = 1. \]

To show this, we may assume that \( c \neq z_j \) for all \( j \). (If \( c = z_j \) then the claim is trivial.) We have

\[ \frac{p'(z)}{p(z)} = \sum_j \frac{1}{z - z_j}, \quad z \in \mathbb{C} \setminus \{z_1, \ldots, z_n\}, \quad (7.2) \]

and the assumption \( p'(c) = 0 \) yields:

\[ 0 = \sum_j \frac{1}{c - z_j}. \]

Therefore,

\[ 0 = \sum_j \frac{c - z_j}{|c - z_j|^2}, \]

thus

\[ c \sum_k \gamma_k = \sum_j \gamma_j z_j \]

with

\[ \gamma_j = |c - z_j|^{-2} > 0. \]

One obtains

\[ c = \frac{\sum_j \gamma_j z_j}{\sum_k \gamma_k} = \sum_j \alpha_j z_j \]

with

\[ \alpha_j = \frac{\gamma_j}{\sum_k \gamma_k}. \]

We have shown:

**Theorem 7.6** Let \( p(z) \) be a polynomial of degree \( n \) with zeros \( z_1, \ldots, z_n \). (The \( z_j \) are not necessarily distinct.) Any zero \( c \) of \( p'(z) \) lies in the convex hull of \( z_1, \ldots, z_n \).
Another simple implication of (7.2) is the following: Assume that $z$ is a complex number with $p(z) \neq 0$ and $p'(z) \neq 0$. Then (7.2) yields

$$\left| \frac{p'(z)}{p(z)} \right| \leq n \max_j \frac{1}{|z - z_j|},$$

thus

$$n \left| \frac{p(z)}{p'(z)} \right| \geq \min_j |z - z_j|.$$

This says that for every $z$ with $p'(z) \neq 0$ the closed disk

$$\bar{D}(z, R) \quad \text{with} \quad R = n \left| \frac{p(z)}{p'(z)} \right|$$

contains at least one zero $z_j$ of $p(z)$. Here $n$ is the degree of $p$.

**Theorem 7.7** Let $p(z)$ be a polynomial of degree $n$. Let $z \in \mathbb{C}$ and $p'(z) \neq 0$. Set

$$R_z = n \left| \frac{p(z)}{p'(z)} \right|.$$

The closed disk

$$\bar{D}(z, R_z)$$

contains at least one zero of $p$. 
8 Morera’s Theorem and Locally Uniform Limits of Holomorphic Functions

8.1 On Connected Sets

If \((X, d)\) is any metric space, one calls \(X\) disconnected if one can write \(X = X_1 \cup X_2\) where \(X_1\) and \(X_2\) are nonempty, disjoint, open subsets of \(X\):

\[
X = X_1 \cup X_2, \quad X_1 \cap X_2 = \emptyset, \quad X_1 \neq \emptyset \neq X_2, \quad X_j \text{ open}.
\]

Otherwise, \(X\) is called connected. The study of a function \(f\) defined on a metric space \(X\) can typically be reduced to the study of \(f\) on the connected components of \(X\). Therefore, without much loss of generality, one may often assume that \(X\) is connected.

For a complicated subset \(X\) of \(\mathbb{R}^2\) it may not be easy to determine if it connected or disconnected. For example, consider the set \(X = X_1 \cup X_2\) where

\[
X_1 = \{(0, y) : -1 \leq y \leq 1\}
\]

and

\[
X_2 = \{(x, \sin(1/x)) : x > 0\}.
\]

One might believe that \(X\) is disconnected, but it is not. Note that the subset \(X_2\) of \(X\) is not closed in \(X\).

Since we will only deal with open subsets \(U\) of \(\mathbb{C}\), the issue of connectedness is simple. One can show that an open subset \(U\) of \(\mathbb{C}\) is connected if and only if for any two points \(P, Q\) in \(U\) there is a smooth curve \(\Gamma\) in \(U\) from \(P\) to \(Q\).

Suppose \(U \subset \mathbb{C}\) is disconnected and \(U = U_1 \cup U_2\) where the \(U_j\) are nonempty, disjoint, and open. Then, if \(g \in H(U_1), h \in H(U_2)\), the function \(f : U \to \mathbb{C}\) defined by

\[
f(z) = g(z) \quad \text{for} \quad z \in U_1, \quad f(z) = h(z) \quad \text{for} \quad z \in U_2,
\]

is holomorphic on \(U\). This says that the behavior of any \(f \in H(U)\) on the set \(U_1\) is completely unrelated to the behavior of \(f\) on \(U_2\). In other words, it suffices to study holomorphic maps on open, connected sets.

8.2 Morera’s Theorem

Morera’s theorem is a converse of Cauchy’s integral theorem. It is very useful when studying convergence of sequences and series of holomorphic functions.

**Theorem 8.1 (Morera)** Let \(U\) be open, connected. Let \(f : U \to \mathbb{C}\) be continuous. Assume that

\[
\int_{\Gamma} f(z) \, dz = 0
\]

for all closed, piecewise smooth curves \(\Gamma\) in \(U\). Then there is a holomorphic function \(F : U \to \mathbb{C}\) with \(F' = f\). In particular, \(f\) is holomorphic.
Proof: Fix $P_0 \in U$, and let $\psi_P$ be a curve in $U$ from $P_0$ to $P$. Define

$$F(P) = \int_{\psi_P} f(z) \, dz.$$  

(Note: Because of the assumption $\int_{\Gamma} f(z) \, dz = 0$ for any closed curve $\Gamma$ in $U$, the value $F(P)$ is well-defined.)

Consider any $P \in U$. We will prove that $F'(P) = f(P)$. There is $r > 0$ with $D(P, r) \subset U$. Let $|h| < r$. Define the curve $\gamma_h$ by

$$\gamma_h(t) = P + th, \quad 0 \leq t \leq 1.$$  

Then the curve $\psi_{P+h} - (\psi_P + \gamma_h)$ is closed. Therefore,

$$F(P + h) - F(P) = \int_{\gamma_h} f(z) \, dz = \int_0^1 f(P + th) h \, dt.$$  

The function $t \to f(P + th)$ converges to $f(P)$ as $h \to 0$, uniformly for $0 \leq t \leq 1$. (This follows from the continuity of $f$ in $P$.) Therefore,

$$\frac{1}{h} (F(P + h) - F(P)) \to f(P) \quad \text{as} \quad h \to 0.$$  

\diamondsuit

8.3 Modes of Convergence of a Sequence of Functions

Let $X$ be any set and let $f_n, n = 0, 1, \ldots$ and $f$ be functions from $X$ to $\mathbb{C}$. What does it mean that the sequence $f_n$ converges to $f$ as $n \to \infty$? Different definitions are used, leading to different notions of convergence. The most commonly used notions are pointwise convergence and uniform convergence. Recall:

Definition 1: The sequence $f_n$ converges to $f$ pointwise on $X$ if for every $z \in X$ and every $\varepsilon > 0$ there is $N = N(\varepsilon, z) \in \mathbb{N}$ so that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$.

Definition 2: The sequence $f_n$ converges to $f$ uniformly on $X$ if for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ so that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$ and all $z \in X$.

It turns out that both theses concepts are not adequate in function theory. The concept of pointwise convergence is too weak; one cannot integrate the limit relation $f_n(z) \to f(z)$ if the convergence is only pointwise. On the other hand, the concept of uniform convergence is too strong, because it typically does not hold on the whole domain. For example, let
\[ f_n(z) = \sum_{j=0}^{n} z^j \quad \text{and} \quad f(z) = \frac{1}{1 - z} \quad \text{for} \quad z \in D(0,1). \]

Then \( f_n \) converges pointwise to \( f \) on \( D(0,1) \), but not uniformly. The convergence is uniform, however, on any subdomain \( D(0,r) \) with \( 0 < r < 1 \).

The following two notions of convergence, which turn out to be equivalent, are appropriate in function theory. Let \( U \subseteq \mathbb{C} \) be an open set and let \( f_n, f : U \to \mathbb{C} \) be functions.

**Definition 3:** The sequence \( f_n \) converges to \( f \) uniformly on compact sets in \( U \) if the following holds: For every compact set \( E \subseteq U \) and for every \( \varepsilon > 0 \) there is \( N = N(\varepsilon,E) \) so that

\[ |f_n(z) - f(z)| < \varepsilon \]

for all \( n \geq N \) and all \( z \in E \).

**Definition 4:** The sequence \( f_n \) converges to \( f \) locally uniformly in \( U \) if the following holds: For every \( z_0 \in U \) there is a neighborhood \( D(z_0, r) \subseteq U \) so that for every \( \varepsilon > 0 \) there is \( N = N(\varepsilon, z_0) \) with

\[ |f_n(z) - f(z)| < \varepsilon \]

for all \( n \geq N \) and all \( z \in D(z_0, r) \).

**Remark:** If one replaces \( U \) by a general metric space, the two notions of uniform convergence on compact sets and locally uniform convergence, may differ from one another.

**Theorem 8.2** Let \( U \) denote an open subset of \( \mathbb{C} \) and let \( f_n, f : U \to \mathbb{C} \) be functions. The sequence \( f_n \) converges to \( f \) uniformly on compact sets in \( U \) if and only if it converges to \( f \) locally uniformly in \( U \).

**Proof:** 1) Assume that \( f_n \) converges to \( f \) uniformly on compact sets in \( U \). Let \( z_0 \in U \). There is \( r > 0 \) with \( D(z_0, r) \subseteq U \). etc

2) Assume that \( f_n \) converges to \( f \) locally uniformly in \( U \). Let \( E \subseteq U \) be compact. For every \( z \in E \) there is \( r_z > 0 \) so that \( f_n \) converges to \( f \) uniformly on \( D(z, r_z) \). The sets \( D(z, r_z) \) for \( z \in E \) form an open cover of \( E \). There are finitely many sets

\[ D_j = D(z_j, r_{z_j}), \quad j = 1, \ldots, J, \]

whose union covers \( E \). For every \( \varepsilon > 0 \) and for every \( j \) there is \( N_j = N(\varepsilon, j) \) with

\[ |f_n(z) - f(z)| < \varepsilon \quad \text{for} \quad n \geq N_j \]

if \( z \in D_j \). If \( N = \max_j N_j \) then

\[ |f_n(z) - f(z)| < \varepsilon \quad \text{for} \quad n \geq N \]

and \( z \in E \). \( \diamond \)
We have seen that local uniform convergence in $U$ of a sequence $f_n: U \to \mathbb{C}$ is equivalent to uniform convergence on compact sets in $U$. In function theory, one calls this normal convergence in $U$. If one considers a series, $\sum_{j=0}^{\infty} u_j(z)$ with functions $u_j: U \to \mathbb{C}$, then one says the series converges normally in $U$ if the sequence of partial sums, $f_n(z) = \sum_{j=0}^{n} u_j(z)$, converges normally in $U$ and the series converges absolutely for every fixed $z \in U$, i.e., $\sum_{j=0}^{\infty} |u_j(z)| < \infty$.

**Definition 5:** Let $U \subset \mathbb{C}$ be an open set and let $f_n, f: U \to \mathbb{C}$ be functions. Then, if $f_n$ converges locally uniformly in $U$ to $f$, one says that $f_n$ converges normally in $U$ to $f$.

**Example:** Let $\sum_{j=0}^{\infty} a_j z^j$ have radius of convergence $r > 0$. Let

$$f_n(z) = \sum_{j=0}^{n} a_j z^j, \quad f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad z \in D(0, r).$$

Then $f_n$ converges to $f$ locally uniformly in $D = D(0, r)$. Proof: Let $z_0 \in D$ and let

$$|z| < r_1 < r_2 < r.$$ 

Let $\delta = r_1 - |z|$. Then $D(z, \delta) \subset D(0, r_1)$. It suffices to show that $f_n$ converges to $f$ uniformly on $D(0, r_1)$. We have

$$|a_j| r_2^j \leq M.$$ 

Therefore, for $z \in D(0, r_1)$:

$$|f_n(z) - f(z)| \leq \sum_{j=n+1}^{\infty} |a_j| r_1^j \leq \sum_{j=n+1}^{\infty} |a_j| r_2^j (r_1/r_2)^j \leq M \sum_{j=n+1}^{\infty} q^j \quad \text{with} \quad q = r_1/r_2 \leq M q^{n+1} \frac{1}{1-q} \leq \varepsilon$$

for $n \geq N(\varepsilon)$. $\diamond$

A simple and important convergence theorem for holomorphic functions is stated next. Its proof is based on Morera’s theorem and the Cauchy estimates.

**Theorem 8.3** Let $U$ be open; let $f_n, f: U \to \mathbb{C}$ be functions. Assume that all $f_n$ are holomorphic. If $f_n$ converges to $f$ locally uniformly in $U$, then $f$ is also holomorphic. Furthermore, $f'_n$ converges to $f'$ locally uniformly in $U$.

**Proof:** 1) Let $D = D(z_0, r)$ be any disk in $U$. Let $\Gamma$ be any closed curve in $D$. Then, by Cauchy’s theorem,
\[ \int_{\Gamma} f_n(z) \, dz = 0 \]

for all \( n \). Since \( \Gamma \) is compact, the \( f_n \) converge to \( f \) uniformly on \( \Gamma \). It follows that

\[ \int_{\Gamma} f(z) \, dz = 0 . \]

By Morera’s theorem, \( f \) is holomorphic in \( D \). Since \( D \) was an arbitrary open disk in \( U \), the function \( f \) is holomorphic in \( U \).

2) Let \( z_0 \in U \). There is \( r > 0 \) so that \( \overline{D}(z_0, 2r) \subset U \). Then, for \( z \in D(z_0, r) \),

\[ |f'_n(z) - f'(z)| \leq \frac{1}{r} \max_{|\zeta - z| = r} |f_n(\zeta) - f(\zeta)| \leq \frac{1}{r} \max_{\zeta \in \overline{D}(z_0, 2r)} |f_n(\zeta) - f(\zeta)| =: M_n . \]

As \( n \to \infty \), the maximum \( M_n \) converges to zero since \( f_n \) converges to \( f \) uniformly on \( D(z_0, 2r) \). Also, \( M_n \) is uniform for all \( z \in D(z_0, r) \). This proves the theorem.

\[ \Box \]

### 8.4 Integration with Respect to a Parameter

**Theorem 8.4** Let \( U \) be an open subset of \( \mathbb{C} \) and let \( F : U \times [a, b] \to \mathbb{C} \) be a function. Here \( [a, b] \) is a compact interval in \( \mathbb{R} \). Assume that \( F \) is continuous on \( U \times [a, b] \) and that \( z \to F(z, t) \) is holomorphic on \( U \) for every fixed \( t \). Then

\[ f(z) = \int_a^b F(z, t) \, dt, \quad z \in U , \]

is holomorphic on \( U \).

**Proof:** Let \( D = D(P, r) \) be any open disk in \( U \) and let \( \Gamma \) be a smooth closed curve in \( D \). By Cauchy’s theorem,

\[ \int_{\Gamma} F(z, t) \, dz = 0 \quad \text{for all} \quad a \leq t \leq b . \]

We have

\[
\int_{\Gamma} f(z) \, dz = \int_{\Gamma} \int_a^b F(z, t) \, dt \, dz
= \int_a^b \int_{\Gamma} F(z, t) \, dz \, dt
= 0 .
\]

Therefore, by Morera’s theorem, the function \( f(z) \) is holomorphic in \( D \). Since \( D \) is an arbitrary disk in \( U \), the function \( f(z) \) is holomorphic in \( U \).
Note that, in the second equation, we have exchanged the order of integration. Let us justify this. If $\Gamma$ has the parametrization $\gamma(s), c \leq s \leq d$, then

$$
\int_{\Gamma} \int_{a}^{b} F(z, t) \, dt \, dz = \int_{c}^{d} \int_{a}^{b} F(\gamma(s), t) \gamma'(s) \, dt \, ds
$$

$$
= \int_{a}^{b} \int_{c}^{d} F(\gamma(s), t) \gamma'(s) \, ds \, dt
$$

$$
= \int_{a}^{b} \int_{\Gamma} F(z, t) \, dz \, dt
$$

Here, in the second step, the continuous function $(s, t) \rightarrow F(\gamma(s), t) \gamma'(s)$ is integrable over $[c, d] \times [a, b]$, and Fubini’s theorem justifies to exchange the order of integration. ⋄

8.5 Application to the $\Gamma$–Function: Analyticity in the Right Half–Plane

Let $H_r = \{z = x + iy : x > 0\}$ denote the open right half–plane. For $z \in H_r$ define Euler’s $\Gamma$–function by

$$
\Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} \, dt.
$$

We want to prove that $\Gamma(z)$ is holomorphic on $H_r$. If $\epsilon > 0$ and $z \in \mathbb{C}$ define

$$
\Gamma_{\epsilon}(z) = \int_{\epsilon}^{1/\epsilon} t^{z-1} e^{-t} \, dt.
$$

Note: If $t > 0$ then

$$
t = e^{\ln t}
$$

and

$$
t^z = e^{z \ln t}.
$$

For every fixed $t > 0$, the function

$$
z \rightarrow t^{z-1} e^{-t} = e^{(z-1) \ln t} e^{-t}
$$

is entire. Also,

$$
(z, t) \rightarrow t^{z-1} e^{-t} = e^{(z-1) \ln t} e^{-t}
$$

is continuous on $\mathbb{C} \times [\epsilon, \frac{1}{\epsilon}]$. Therefore, by Theorem 8.4, each function $\Gamma_{\epsilon}(z)$ is entire.

Fix $0 < a < b < \infty$ and consider the vertical strip

$$
S_{a,b} = \{z = x + iy : a \leq x \leq b, y \in \mathbb{R}\}.
$$
For $z = x + iy \in S_{a,b}$ and $0 < \varepsilon \leq 1$ we have

$$|\Gamma(z) - \Gamma_\varepsilon(z)| \leq \int_0^\varepsilon t^{x-1}e^{-t} dt + \int_{1/\varepsilon}^\infty t^{x-1}e^{-t} dt$$

$$\leq \int_0^\varepsilon t^{a-1} dt + \int_{1/\varepsilon}^\infty t^{b-1}e^{-t} dt$$

$$=: R(\varepsilon)$$

It is clear that $R(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore,

$$\sup_{z \in S_{a,b}} |\Gamma(z) - \Gamma_\varepsilon(z)| \to 0 \text{ as } \varepsilon \to 0.$$  

If $E \subset H_r$ is an arbitrary compact set, then there exist $0 < a < b < \infty$ with $E \subset S_{a,b}$. It follows that $\Gamma_\varepsilon(z)$ converges to $\Gamma(z)$ uniformly on compact subsets of $H_r$. This implies that $\Gamma(z)$ is holomorphic on $H_r$.

**Remarks:**

1) We will see later that $\Gamma(z)$ can be continued as a holomorphic function defined for $z \in U$ with

$$U = \mathbb{C} \setminus \{0, -1, -2, \ldots\}.$$  

The extended function, also denoted by $\Gamma(z)$, has a simple pole at each $n \in \{0, -1, -2, \ldots\}$.

The integral representation (8.1) for $\Gamma(z)$ only holds for $\Re z > 0$, however, since the integral does not exist if $\Re z \leq 0$. (The singularity of the function $t \to t^{z-1}$ at $t = 0$ is not integrable if $\Re z \leq 0$.)

2) Consider

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} dt \quad \text{for} \quad 0 < x < \infty.$$  

For $0 < x < 1$ we have

$$\Gamma(x) \sim \int_0^1 t^{x-1} dt = \frac{1}{x} t^x \bigg|_{t=0}^{t=1} = \frac{1}{x}.$$  

This suggests that, for $z \sim 0$,

$$\Gamma(z) = \frac{1}{z} + \sum_{j=0}^\infty a_j z^j$$

where the series converges for $z \sim 0$. This can in fact be shown. The above representation holds for $|z| < 1$. The function $\Gamma(z)$ has a simple pole at $z = 0$ with

$$\text{Res}(\Gamma, 0) = 1.$$  

3) Consider

$$\Gamma(x + 1) = \int_0^\infty t^x e^{-t} dt \quad \text{for} \quad x > 1.$$
The term $t^x$ is very large for large $t$. In fact, one can show that $\Gamma(x + 1)$ grows faster than $e^{\alpha x}$ as $x \to \infty$, for any $\alpha > 0$.

Stirling’s formula says that

$$\frac{\Gamma(x + 1)}{\left(x\frac{e}{x}\right)^x \sqrt{2\pi x}} \to 1 \quad \text{as} \quad x \to \infty \quad \text{(8.2)}$$

For any $\alpha > 0$,

$$\ln \left(\frac{x^x}{e^{\alpha x}}\right) = x(\ln x - \alpha) \to \infty \quad \text{as} \quad x \to \infty$$

Therefore, using (8.2),

$$\frac{\Gamma(x + 1)}{e^{\alpha x}} \to \infty \quad \text{as} \quad x \to \infty$$

Thus, $\Gamma(x)$ grows faster than any exponential $e^{\alpha x}$. On the other hand, if $\varepsilon > 0$, then

$$\ln \left(\frac{x^x}{e^{x^{1+\varepsilon}}}\right) = x(\ln x - x^\varepsilon) \to -\infty \quad \text{as} \quad x \to \infty$$

Therefore, using (8.2),

$$\frac{\Gamma(x + 1)}{e^{x^{1+\varepsilon}}} \to 0 \quad \text{as} \quad x \to \infty$$

Thus, for any $\varepsilon > 0$ the function $e^{x^{1+\varepsilon}}$ grows faster than $\Gamma(x)$ as $x \to \infty$.

4) Let $z = x + iy, x > 0, y \in \mathbb{R}$. We have

$$\Gamma(z) = \int_0^\infty t^{x-1}i^y e^{-t} \, dt$$

where

$$i^y = e^{iy \ln t} = \cos(y \ln t) + i \sin(y \ln t)$$

Let us try to understand

$$\text{Re}\Gamma(z) = \int_0^\infty t^{x-1} \cos(y \ln t)e^{-t} \, dt$$

For $y \neq 0$ the function $\cos(y \ln t)$ varies rapidly in the interval $0 < t < \infty$ as $t \to 0$ and as $t \to \infty$.

Fix $y \neq 0$ and let $x = 0$. The integral

$$\int_0^\infty t^{-1} \cos(y \ln t)e^{-t} \, dt$$

does not exist since the singularity at $t = 0$ is not integrable. However, the integrand varies rapidly as $t \to 0$, leading to cancellations. This is the intuitive reason why $\Gamma(z)$ can be continued analytically into parts of the left half–plane, i.e., into

$$\{ z = x + iy : x > -\varepsilon, z \neq 0 \}.$$
8.6 Stirling’s Formula

Consider the Gamma–function for real positive $x$,

$$\Gamma(x + 1) = \int_0^\infty t^x e^{-t} dt.$$  

Stirling’s formula,

$$\Gamma(x + 1) \sim \left(\frac{x}{e}\right)^x \sqrt{2\pi x},$$

gives an approximation for $\Gamma(x + 1)$ which is valid for large $x$. Precisely:

**Theorem 8.5** As $x \to \infty$ we have

$$\Gamma(x + 1) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x} \left(1 + O(x^{-1})\right). \quad (8.3)$$

**Proof:** We first make simple linear substitutions:

$$\Gamma(x + 1) = \int_0^\infty t^x e^{-t} dt \quad \text{(substitute } t = xs)$$

$$= x^{x+1} \int_0^\infty s^x e^{-sx} ds$$

$$= x^{x+1} \int_0^\infty e^{x(\ln s - s)} ds \quad \text{(substitute } s = 1 + u)$$

$$= x^{x+1} \int_{-1}^\infty e^{x(\ln(1+u) - 1 - u)} du$$

$$= \left(\frac{x}{e}\right)^x x \int_{-1}^\infty e^{x(\ln(1+u) - u)} du.$$  

We have to analyze the integral

$$J(x) = \int_{-1}^\infty e^{x\phi(u)} \, du$$

where

$$\phi(u) = \ln(1 + u) - u \quad u > -1.$$

We must show that

$$J(x) = \sqrt{\frac{2\pi}{x}} + O(x^{-3/2}).$$

Clearly,

$$\phi'(u) = \frac{1}{1+u} - 1, \quad \phi''(u) = -\frac{1}{(1+u)^2} < 0.$$  

Therefore, the function $\phi(u)$ attains its maxium at $u = 0$. Since
\[
\ln(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \ldots \quad \text{for} \quad |u| < 1
\]

we have

\[
\phi(u) = -\frac{u^2}{2} \left(1 - \frac{2u}{3} + Q(u)\right).
\]

Here

\[
|Q(u)| \leq Cu^2 \quad \text{for} \quad |u| \leq \frac{1}{2}.
\]

We choose a small constant \(c > 0\) so that the term

\[
R(u) = -\frac{2u}{3} + Q(u)
\]

in the above formula for \(\phi(u)\) satisfies

\[
|R(u)| \leq \frac{1}{2} \quad \text{for} \quad |u| \leq c.
\]

With some \(\kappa > 0\) we can write

\[
J(x) = \int_{-c}^{c} e^{x\phi(u)} \, du + \mathcal{O}(e^{-\kappa x}) \quad \text{as} \quad x \to \infty.
\]

(For example, if \(u \geq c\), then

\[
\phi(u) \leq -c_1 - c_2(u - c), \quad c_j > 0,
\]

and

\[
\int_{c}^{\infty} e^{x\phi(u)} \, du \leq \int_{0}^{\infty} e^{-c_1 x - c_2 xu} \, du
\]

where the right–hand side converges to zero exponentially as \(x \to \infty\).)

It remains to discuss the integral

\[
J_1(x) = \int_{-c}^{c} e^{x\phi(u)} \, du
\]

\[
= \int_{-c}^{c} \exp \left( x \left( -\frac{u^2}{2} (1 - \frac{2u}{3} + Q(u)) \right) \right) \, du.
\]

Since

\[
\sqrt{1 + \varepsilon} = 1 + \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2)
\]

we can write

\[
(1 - \frac{2u}{3} + Q(u))^{1/2} = 1 - \frac{u}{3} + Q_1(u), \quad |Q_1(u)| \leq Cu^2.
\]

Then, in the integral
\[ J_1(x) = \int_{-c}^{c} \exp \left( -\frac{x}{2} u^2 (1 - \frac{u}{3} + Q_1(u))^2 \right) du \]

we make the substitution

\[ u \left( 1 - \frac{u}{3} + Q_1(u) \right) = y. \]

Note that, to leading order, \( y \) equals \( u \). We have

\[ u = \frac{y}{1 - \frac{u}{3} + Q_1(u)} \]
\[ = y(1 + \frac{u}{3} + Q_2(u)) \]
\[ = y(1 + \frac{y}{3} + Q_3(y)) \]

This implies

\[ du = dy \left( 1 + \frac{2y}{3} + Q_4(y) \right). \]

We find that

\[ J_1(x) = \int_{-c}^{c} \exp \left( -\frac{x}{2} u^2 (1 - \frac{u}{3} + Q_1(u))^2 \right) du \]
\[ = \int_{-c'}^{c'} e^{-x\rho^2/2} \left( 1 + \frac{2\rho}{3} + Q_4(y) \right) dy \]

where

\[ c' = c \left( 1 - \frac{c}{3} + Q_1(c) \right) \sim c. \]

In the integral

\[ J_2(x) = \int_{-c'}^{c'} e^{-x\rho^2/2} d\rho \]

we substitute

\[ \sqrt{x/2} y = \rho, \quad dy = \sqrt{2/x} d\rho \]

and find that

\[ J_2(x) = \sqrt{2/x} \int_{-c'}^{c'} e^{-\rho^2} d\rho = \sqrt{2\pi/x} + O(e^{-\kappa x}). \]

The integral

\[ J_3(x) = \int_{-c'}^{c'} ye^{-x\rho^2/2} d\rho \]
equals zero and

\[ J_4(x) = \int_{-c'}^{c'} y^2 e^{-xy^2/2} \, dy \]

can be estimated by \( Cx^{-3/2} \). This proves the theorem. \( \diamondsuit \)

**Remark:** According to [Whittaker, Watson, p. 253]:

\[
\Gamma(x+1) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + O(x^{-5}) \right) \quad \text{as} \quad x \to \infty.
\]
9 Zeros of Holomorphic Functions and the Identity Theorem

We first prove a result on the connectedness of the interval $[0, 1]$.

**Lemma 9.1** Let $A \subset [0, 1]$. Assume:

a) $0 \in A$.

b) $A$ is open in $[0, 1]$, i.e., for every $t \in A$ there is $\varepsilon > 0$ such that

$$\{s \in [0, 1] : |s - t| < \varepsilon\} \subset A.$$  

c) $A$ is closed in $[0, 1]$, i.e., if $t_n \in A$ converges to $t \in [0, 1]$, then $t \in A$.

Under these assumptions we have $A = [0, 1]$.

**Proof:** Suppose $B = A^c = [0, 1] \setminus A$ is not empty. Then let

$$\beta = \inf B.$$  

Since $0 \in A$ and $A$ is open in $[0, 1]$ we have that $\beta > 0$. Also, $[0, \beta) \subset A$. Since $A$ is closed, it follows that $\beta \in A$. Therefore, $[0, \beta] \subset A$. If $\beta = 1$, then $A^c$ is empty, which contradicts our assumption. Thus, $\beta < 1$. But then, since $A$ is open in $[0, 1]$, there is $\varepsilon > 0$ such that $[0, \beta + \varepsilon) \subset A$. This contradicts the definition, $\beta = \inf A^c$. \(\diamondsuit\)

**Definition:** Let $S \subset \mathbb{C}$ be non–empty. Let $P \in \mathbb{C}$. The point $P$ is called an accumulation point of $S$ if there is a sequence of points $z_n \in S \setminus \{P\}$ with $z_n \to P$. Here $P$ may or may not be a point of $S$. If $P \in S$ and $P$ is not an accumulation point of $S$, then $P$ is called an isolated point of $S$.

The following theorem is called the **Identity Theorem**. It implies that two holomorphic functions, $f, g \in H(U)$, are identical on $U$ if $U$ is connected and if the set of all $z \in U$ with $f(z) = g(z)$ has an accumulation point in $U$. In particular, if $f(z) = g(z)$ for all $z$ in an open disk in $U$ or if $f(z) = g(z)$ for all $z$ on a line segment of positive length, then $f$ and $g$ are identical on $U$.

**Theorem 9.1** Let $U$ be open and connected. Let $f : U \to \mathbb{C}$ be holomorphic. Let

$$Z = \{z \in U : f(z) = 0\}$$

be the set of points in $U$ where $f$ is zero. If $Z$ has an accumulation point belonging to $U$, then $f \equiv 0$.

**Proof:** a) Let $P \in U$ be an accumulation point of $Z$ and let $z_n \in Z$ with $z_n \to P$, $z_n \neq P$. Let

$$f(z) = \sum_{j=0}^{\infty} a_j(z - P)^j, \quad |z - P| < r.$$  

We claim that $a_j = 0$ for all $j$. Otherwise, let

$$a_0 = a_1 = \ldots = a_J = 0, \quad a_{J+1} \neq 0.$$  

Then we have

\[ f(z) = (z - P)^{J+1}(a_{J+1} + a_{J+2}(z - P) + \ldots) \]

\[ = (z - P)^{J+1}g(z) \]

with \( g(z) \) holomorphic in \( D(P, r) \) and \( g(P) \neq 0 \). There is \( \varepsilon > 0 \) so that \( g(z) \neq 0 \) for \( |z - P| < \varepsilon \). Therefore,

\[ f(z) \neq 0 \quad \text{for} \quad 0 < |z - P| < \varepsilon \, . \]

This contradicts

\[ z_n \to P, \quad z_n \neq P, \quad f(z_n) = 0 \, . \]

b) Let

\[ V = \{ z \in U : f^{(j)}(z) = 0 \quad \text{for all} \quad j \} \, . \]

We have shown that \( P \in V \) and claim that \( V = U \). To show this, let \( Q \in U \) be arbitrary. Let \( \gamma : [0, 1] \to U \) be a curve with

\[ \gamma(0) = P, \quad \gamma(1) = Q \, . \]

Let

\[ A = \{ t \in [0, 1] : \gamma(t) \in V \} \, . \]

We have that \( 0 \in A \) since \( P \in V \). If \( t \in A \) then \( \gamma(t) \in V \), and therefore \( f \equiv 0 \) in a neighborhood of \( \gamma(t) \). This implies that \( A \) is open in \([0, 1]\). If \( t_n \in A \) and \( t_n \to t \), then

\[ f^{(j)}(\gamma(t)) = 0 \]

for all \( n \) and all \( j \). This yields that

\[ f^{(j)}(\gamma(t)) = 0 \]

for all \( j \). Therefore, \( t \in A \). By the previous lemma, we have \( A = [0, 1] \). Therefore, \( Q \in V \).

**Remark:** We can use a different argument for part b) of the proof if we use the definition of connectedness of \( U \) from topology. The set \( V \) is closed in \( U \) since all \( f^{(j)} \) are continuous. Also, if \( z \in V \) then \( f \) is zero in a neighborhood of \( z \). Therefore, \( V \) is open in \( U \). Since \( P \in V \) we have that \( V \neq \emptyset \). The connectedness of \( U \) then implies that \( V = U \) showing that \( f \) is zero on \( U \).
10 Isolated Singularities and Laurent Expansion

10.1 Classification of Isolated Singularities

Let \( P \in \mathbb{C} \) and let \( r > 0 \). Then the set

\[
D(P, r) \setminus \{P\}
\]

is a punctured disk, a disk where the center is removed. If \( U \) is an open set containing \( D(P, r) \setminus \{P\} \) and if \( f \in H(U) \), then one says that \( f \) has an isolated singularity at \( P \).

For simplicity of notation, let \( P = 0 \). There are three cases:

**Case 1:** There is \( \varepsilon > 0 \) and \( M > 0 \) with

\[
|f(z)| \leq M \quad \text{for} \quad 0 < |z| \leq \varepsilon .
\]

**Case 2:** \( |f(z)| \to \infty \) as \( z \to 0 \), i.e., for all \( R > 0 \) there is \( \varepsilon > 0 \) with

\[
|f(z)| \geq R \quad \text{for} \quad 0 < |z| \leq \varepsilon .
\]

**Case 3:** Neither case 1 nor case 2 holds.

**Terminology:** Assume that \( f \) has an isolated singularity at \( P \), i.e., \( f \) is a holomorphic function in a set that contains \( D(P, r) \setminus \{P\} \) for some \( r > 0 \). In case 1, one says that \( f \) has a removable singularity at \( P \). This terminology is justified by Riemann’s removability theorem, which we prove below. In case 2, one says that \( f \) has a pole at \( P \). In case 3 one says that \( f \) has an essential singularity at \( P \).

**Example 1:** The function

\[
f(z) = \frac{z^2 - 9}{z - 3} \quad \text{for} \quad z \neq 3
\]

has an isolated singularity at \( z = 3 \). For \( z \neq 3 \) we have

\[
f(z) = z + 3 .
\]

Case 1 holds. By setting \( f(3) = 6 \) we can remove the singularity of \( f \) at \( z = 3 \).

**Example 2:** The function

\[
f(z) = \frac{1}{z^2} \quad \text{for} \quad z \neq 0
\]

has an isolated singularity at \( z = 0 \). Case 2 holds.

**Example 3:** The function

\[
f(z) = e^{1/z} \quad \text{for} \quad z \neq 0
\]

has an isolated singularity at \( z = 0 \). We claim that case 3 holds. To see this, let

\[
a_n = \frac{1}{m}, \quad b_n = \frac{1}{n} \quad \text{for} \quad n \in \mathbb{Z}, \quad n \neq 0 .
\]

Then we have
\[ f(a_n) = e^{i\alpha}, \quad |f(a_n)| = 1, \]

and

\[ f(b_n) = e^{\alpha}. \]

Since \( a_n \to 0 \) and \( b_n \to 0 \) neither case 1 nor case 2 holds.

### 10.2 Removable Singularities

**Theorem 10.1 (Riemann’s removability theorem)** Let \( f \in H(U) \) where \( U \) is an open set containing \( D(0, r) \setminus \{0\} \) for some \( r > 0 \). Assume that case 1 holds, i.e., \( f \) is bounded near the isolated singularity at \( P = 0 \): \( |f(z)| \leq M \) for \( 0 < |z| \leq \varepsilon \).

Then

\[
\lim_{z \to 0} f(z) =: f_0
\]

exists and the extended function, \( f_e(z) \), defined by

\[ f_e(z) = f(z) \quad \text{for} \quad 0 < |z| < r, \quad f_e(0) = f_0, \]

is holomorphic in \( U \cup \{0\} \).

**Proof:** Set

\[ g(z) = z^2 f(z) \quad \text{for} \quad z \in U, \quad g(0) = 0. \]

Clearly, \( g \) is holomorphic in \( U \). We want to show that \( g \) is holomorphic in \( U \cup \{0\} \) and must show that \( g \) is complex differentiable in \( P = 0 \).

For \( 0 < |h| < \varepsilon \) we have

\[
\left| \frac{1}{h} (g(h) - g(0)) \right| = \left| \frac{1}{h} g(h) \right| = |h| |f(h)| \leq M |h|. \]

Therefore, \( g'(0) \) exists and is zero. Since \( g \) is holomorphic in \( D(0, r) \) we can write

\[ g(z) = a_0 + a_1 z + a_2 z^2 \cdots \quad \text{for} \quad |z| < r. \]

Also, since \( g(0) = g'(0) = 0 \), we have \( a_0 = a_1 = 0 \). Therefore,

\[ g(z) = z^2 (a_2 + a_3 z + \cdots) \quad \text{for} \quad |z| < r. \]

Here the power series converges for \( |z| < r \). Since \( g(z) = z^2 f(z) \) for \( 0 < |z| < r \) it follows that

\[ f(z) = a_2 + a_3 z + \cdots \quad \text{for} \quad 0 < |z| < r. \]

This implies that \( \lim_{z \to 0} f(z) \) exists, is equal to \( f_0 := a_2 \), and that the extended function \( f_e(z) \) is holomorphic in \( U \cup \{0\} \). This proves the theorem. \( \diamond \)
10.3 Theorem of Casorati–Weierstrass on Essential Singularities

The following result is known as the Casorati–Weierstrass Theorem:

**Theorem 10.2** Let $f$ be a holomorphic function defined on $D(P, r) \setminus \{P\}$ and assume that $f$ has an essential singularity at $P$. Then, for any $0 < \delta < r$, the set

$$f\left(D(P, \delta) \setminus \{P\}\right)$$

is dense in $\mathbb{C}$.

**Proof:** Suppose this does not hold. Then fix $0 < \delta < r$ so that the set

$$f\left(D(P, \delta) \setminus \{P\}\right)$$

is not dense in $\mathbb{C}$. This means that there exists $Q \in \mathbb{C}$ and $\varepsilon > 0$ with

$$|f(z) - Q| \geq \varepsilon \quad \text{for} \quad 0 < |z - P| < \delta .$$

Set

$$g(z) = \frac{1}{f(z) - Q} \quad \text{for} \quad 0 < |z - P| < \delta .$$

We have $|g(z)| \leq \frac{1}{\varepsilon}$. By Riemann’s theorem,

$$\lim_{z \to P} g(z) =: g_0$$

exists.

a) $g_0 \neq 0$. In this case,

$$\lim_{z \to P} (f(z) - Q) = \frac{1}{g_0} .$$

This implies that $f(z)$ is bounded near $P$, which contradicts our assumption.

a) $g_0 = 0$. In this case,

$$\lim_{z \to P} |f(z) - Q| = \infty .$$

It follows that $f$ has a pole at $P$, which contradicts our assumption. \diamond

**Remark:** A much deeper result is Picard’s big theorem:

**Theorem 10.3** Under the same assumptions as in the Casorati–Weierstrass theorem, we have

$$f\left(D(P, \delta) \setminus \{P\}\right) = \mathbb{C} \quad \text{for} \quad 0 < \delta < r$$

or, for some $Q \in \mathbb{C}$,

$$f\left(D(P, \delta) \setminus \{P\}\right) = \mathbb{C} \setminus \{Q\} \quad \text{for} \quad 0 < \delta < r .$$
In other words, only the following two possibilities exist:

Possibility 1:
For any \( w \in \mathbb{C} \) and any \( 0 < \delta < r \) the equation \( f(z) = w \) has infinitely many solutions \( z = z_n \) with \( 0 < |z_n - P| < \delta \).

Possibility 2:
There is a point \( Q \in \mathbb{C} \) so that for any \( w \in \mathbb{C} \setminus \{Q\} \) and any \( 0 < \delta < r \) the equation \( f(z) = w \) has infinitely many solutions \( z = z_n \) with \( 0 < |z_n - P| < \delta \).

Example 1: Let \( f(z) = e^{1/z}, z \neq 0 \). Clearly, \( f \) has an essential singularity at \( P = 0 \). Here we can directly verify that possibility 2 holds with \( Q = 0 \). If \( w \in \mathbb{C}, w \neq 0 \), is given, then we can write
\[
 w = re^{i\theta} = e^{\ln r + i\theta + 2\pi in}
\]
for any \( n \in \mathbb{Z} \). If
\[
 z_n = \frac{1}{\ln r + i\theta + 2\pi in}
\]
then \( f(z_n) = w \) and \( |z_n| < \delta \) for all large \( |n| \). This shows that the function
\[
 f(z) = e^{1/z}, \quad z \in \mathbb{C} \setminus \{0\},
\]
has the following property: Given any \( w \in \mathbb{C} \setminus \{0\} \) and given any \( \delta > 0 \), there are infinitely many points \( z_n \) with \( 0 < |z_n| < \delta \) and \( f(z_n) = w \). In other words: In any neighborhood of its essential singularity at \( P = 0 \), the function \( f(z) = e^{1/z} \) attains every value \( w \in \mathbb{C} \), except for \( w = 0 \), infinitely many times.

Example 2: \( f(z) = \sin(1/z), z \neq 0 \). Again, \( f \) has an essential singularity at \( P = 0 \). In this case, for any \( w \in \mathbb{C} \) and any \( \delta > 0 \) the equation \( f(z) = w \) has infinitely many solutions \( z = z_n \) with \( 0 < |z_n| < \delta \). Proof: We solve
\[
 \sin \alpha = \frac{1}{2i} \left( e^{i\alpha} - e^{-i\alpha} \right) = w
\]
by setting
\[
 q = e^{i\alpha}.
\]
The equation becomes
\[
 q - \frac{1}{q} = 2iw \quad \text{or} \quad q^2 - 2iwq - 1 = 0.
\]
Clearly, given any \( w \in \mathbb{C} \) there is a solution \( q \in \mathbb{C}, q \neq 0 \). The equation
\[
 e^{i\alpha} = q
\]
has solutions
\[
 \alpha_n = \alpha_{par} + 2\pi n, \quad n \in \mathbb{Z}.
\]
For all large \( n \) let
and obtain that

$$\sin \alpha_n = w, \quad |z_n| < \delta.$$  

10.4 Laurent Series

10.4.1 Terminology

An expression

$$\sum_{j=-\infty}^{\infty} a_j (z - P)^j$$  

(10.1)

is called a Laurent series centered at $P$. The series (10.1) is called convergent at $z$ if the limits

$$\lim_{n \to \infty} \sum_{j=0}^{n} a_j (z - P)^j =: L_1$$

and

$$\lim_{n \to \infty} \sum_{j=-n}^{-1} a_j (z - P)^j =: L_2$$

exist. In this case the value of (10.1) is $L_1 + L_2$.

Typically, such series converge in annuli (plus parts of the boundary). Here, if $P \in \mathbb{C}$ and $0 \leq r_1 < r_2 \leq \infty$, the set

$$A = A(P, r_1, r_2) = \{z : r_1 < |z - P| < r_2\}$$

is the annulus centered at $P$ with inner radius $r_1$ and outer radius $r_2$. 

Figure 10.1: The annulus $A(P, r_1, r_2)$

$$z_n = \frac{1}{\alpha_n} = \frac{1}{\alpha_{par} + 2\pi n}$$
10.4.2 Characterization of Isolated Singularities in Terms of Laurent Expansions

If the holomorphic function $f$ has an isolated singularity at $P$ then $P$ is removable or a pole or an essential singularity. We will prove that these three possibilities have a simple characterization in terms of the Laurent expansion of $f$ in $D(P, r) \setminus \{ P \}$.

Let $A = A(P, 0, r) = D(P, r) \setminus \{ P \}$. We will show: If $f : A \rightarrow \mathbb{C}$ is holomorphic, then $f$ has a unique Laurent expansion in $A$,

$$f(z) = \sum_{j=-\infty}^{\infty} a_j(z - P)^j, \quad z \in A.$$  

Clearly, there are three cases:

**Case A:** $a_j = 0$ for all $j < 0$.

**Case B:** There is $J < 0$ with $a_j \neq 0$ and $a_j = 0$ for all $j < J$. (We will see below that this case holds if and only if $f$ has a pole at $P$; one says that $f$ has a pole of order $|J|$.)

**Case C:** There are infinitely many $j < 0$ with $a_j \neq 0$.

We will prove:

**Theorem 10.4** Under the above assumptions, $f$ has a removable singularity at $P$ if and only if Case A holds; $f$ has a pole at $P$ if and only if Case B holds; $f$ has an essential singularity at $P$ if and only if Case C holds.

10.4.3 Convergence of Laurent Series

**Theorem 10.5** Assume the Laurent series $\sum_j a_j(z - P)^j$ converges for $z = z_1$ and $z = z_2$ with

$$r_1 = |z_1 - P| < r_2 = |z_2 - P|.$$  

Then the series converges for all $z$ with

$$r_1 < |z - P| < r_2.$$  

Furthermore, the series

$$\sum_{j=0}^{\infty} a_j(z - P)^j =: g(z)$$  

converges absolutely for $|z - P| < r_2$ and the series

$$\sum_{j=-\infty}^{-1} a_j(z - P)^j =: h(z)$$  

converges absolutely for $|z - P| > r_1$. Also,

$$\sum_{j=0}^{n} a_j(z - P)^j \rightarrow g(z) \quad \text{as} \quad n \rightarrow \infty.$$
normally in $D(P, r_2)$ and

$$\sum_{j=-n}^{-1} a_j (z - P)^j \to h(z) \quad \text{as} \quad n \to \infty$$

normally for $|z - P| > r_1$, i.e., in $A(P, r_1, \infty)$.

**Proof:** This follows, essentially, from Abel’s Lemma for power series. ◊

### 10.4.4 Examples

1) The series

$$f(z) = \sum_{j=-10}^{\infty} \frac{z^j}{j^2 + 1}$$

converges for $0 < |z| < 1$. The annulus of convergence is $A(0, 0, 1)$. The function $f(z)$ has a pole of order 10 at $z = 0$.

2) The series

$$f(z) = \sum_{j=-\infty}^{50} 2^j z^j$$

converges for $|z| > \frac{1}{2}$. The annulus of convergence is $A(0, \frac{1}{2}, \infty)$. The function $f(z)$ does not have an isolated singularity at $z = 0$.

3) In the following example we show that the Laurent expansion of a function $f(z)$ in an annulus $A(P, r_1, r_2)$ not only depends on $P$, but also on $r_1$ and $r_2$. Consider the function

$$f(z) = \frac{1}{(1-z)(2-z)} = \frac{1}{1-z} - \frac{1}{2-z}, \quad z \in \mathbb{C} \setminus \{1, 2\}.$$

It can be written as a Laurent series, centered at $z = 0$, in

- $A_1 = A(0, 0, 1)$
- $A_2 = A(0, 1, 2)$
- $A_3 = A(0, 2, \infty)$

a) The expansion in $A_1$ is the Taylor expansion about 0: We have

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j, \quad |z| < 1,$$

and

$$\frac{1}{2-z} = \frac{1}{2(1-z/2)}
= \frac{1}{2} \sum_{j=0}^{\infty} 2^{-j} z^j, \quad |z| < 2.$$
Therefore,
\[ f(z) = \sum_{j=0}^{\infty} (1 - 2^{-j-1})z^j, \quad |z| < 1. \]

b) To obtain the Laurent expansion in \( A_2 \) we write
\[
\frac{1}{1 - z} = -\frac{1}{z} \frac{1}{1 - 1/z} = -\frac{1}{z} \sum_{j=0}^{\infty} z^{-j}
\]
for \(|z| > 1\). Together with the expansion of \( 1/(2 - z) \) of the previous case:
\[
f(z) = -\frac{1}{z} \sum_{j=0}^{\infty} z^{-j} - \sum_{j=0}^{\infty} 2^{-j-1}z^j \quad \text{for} \quad 1 < |z| < 2.
\]

c) To obtain the Laurent expansion in \( A_3 \) we write for \(|z| > 2\):
\[
-\frac{1}{2 - z} = \frac{1}{z} \frac{1}{1 - 2/z} = \frac{1}{z} \sum_{j=0}^{\infty} 2^j z^{-j}
\]
Therefore,
\[
f(z) = \frac{1}{z} \sum_{j=0}^{\infty} (2^j - 1)z^{-j}.
\]

10.4.5 Laurent Expansion: Uniqueness

Let \( P = 0 \), for simplicity. Let \( 0 \leq r_1 < r_2 \leq \infty \) and let
\[
\mathcal{A} = \mathcal{A}(0, r_1, r_2)
\]
denote an annulus. Assume that
\[
f(z) = \sum_{j=-\infty}^{\infty} a_j z^j, \quad z \in \mathcal{A}.
\]

Since the convergence is normal in \( \mathcal{A} \), the function \( f(z) \) is holomorphic in \( \mathcal{A} \). Let \( r_1 < r < r_2 \) and let
\[
\gamma(t) = re^{it}, \quad 0 \leq t \leq 2\pi.
\]

We claim that
\[ a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z^{n+1}} \, dz, \quad n \in \mathbb{Z}. \]

The proof is easy: Since the convergence of the series (10.2) is uniform on \( \gamma \), we can exchange summation and integration. Therefore,

\[ \int_{\gamma} \frac{f(z)}{z^{n+1}} \, dz = \sum_j a_j \int_{\gamma} \frac{z^j}{z^{n+1}} \, dz = 2\pi i a_n. \]

This result shows that the coefficients \( a_j \) of the expansion (10.2) are uniquely determined by the function \( f(z) \).

### 10.4.6 Laurent Expansion: Existence

Let \( A \) be as above and let \( f : A \to \mathbb{C} \) be holomorphic. Let \( z \in A \) be arbitrary. Choose \( s_1 \) and \( s_2 \) with

\[ r_1 < s_1 < |z| < s_2 < r_2. \]

Let

\[ \gamma_1(t) = s_1 e^{it}, \quad \gamma_2(t) = s_2 e^{it}, \quad 0 \leq t \leq 2\pi. \]

We claim that

\[ 2\pi i f(z) = \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta =: \text{r.h.s.}. \]

In order to show this, we deform the curves \( \gamma_1 \) and \( \gamma_2 \) so that the right-hand side becomes

\[ \text{r.h.s.} = \int_{\gamma_{\varepsilon}} \frac{f(\zeta)}{\zeta - z} \, d\zeta \]

with

\[ \gamma_{\varepsilon} = z + \varepsilon e^{it}, \quad 0 \leq t \leq 2\pi. \]

Writing

\[ f(\zeta) = f(z) + (f(\zeta) - f(z)) \]

and taking the limit \( \varepsilon \to 0 \) one finds that

\[ \text{r.h.s.} = 2\pi i f(z). \]

The Laurent expansion of \( f(z) \) can now be obtained by employing the geometric sum. The details are as follows. We have

\[ 2\pi i f(z) = \text{Int}_2 - \text{Int}_1 \]

with
\[ Int_k = \int_{\gamma_k} \frac{f(\zeta)}{\zeta - z} \, d\zeta, \quad k = 1, 2. \]

Consider \( Int_2 \) first. We have \(|\zeta| > |z|\), thus

\[ \frac{1}{\zeta - z} = \frac{1}{\zeta (1 - z/\zeta)} = \frac{1}{\zeta} \sum_{j=0}^{\infty} \left( \frac{z}{\zeta} \right)^j. \]

Therefore,

\[ Int_2 = \sum_{j=0}^{\infty} a_j z^j \]

with

\[ a_j = \int_{\gamma_2} \frac{f(\zeta)}{\zeta^{j+1}} \, d\zeta. \]

When considering \( Int_1 \), we note that \(|\zeta| < |z|\). Therefore,

\[ \frac{1}{\zeta - z} = -\frac{1}{z} \frac{1}{1 - \zeta/z} = -\frac{1}{z} \sum_{j=0}^{\infty} \left( \frac{\zeta}{z} \right)^j. \]

This yields

\[ Int_1 = \sum_{j=0}^{\infty} b_j z^{-j-1} \]

with

\[ b_j = -\int_{\gamma_1} f(\zeta) \zeta^j \, d\zeta. \]

We summarize:

**Theorem 10.6** Let \( A = A(P, r_1, r_2) \) denote an open annulus and let \( f \in H(A) \). There are uniquely determined coefficients \( a_j, j \in \mathbb{Z} \), so that

\[ f(z) = \sum_{j=-\infty}^{\infty} a_j (z - P)^j \quad \text{for} \quad z \in A. \]  \hspace{1cm} (10.3)

This series representation of \( f \) is called the Laurent expansion of \( f \) in \( A \).
10.4.7 Local Behavior and Laurent Expansion

Assume that \( f \) has an isolated singularity at \( P \). There are three cases:

- \( P \) is a removable singularity;
- \( P \) is a pole; or
- \( P \) is an essential singularity.

These notions have been defined in Section 10.1 in terms of the local behavior of \( f \) near \( P \).

We can now characterize the three cases in terms of the Laurent expansion of \( f \) near \( P \).

**Theorem 10.7** Let \( f \) be a holomorphic function defined in \( D(P, r) \setminus \{P\} \),

\[
f(z) = \sum_{j=-\infty}^{\infty} a_j(z - P)^j, \quad 0 < |z - P| < r.
\]

a) The point \( P \) is a removable singularity of \( f \) if and only if \( a_j = 0 \) for all \( j < 0 \).

b) The point \( P \) is a pole of \( f \) if and only if there exists \( J < 0 \) with

\[
a_J \neq 0 \quad \text{and} \quad a_j = 0 \quad \text{for all} \quad j < J.
\]

c) The point \( P \) is an essential singularity of \( f \) if and only if there are infinitely many \( j < 0 \) with \( a_j \neq 0 \).

**Proof:**

a) If \( P \) is removable, then \( a_j = 0 \) for all \( j < 0 \) by Riemann’s removability theorem. The converse is trivial.

b) First assume that \( J \) exists, i.e., with \( J = -k \),

\[
f(z) = z^{-k}(a_{-k} + a_{-k+1}z + \ldots) = z^{-k}g(z).
\]

The function \( g(z) \) has a removable singularity at \( z = P \) and \( |g(z)| \geq \frac{1}{2}|a_{-k}| \) for \( |z - P| < \varepsilon \). It follows that \( f(z) \) has a pole at \( z = P \). Conversely, let \( |f(z)| \to \infty \) as \( z \to P \). Set \( g(z) = 1/f(z) \) for \( 0 < |z - P| < \varepsilon \) and apply Riemann’s theorem to \( g(z) \). Obtain that, for some \( m \geq 0 \),

\[
g(z) = z^m(b_m + b_{m+1}z + \ldots), \quad b_m \neq 0.
\]

This yields that

\[
f(z) = z^{-m}Q(z)
\]

where \( Q(z) \) has a holomorphic extension to \( z = P \). The statement c) now follows trivially.

**Terminology:** If \( f(z) \) is as above, then

\[
\sum_{j=-\infty}^{-1} a_j(z - P)^j
\]

is the principle part of \( f \) (about \( P \)). The coefficient

\[
a_{-1} = \text{Res}(f, P)
\]

is called the residue of \( f \) at \( P \).
11 The Calculus of Residues; Applications to the Evaluation of Integrals

11.1 Computation of Residues

Let $f$ be holomorphic in

$$D'(P, r) = D(P, r) \setminus \{P\},$$

i.e., $f$ has an isolated singularity at $P$. We have shown that $f$ has a Laurent expansion in $D'(P, r)$,

$$f(z) = \sum_{j=-\infty}^{\infty} a_j(z - P)^j, \quad 0 < |z - P| < r,$$

where the coefficients $a_j$ are uniquely determined. The coefficient

$$a_{-1} = \text{Res}(f, P)$$

is called the residue of $f$ at $P$.

11.1.1 The Case of a Simple Pole

If $f$ has a simple pole at $P$ then one can write

$$f(z) = \frac{g(z)}{z - P}, \quad 0 < |z - P| < r,$$

where, after extension, $g$ is holomorphic in $D(P, r)$. In this case,

$$a_{-1} = \text{Res}(f, P) = g(P).$$

Example: Let

$$f(z) = \frac{e^z}{(z - 1)(z - 2)}.$$

To determine $\text{Res}(f, 1)$ we write

$$f(z) = \frac{g(z)}{z - 1} \quad \text{with} \quad g(z) = \frac{e^z}{z - 2}.$$

Therefore,

$$\text{Res}(f, 1) = g(1) = -e.$$

To determine $\text{Res}(f, 2)$ we write

$$f(z) = \frac{g(z)}{z - 2} \quad \text{with} \quad g(z) = \frac{e^z}{z - 1}.$$

Therefore,

$$\text{Res}(f, 2) = g(2) = e^2.$$

Another result about the residue at a simple pole is the following:
Lemma 11.1 Let \( f, g \in H(D(P, r)) \) with
\[
g(P) = 0, \quad g'(P) \neq 0, \quad f(P) \neq 0.
\]
Then the function
\[
g(z) = \frac{f(z)}{g(z)}, \quad 0 < |z - P| < \varepsilon,
\]
has a simple pole at \( z = P \) and
\[
\text{Res}(q, P) = \frac{f(P)}{g'(P)}.
\]

Proof: We have
\[
g(z) = g'(P)(z - P) + \mathcal{O}((z - P)^2)
\]
\[
= (z - P)g'(P) \left( 1 + \mathcal{O}(z - P) \right).
\]
Therefore,
\[
q(z) = \frac{f(z)}{g(z)} = \frac{1}{z - P} \left( \frac{f(P)}{g'(P)} + \mathcal{O}(z - P) \right).
\]

Example: Let \( a \in \mathbb{C} \setminus \mathbb{Z} \). We apply the lemma to
\[
q(z) = \frac{\cot(\pi z)}{(z - a)^2} = \frac{\cos(\pi z)}{(z - a)^2 \sin(\pi z)}.
\]
The denominator
\[
g(z) = (z - a)^2 \sin(\pi z)
\]
has a simple zero at each integer \( z = j \in \mathbb{Z} \) and we have
\[
g'(j) = \pi (j - a)^2 \cos(\pi j),
\]
thus
\[
\text{Res}(q, j) = \frac{1}{\pi (j - a)^2}, \quad j \in \mathbb{Z}.
\]

11.1.2 Poles of Order \( k \)
Assume that \( f \) has a pole of order \( k \geq 1 \) at \( P \),
\[
f(z) = \sum_{j=-k}^{\infty} a_j (z - P)^j, \quad 0 < |z - P| < r.
\]
Then we have
and $g(z)$ has a removable singularity at $P$. We have
\[
(d/dz)^l g(z)_{|z=P} = l! a_{-k+l}.
\]
Setting $l = k + j$ we obtain:
\[
(d/dz)^{k+j} \left( (z-P)^k f(z) \right)_{|z=P} = (k+j)! a_j.
\]
For $j = -1$:
\[
(k-1)! a_{-1} = (d/dz)^{k-1} \left( (z-P)^k f(z) \right)_{|z=P}.
\]
One obtains:

**Lemma 11.2** If $f(z)$ has a pole of order $k$ at $P$ then
\[
\text{Res}(f,P) = \frac{1}{(k-1)!} \left( \frac{d}{dz} \right)^{k-1} \left( (z-P)^k f(z) \right)_{|z=P}.
\]

**Example:** Consider the same function as in the previous example,
\[
q(z) = \frac{\cot(\pi z)}{(z-a)^2}
\]
where $a \in \mathbb{C} \setminus \mathbb{Z}$. The function $q(z)$ has a double pole at $z = a$. We use the previous lemma with $k = 2$ to compute the residue of $q(z)$ at $z = a$: We have
\[
\frac{d}{dz} \left( (z-a)^2 q(z) \right) = \frac{d}{dz} \cot(\pi z)
\]
\[
= -\frac{\pi}{\sin^2(\pi z)}.
\]
Therefore,
\[
\text{Res}(q,a) = -\frac{\pi}{\sin^2(\pi a)}.
\]

It may be difficult to remember the previous lemma. One can proceed more directly, as in the following example.

**Example:** Let
\[
f(z) = \frac{e^z}{(z-1)^3}.
\]
The function has a pole of order 3 at $P = 1$. We determine the principle part as follows: Let
\[
g(z) = (z-1)^3 f(z) = e^z.
\]
Then we make a Taylor expansion of $g(z) = e^z$ about $z = 1$:
\[ g(z) = g(1) + g'(1)(z-1) + \frac{1}{2} g''(1)(z-1)^2 + \ldots \]

In our case, \( g^{(j)}(1) = e \) for all \( j \). Therefore,

\[ f(z) = (z-1)^{-3} \left( e + e(z-1) + \frac{e}{2} (z-1)^2 + \ldots \right) \]
\[ = e(z-1)^{-3} + e(z-1)^{-2} + \frac{e}{2} (z-1)^{-1} + \ldots \]

In particular,

\[ \text{Res}(f, 1) = \frac{e}{2}. \]

### 11.2 Calculus of Residues

Suppose that \( f \in H(U) \) has an isolated singularity at \( P \in U \) and let

\[ \gamma_{\varepsilon}(t) = P + \varepsilon e^{it}, \quad 0 \leq t \leq 2\pi. \]

Assume that \( \varepsilon \) is so small that the curve \( \gamma_{\varepsilon} \) encircles only the singularity \( P \) of \( f \), but no other singularities. In this case,

\[ \int_{\gamma_{\varepsilon}} f(z) \, dz = 2\pi i \, a_{-1} \]

with

\[ a_{-1} = \text{Res}(f, P). \]

Together with Cauchy’s theorem, which allows the deformation of paths in regions where \( f \) is holomorphic, this yields a very powerful tool for the evaluation of integrals. We formalize this in the residue theorem.

**Theorem 11.1** (Residue Theorem) Let \( U \subset \mathbb{C} \) be an open set and let \( \Gamma \subset U \) be a simple closed curve which is positively oriented. Let \( V \) denote the region encircled by \( \Gamma \) and assume that \( V \subset U \). Let \( P_1, \ldots, P_k \in V \) and let \( f \in H(U \setminus \{P_1, \ldots, P_k\}) \). Then we have

\[ \int_{\Gamma} f(z) \, dz = 2\pi i \sum_{j=1}^k \text{Res}(f, P_j). \]

### 11.2.1 Direct Applications of the Residue Theorem

In Examples 1 to 3 we evaluate integrals directly using residue calculus.

**Example 1:** Let \( \gamma(t) = e^{it}, 0 \leq t \leq 2\pi \), denote the parametrized unit circle.

We want to evaluate

\[ I = \int_{\gamma} z^2 \sin(1/z) \, dz. \]
We have
\[ \sin w = w - \frac{1}{6}w^3 + \ldots, \]
thus
\[ \sin(1/z) = z^{-1} - \frac{1}{6}z^{-3} + \ldots, \]
thus
\[ z^2 \sin(1/z) = z - \frac{1}{6}z^{-1} + \ldots. \]
Therefore,
\[ I = -\frac{2\pi i}{6} = -\frac{\pi i}{3}. \]

**Example 2:** Let \( \gamma(t) = 2e^{it}, 0 \leq t \leq 2\pi \). We want to evaluate
\[ I = \int_{\gamma} \frac{5z - 2}{z(z - 1)} \, dz. \]
We have
\[ f(z) = \frac{1}{z} \cdot \frac{5z - 2}{z - 1} = \frac{1}{z - 1} \cdot \frac{5z - 2}{z}, \]
thus
\[ Res(f, 0) = 2 \]
and
\[ Res(f, 1) = 3. \]
It follows that
\[ I = (2 + 3)2\pi i = 10\pi i. \]

**Example 3:** Let \( \gamma(t) = 2e^{it}, 0 \leq t \leq 2\pi \). We want to evaluate
\[ I = \int_{\gamma} \frac{\sinh z}{z^4} \, dz. \]
We have
\[ \sinh z = \frac{1}{2}(e^z - e^{-z}) = z + \frac{z^3}{6} + \frac{z^5}{5!} + \ldots \]
Therefore,
\[ \frac{\sinh z}{z^4} = z^{-3} + \frac{z^{-1}}{6} + \frac{z}{5!} + \ldots. \]
This yields that

\[ I = \frac{\pi i}{3}. \]

### 11.2.2 Substitution of \( z = e^{it} \)

Integrals involving trigonometric functions can sometimes be rewritten as complex line integrals and then be evaluated using the calculus of residues.

In the following example we use the substitution

\[ z(t) = e^{it}, \quad 0 \leq t \leq 2\pi, \]

to turn an integral involving a trigonometric function into an integral along the unit circle, \( C_1 \).

**Example 4:** For \( a > 1 \) evaluate

\[ I = \int_0^\pi \frac{dt}{a + \cos t}. \]

We have

\[ 2I = \int_0^{2\pi} \frac{dt}{a + (e^{it} + e^{-it})/2}. \]

In general, if \( z(t) = e^{it} \) parametrizes the unit circle, then

\[ \int_{C_1} f(z) \, dz = \int_0^{2\pi} f(e^{it})ie^{it} \, dt. \]

Thus we define \( f(z) \) by

\[ f(z)iz = \frac{1}{a + (z + 1/z)/2} \]

and obtain

\[ f(z) = \frac{2}{i} \cdot \frac{1}{z^2 + 2az + 1}. \]

This yields

\[ I = \frac{1}{i} \int_{C_1} \frac{dz}{z^2 + 2az + 1}. \]

Thus we have rewritten the given integral \( I \) as a complex line integral. We now evaluate \( I \) using residue calculus. The solutions of

\[ z^2 + 2az + 1 = 0 \]

are

\[ z_1 = -a + \sqrt{a^2 - 1}, \quad z_2 = -a - \sqrt{a^2 - 1} \]
with $z_1 z_2 = 1$, thus

$$z_2 < -1 < z_1 < 0.$$ 

Also,

$$g(z) = \frac{1}{z^2 + 2az + 1} = \frac{1}{(z - z_1)(z - z_2)},$$

thus

$$\text{Res}(g, z_1) = \frac{1}{z_1 - z_2} = \frac{1}{2\sqrt{a^2 - 1}}.$$ 

Therefore,

$$I = \frac{\pi}{\sqrt{a^2 - 1}}.$$ 

11.2.3 Integrals over $-\infty < x < \infty$

**Example 5:** We know from calculus that

$$I := \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi . \tag{11.1}$$

(In calculus, one uses that $(d/dx) \arctan x = (1 + x^2)^{-1}$.) Let us obtain (11.1) using the calculus of residues. Let

$$\Gamma_1(x) = x, \quad -R \leq x \leq R$$

and

$$\Gamma_2(t) = \Re e^{it}, \quad 0 \leq t \leq \pi .$$

Then $\Gamma_R = \Gamma_1 + \Gamma_2$ is a closed curve, consisting of the part $-R \leq x \leq R$ of the $x$-axis and a semi-circle in the upper half-plane.

We assume $R > 1$. Then, by residue calculus,
\[
\int_{\Gamma_R} \frac{dz}{z^2 + 1} = \int_{\Gamma_R} \frac{dz}{(z-i)(z+i)} = \frac{2\pi i}{2i} = \pi
\]

We have

\[
\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{R \to \infty} \int_{\Gamma_{1R}} \frac{dz}{z^2 + 1}
\]

and the corresponding integral along \(\Gamma_{2R}\) tends to zero as \(R \to \infty\). Therefore, \(I = \pi\).

**Example 6:** We claim that, for \(a > 0\),

\[
I = \int_{-\infty}^{\infty} \frac{\cos x}{a^2 + x^2} \, dx = \frac{\pi}{a} e^{-a}.
\]

A simple bound for the integral follows from

\[
|I| \leq \int_{-\infty}^{\infty} \frac{1}{a^2 + x^2} = \frac{\pi}{a}.
\]

Let \(\Gamma_{1R}, \Gamma_{2R}\), and \(\Gamma_R\) be defined as in Example 5. One should note that

\[
\cos z = \frac{1}{2} (e^{iz} + e^{-iz})
\]

becomes exponentially large in the upper half-plane: If \(z = x + iy\), then

\[
|e^{-iz}| = e^y, \quad y \geq 0.
\]

Thus we cannot directly proceed as in the previous example, because the integral of \(\cos z/(a^2 + z^2)\) along \(\Gamma_{2R}\) does not converge to zero as \(R \to \infty\). Instead, we note that

\[
\cos z = \text{Re} \, e^{iz} \quad \text{for} \quad z = x \in \mathbb{R}.
\]

Therefore,

\[
I = \text{Re} \int_{-\infty}^{\infty} \frac{e^{iz}}{a^2 + z^2} \, dz.
\]

We have

\[
g(z) := \frac{e^{iz}}{a^2 + z^2} = \frac{e^{iz}}{(z-ia)(z+ia)}
\]

with

\[
\text{Res}(g, ia) = \frac{e^{-a}}{2ia}.
\]

Therefore, for \(R > a\),
\[
\int_{\Gamma_R} g(z) \, dz = 2\pi i \text{Res}(g, ia) = \frac{\pi}{a} e^{-a}.
\]

It remains to show that
\[
\int_{\Gamma_{2R}} g(z) \, dz \to 0 \quad \text{as} \quad R \to \infty. \tag{11.2}
\]

Note that \(|e^{iz}| \leq 1\) in the upper half-plane. Also, if \(|z| = R \geq 2a\) then
\[
|a^2 + z^2| \geq |z|^2 - a^2 \geq \frac{3}{4} R^2,
\]
thus
\[
|g(z)| \leq \frac{4}{3} R^{-2}.
\]

This implies (11.2).

### 11.2.4 Extensions Using Jordan’s Lemma

**Example 7:** We claim that, for \(a > 0\),
\[
I = \int_{-\infty}^{\infty} \frac{x \sin x}{a^2 + x^2} \, dx = \pi e^{-a}.
\]

Here, by definition,
\[
I = \lim_{R \to \infty} \int_{-R}^{R} \frac{x \sin x}{a^2 + x^2} \, dx.
\]

Let \(\Gamma_1, \Gamma_2,\) and \(\Gamma_R\) be defined as in Example 5. We have
\[
\sin z = \text{Im} e^{iz}.
\]

Setting
\[
g(z) = \frac{ze^{iz}}{a^2 + z^2} = \frac{ze^{iz}}{(z - ia)(z + ia)}, \tag{11.3}
\]

we have
\[
I = \text{Im} \lim_{R \to \infty} \int_{\Gamma_1} g(z) \, dz.
\]

In this example,
\[
\text{Res}(g, ia) = \frac{1}{2} e^{-a},
\]
thus, for \(R > a\),

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It remains to prove (11.2) for the function $g(z)$ defined in (11.3). Note that the estimate of the previous example, $|g(z)| \leq CR^{-2}$ for $z \in \Gamma_{2R}$, does not hold here. We must estimate the integral along $\Gamma_{2R}$ more carefully.

**Theorem 11.2** (Jordan’s Lemma) Recall that $\Gamma_{2R}$ denotes the semi-circle with parametrization

$$z(t) = Re^{it}, \quad 0 \leq t \leq \pi.$$ 

Let $\overline{\mathbb{H}}$ denote the closed upper half-plane and let $f : \overline{\mathbb{H}} \to \mathbb{C}$ be a continuous function. Let

$$M_R = \max \{|f(z)| : z \in \Gamma_{2R}\}$$

and assume that $M_R \to 0$ as $R \to \infty$. Then we have

$$I_R := \int_{\Gamma_{2R}} f(z)e^{iz} \, dz \to 0 \quad \text{as} \quad R \to \infty.$$ 

**Proof:** Noting that

$$z(t) = R(\cos t + i \sin t) \quad \text{and} \quad |z'(t)| = R$$

we have

$$|I_R| \leq M_R \int_0^\pi |e^{iz(t)}| R \, dt$$

$$= RM_R \int_0^\pi e^{-R \sin t} \, dt$$

$$= 2RM_R \int_0^{\pi/2} e^{-R \sin t} \, dt.$$ 

Since
\[
\sin t \geq \frac{2t}{\pi} \quad \text{for} \quad 0 \leq t \leq \frac{\pi}{2}
\]
we have, with \( c = 2R/\pi \):
\[
\int_0^{\pi/2} e^{-R \sin t} \, dt \leq \int_0^{\pi/2} e^{-ct} \, dt \\
\leq \frac{1}{c} \frac{\pi}{2R}
\]
Therefore,
\[
|I_R| \leq \pi M_R \to 0 \quad \text{as} \quad R \to \infty.
\]

11.2.5 A Pole on the Real Axis

Example 8: We want to show
\[
\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \pi.
\]
The integral exists as an improper Riemann integral.

We first discuss the existence of the integral. The integral \( \int_1^{\infty} \frac{\sin x}{x} \, dx \) does not exist as a proper Riemann or Lebesgue integral since the integrand decays too slowly. To see this, note that, for \( j = 1, 2, \ldots \)
\[
|\sin x| \geq \frac{1}{\sqrt{2}} \quad \text{for} \quad \pi(j + \frac{1}{4}) \leq x \leq \pi(j + \frac{3}{4})
\]
Therefore,
\[
\frac{|\sin x|}{x} \geq \frac{1}{\sqrt{2}} \frac{1}{\pi(j + 1)} =: \frac{c}{j + 1} \quad \text{for} \quad \pi(j + \frac{1}{4}) \leq x \leq \pi(j + \frac{3}{4})
\]
It follows that
\[
\int_{\pi j}^{\pi(j+1)} \frac{|\sin x|}{x} \, dx \geq \frac{\pi c}{j + 1}.
\]
Since \( \sum_{j=1}^{\infty} \frac{1}{j+1} = \infty \) one obtains that
\[
\int_{\pi}^{\infty} \frac{|\sin x|}{x} \, dx = \infty.
\]
A theorem of integration theory implies that
\[
\int_{\pi}^{\infty} \frac{\sin x}{x} \, dx
\]
does not exist.

However, for $1 < R < \infty$:

$$\int_1^R \frac{\sin x}{x} \, dx = -\frac{1}{x} \cos x \bigg|_1^R - \int_1^R \frac{1}{x^2} \cos x \, dx$$

$$= -\frac{1}{R} \cos R + \cos 1 - \int_1^R \frac{1}{x^2} \cos x \, dx$$

Therefore,

$$\lim_{R \to \infty} \int_1^R \frac{\sin x}{x} \, dx$$

does exist. By definition,

$$I := P.V. \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{\sin x}{x} \, dx \quad (11.4)$$

where $P.V.$ stands for principle value. It is common to drop the $P.V.$ notation and to say that the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx$$

exists as an improper integral, defined by (11.4).

Computation of $I$: We have

$$I = \lim_{R \to \infty, \varepsilon \to 0} I(R, \varepsilon)$$

with

$$I(R, \varepsilon) = \int_{-\varepsilon}^{-R} \frac{\sin x}{x} \, dx + \int_{\varepsilon}^{R} \frac{\sin x}{x} \, dx.$$  

Also, for $x = z \in \mathbb{R}$:

$$\frac{\sin x}{x} = \text{Im} \left( \frac{e^{iz}}{z} \right),$$

thus

$$I(R, \varepsilon) = \text{Im} \left( \int_{-\varepsilon}^{-R} \frac{e^{iz}}{z} \, dz + \int_{\varepsilon}^{R} \frac{e^{iz}}{z} \, dz \right).$$

The term in brackets is

$$K(R, \varepsilon) := \int_{\Gamma_{-R,-\varepsilon} + \Gamma_{\varepsilon,R}} \frac{e^{iz}}{z} \, dz.$$  

Let $\Gamma$ denote the closed curve shown in Figure 4:

$$\Gamma = \Gamma_{-R,-\varepsilon} + \Gamma_{-\varepsilon,\varepsilon} + \Gamma_{\varepsilon,R} + \Gamma_{2R}.$$
By Cauchy’s theorem,
\[ \int_{\Gamma} \frac{e^{iz}}{z} \, dz = 0. \]

Therefore,
\[ K(R, \varepsilon) := -\int_{\Gamma_{-\varepsilon, \varepsilon} + \Gamma_{2R}} \frac{e^{iz}}{z} \, dz. \]

By Jordan’s lemma, the integral along \( \Gamma_{2R} \) tends to zero as \( R \to \infty \). Also,
\[ \frac{e^{iz}}{z} = \frac{1}{z} + g(z) \]

where \( g(z) \) is holomorphic near \( z = 0 \). Therefore,
\[ \lim_{\varepsilon \to 0} \int_{\Gamma_{-\varepsilon, \varepsilon}} \frac{e^{iz}}{z} \, dz = -\pi i. \]

One obtains that
\[ \lim_{R \to \infty, \varepsilon \to 0} K(R, \varepsilon) = \pi i, \]

thus \( I = \pi \).

**Remarks on Fourier transforms:** Let \( \chi_J(x) \) denote the characteristic function of the interval \( J = [-1, 1] \). Its Fourier transform is
\[
\hat{\chi}_J(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_J(x) e^{-ikx} \, dx \\
= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ikx} \, dx \\
= \frac{1}{\sqrt{2\pi}} \frac{1}{-ik} (e^{-ik} - e^{ik}) \\
= \frac{2}{\sqrt{\pi}} \frac{\sin k}{k}, \quad k \neq 0.
\]

The function \(\hat{\chi}_J(k)\) is not integrable over \(\mathbb{R}\). The inverse Fourier transform of \(\hat{\chi}_J(k)\) exists only in the sense of principle values. We have for the inverse Fourier transform of \(\hat{\chi}_J(k)\):

\[
g(x) := \frac{1}{\sqrt{2\pi}} \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\sin k}{k} e^{ikx} \, dk.
\]

We have shown that \(\int_{-\infty}^{\infty} (\sin k)/k \, dk = \pi\) and obtain

\[
g(0) = \frac{1}{\sqrt{2\pi}} \frac{2}{\sqrt{\pi}} \pi = 1.
\]

This is to be expected since \(\chi_J(0) = 1\). One can also say that the formula \(\int_{-\infty}^{\infty} (\sin k)/k \, dk = \pi\) is a special case of the Fourier inversion theorem.

### 11.2.6 Use of a Second Path

**Example 9:** For \(0 < a < 1\):

\[
\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \, dx = \frac{\pi}{\sin(\pi a)}.
\]

This integral will be used below to show the reflection property of the \(\Gamma\)-function.

Let \(f(z) = \frac{e^{az}}{1 + e^z}\). Consider the rectangle \(\mathcal{R}\) with corners at

\[-R, \quad R, \quad R + 2\pi i, \quad -R + 2\pi i.\]

Denote the positively oriented boundary curve of \(\mathcal{R}\) by

\(\Gamma_R = \Gamma_{1R} + \Gamma_{2R} + \Gamma_{3R} + \Gamma_{4R}\).

The pieces have parametrizations

\[
\Gamma_{1R} : z(x) = x, \quad -R \leq x \leq R \\
-\Gamma_{3R} : z(x) = x + 2\pi i, \quad -R \leq x \leq R \\
\Gamma_{2R} : z(y) = R + iy, \quad 0 \leq y \leq 2\pi \\
-\Gamma_{4R} : z(y) = -R + iy, \quad 0 \leq y \leq 2\pi
\]
The function $f(z)$ has one singularity in the rectangle $\mathcal{R}$. The singularity is a simple pole at $P = \pi i$ and

$$Res(f, \pi i) = -e^{a\pi i}.$$ 

By the residue theorem:

$$\int_{\Gamma_{R}} f(z)\,dz = -2\pi i e^{a\pi i}.$$ 

It is not difficult to show that

$$Q_{R} := \int_{\Gamma_{2R}+\Gamma_{4R}} f(z)\,dz \to 0 \text{ as } R \to \infty.$$ 

Set

$$I_{R} = \int_{\Gamma_{1R}} f(z)\,dz = \int_{-R}^{R} f(x)\,dx.$$ 

The main trick of the whole approach is that the integral $I_{R}$ occurs again when one integrates along $\Gamma_{3R}$:

$$\int_{-\Gamma_{3R}} f(z)\,dz = e^{2\pi ai} \int_{-R}^{R} f(x)\,dx = e^{2\pi ai} I_{R}.$$ 

Therefore,

$$-2\pi i e^{a\pi i} = I_{R}(1 - e^{2\pi ai}) + Q_{R}.$$ 

This implies that

$$I_{R} = \frac{2\pi i}{e^{\pi ai} - e^{-\pi ai}} + \tilde{Q}_{R} = \frac{\pi}{\sin(\pi a)} + \tilde{Q}_{R}.$$ 

As $R \to \infty$ one obtains (11.5).

11.3 Derivation of a Partial Fraction Decomposition via Integration

**Example 10:** Let $a \in \mathbb{C} \setminus \mathbb{Z}$ and consider the function

$$q(z) = \frac{\cot(\pi z)}{(z - a)^2}.$$ 

We have seen that $q$ has a simple pole at each integer $j$ and a double pole at $z = a$. Also,

$$Res(q, j) = \frac{1}{\pi(j - a)^2}, \quad j \in \mathbb{Z},$$

and

$$Res(q, a) = -\frac{\pi}{\sin^2(\pi a)}.$$
For positive integers \( n \), let \( \gamma_n \) denote the boundary curve of the rectangle in Figure 6. Assume that \( n > |a| \). By the residue theorem,

\[
\frac{1}{2\pi i} \int_{\gamma_n} q(z) \, dz = \sum_{j=-n}^{n} \frac{1}{\pi(j-a)^2} - \frac{\pi}{\sin^2(\pi a)}.
\]

(11.6)

By estimating the integrand \( q(z) \) on \( \gamma_n \) we will prove that

\[
\int_{\gamma_n} q(z) \, dz \to 0 \quad \text{as} \quad n \to \infty.
\]

The following lemma will be used to bound \( \cot(\pi z) \) on \( \gamma_n \). If one sets \( w = \pi z, \quad Q = e^{2\pi i z} \) then

\[
\cot(\pi z) = \frac{1}{2i} \frac{e^{iw} + e^{-iw}}{e^{iw} - e^{-iw}} = i \frac{Q + 1}{Q - 1}.
\]

Therefore, in order to bound \( |\cot(\pi z)| \) for \( z \in \gamma_n \), we need to bound \( |Q - 1| \) away from zero. We show:

**Lemma 11.3** For \( n = 1, 2, \ldots \) let \( z \in \gamma_n \) and set \( Q = e^{2\pi i z} \). Then we have

\[
|Q - 1| \geq \frac{1}{2}.
\]

**Proof:**

a) Let \( z = (n + \frac{1}{2}) + iy, \ y \in \mathbb{R} \). We have

\[
Q = e^{2\pi i (n+\frac{1}{2})} e^{-2\pi y} = -e^{-2\pi y} < 0,
\]

thus \( |Q - 1| > 1 \).

The same argument works for \( z = -(n + \frac{1}{2}) + iy, \ y \in \mathbb{R} \).

b) Let \( z = x + ni, \ x \in \mathbb{R} \). We have

\[
Q = e^{2\pi i x} e^{-2\pi n}, \quad |Q| \leq e^{-2\pi} < \frac{1}{2}.
\]
c) Let \( z = x - ni, x \in \mathbb{R} \). We have

\[
Q = e^{2\pi ix}e^{2\pi n}, \quad |Q| \geq e^{2\pi} > 2.
\]

This proves the lemma. \( \diamond \)

**Lemma 11.4** For \( n = 1, 2, \ldots \) we have

\[
|\cot(\pi z)| \leq 6 \quad \text{for all} \quad z \in \gamma_n.
\]

**Proof:** Let \( w = \pi z, Q = e^{2iw} = e^{2\pi iz} \). We have

\[
\cot(\pi z) = \frac{1}{2} \frac{e^{iw} + e^{-iw}}{e^{iw} - e^{-iw}} = \frac{1}{2} \frac{Q + 1}{Q - 1}.
\]

By the previous lemma, \(|Q - 1| \geq \frac{1}{2} \). Case 1: \(|Q| \geq 2\), thus

\[
1 \leq \frac{1}{2} |Q|.
\]

We have

\[
|Q + 1| \leq |Q| + 1 \leq \frac{3}{2} |Q|
\]

\[
|Q - 1| \geq |Q| - 1 \geq \frac{1}{2} |Q|
\]

thus

\[
\left| \frac{Q + 1}{Q - 1} \right| \leq 3.
\]

Case 2: \(|Q| \leq 2\). Recall that \(|Q - 1| \geq \frac{1}{2} \). We have

\[
\left| \frac{Q + 1}{Q - 1} \right| \leq \frac{3}{2} = 6.
\]

This proves the lemma. \( \diamond \)

Let \( \Omega \) be a compact subset of the open set \( U = \mathbb{C} \setminus \mathbb{Z} \). Let \( a \in \Omega \). There is a constant \( C \), depending on \( \Omega \) but not on \( a \), so that

\[
|q(z)| \leq \frac{C}{n^2} \quad \text{for} \quad z \in \gamma_n, \quad a \in \Omega,
\]

for \( n \geq N = N(\Omega) \). The detailed argument is as follows: If \( z \in \gamma_n \), then \(|z| \geq n\). Since \( \Omega \) is bounded, there is \( N(\Omega) \) with

\[
2|a| \leq N(\Omega) \quad \text{for all} \quad a \in \Omega.
\]

If \( n \geq N(\Omega) \) then \( n \geq 2|a| \), thus

\[
|z - a| \geq |z| - |a| \geq n - \frac{n}{2} = \frac{n}{2}.
\]

This implies that

\[
\frac{1}{|(z - a)^2|} \leq \frac{4}{n^2} \quad \text{for} \quad n \geq N(\Omega) \quad \text{and} \quad a \in \Omega.
\]
One obtains that
\[ \left| \int_{\gamma_n} q(z) \, dz \right| \leq \frac{C_1}{n} \text{ for } n \geq N(\Omega). \]

This proves that
\[ \left| \sum_{j=-n}^{n} \frac{1}{(j-a)^2} - \frac{\pi^2}{\sin^2(\pi a)} \right| \leq \frac{\pi C_1}{n} \text{ for } n \geq N(\Omega). \]

We now write \( a = -z \) and obtain:

**Theorem 11.3** We have
\[ \lim_{n \to \infty} \sum_{j=-n}^{n} \frac{1}{(j+z)^2} = \frac{\pi^2}{\sin^2(\pi z)} \text{ for } z \in \mathbb{C} \setminus \mathbb{Z}. \]

The convergence is uniform on compact subsets of \( \mathbb{C} \setminus \mathbb{Z} \).

The above formula is also written as
\[ \sum_{j=-\infty}^{\infty} \frac{1}{(j+z)^2} = \frac{\pi^2}{\sin^2(\pi z)}, \quad z \in \mathbb{C} \setminus \mathbb{Z}. \quad (11.7) \]

The left–hand side is called the partial fraction decomposition of the meromorphic function
\[ f(z) = \frac{\pi^2}{\sin^2(\pi z)}, \quad z \in \mathbb{C} \setminus \mathbb{Z}. \]

**The Special Value** \( z = \frac{1}{2} \). By substituting special values for \( z \) into (11.7) one can obtain interesting (and uninteresting) results. For \( z = \frac{1}{2} \) obtain:
\[
\pi^2 = 4 \sum_{0}^{\infty} \frac{1}{(2j+1)^2} + 4 \sum_{-\infty}^{-1} \frac{1}{(2j+1)^2} \\
= 8 \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \right),
\]
thus
\[ \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}. \]

With a trick we can also evaluate the following series:
\[
S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \\
= \frac{\pi^2}{8} + \frac{1}{4} \left( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \right) \\
= \frac{\pi^2}{8} + \frac{1}{4} S
\]

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Therefore, $S = \frac{\pi^2}{6}$. We have shown that

$$\zeta(2) = \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}.$$ 

Here the Riemann zeta–function is defined by

$$\zeta(z) = \sum_{j=1}^{\infty} \frac{1}{j^z}, \quad \text{Re} \, z > 1.$$ 

11.4 The Partial Fraction Decomposition of $\pi \cot(\pi z)$

We want to show here that the partial fraction decomposition of

$$f(z) = \pi \cot(\pi z), \quad z \in U := \mathbb{C} \setminus \mathbb{Z},$$

can be obtained by integrating (11.7). First note that

$$\frac{d}{dz} \pi \cot(\pi z) = -\frac{\pi^2}{\sin^2(\pi z)}$$
$$\frac{d}{dz} (j + z)^{-1} = -(j + z)^{-2}$$

We define

$$t_n(z) = \sum_{j=-n}^{n} (j + z)^{-1}.$$ 

By Theorem 11.3 we have

$$\lim_{n \to \infty} \frac{d}{dz} t_n(z) = \frac{d}{dz} \pi \cot(\pi z), \quad z \in U,$$

where the convergence is uniform on compact sets in $U$.

Fix any $z_0 \in U$ and let $\Gamma(z_0)$ denote a curve in $U$ from $P = \frac{1}{2}$ to $z_0$. Since $\cot(\pi/2) = 0$ we obtain

$$\int_{\Gamma(z_0)} \frac{d}{dz} (\pi \cot(\pi z)) \, dz = \pi \cot(\pi z_0)$$
$$\int_{\Gamma(z_0)} \frac{d}{dz} t_n(z) \, dz = t_n(z_0) - t_n(1/2)$$

Integrate equation (11.8) along $\Gamma(z_0)$ to obtain

$$\lim_{n \to \infty} \left( t_n(z_0) - t_n(1/2) \right) = \pi \cot(\pi z_0).$$

Note that we are allowed to exchange the limit, as $n \to \infty$, with integration along $\Gamma(z_0)$ since the convergence in (11.8) is uniform on compact sets.) 

We have
\[ t_n(1/2) = \sum_{j=0}^{n} \frac{1}{j + \frac{1}{2}} + \sum_{j=1}^{n} \frac{1}{-j + \frac{1}{2}} \]
\[ = \left( \frac{1}{1/2} + \frac{1}{1 + 1/2} + \frac{1}{2 + 1/2} + \ldots + \frac{1}{n + 1/2} \right) + \left( \frac{1}{-1 + 1/2} + \frac{1}{-2 + 1/2} + \frac{1}{-3 + 1/2} + \ldots + \frac{1}{-n + 1/2} \right) \]
\[ = \left( \frac{1}{1/2} + \frac{1}{1 + 1/2} + \frac{1}{2 + 1/2} + \ldots + \frac{1}{n + 1/2} \right) - \left( \frac{1}{1/2} + \frac{1}{1 + 1/2} + \frac{1}{2 + 1/2} + \ldots + \frac{1}{n - 1 + 1/2} \right) \]
\[ = \frac{1}{n + 1/2} \]

This shows that \( t_n(1/2) \to 0 \) as \( n \to \infty \). We have shown that

\[ \lim_{n \to \infty} \sum_{j=-n}^{n} \frac{1}{j + z} = \pi \cot(\pi z), \quad z \in U. \]

In this case, it is not good to write this result as

\[ \sum_{j=-\infty}^{\infty} \frac{1}{j + z} = \pi \cot(\pi z), \quad z \in U, \]

since the series

\[ \sum_{j=0}^{\infty} \frac{1}{j + z} \]

does not converge. However,

\[ t_n(z) = \sum_{j=-n}^{n} \frac{1}{j + z} \]
\[ = \frac{1}{z} + \sum_{j=1}^{n} \frac{2z}{z^2 - j^2}. \]

One obtains

\[ \frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2} = \pi \cot(\pi z), \quad z \in U. \]

This is the partial fraction decomposition of \( \pi \cot(\pi z) \).

### 11.5 Summary of Examples

**Example 1:** Let \( \gamma(t) = e^{it}, 0 \leq t \leq 2\pi \), denote the parametrized unit circle. Then we have

\[ \int_{\gamma} z^2 \sin(1/z) \, dz = -\frac{\pi i}{3}. \]
Example 2: Let $\gamma(t) = 2e^{it}$. We have
\[ \int_{\gamma} \frac{5z - 2}{z(z - 1)} \, dz = 10\pi i. \]

Example 3: Let $\gamma(t) = 2e^{it}$. We have
\[ \int_{\gamma} \frac{\sinh z}{z^4} \, dz = \frac{\pi i}{3}. \]

Example 4: For $a > 1$:
\[ \int_{0}^{\pi} \frac{dt}{a + \cos t} = \frac{\pi}{\sqrt{a^2 - 1}}. \]

Example 5: We know from calculus that
\[ \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi, \]
which can also be obtained using residues.

Example 6: For $a > 0$:
\[ \int_{-\infty}^{\infty} \frac{\cos x}{a^2 + x^2} \, dx = \frac{\pi}{a} e^{-a}. \]

Example 7: For $a > 0$:
\[ \int_{-\infty}^{\infty} \frac{x \sin x}{a^2 + x^2} \, dx = \pi e^{-a}. \]
(This requires Jordan’s lemma.)

Example 8: We have:
\[ \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \pi. \]
The integral exists as an improper Riemann integral.

Example 9: For $0 < a < 1$:
\[ \int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \, dx = \frac{\pi}{\sin(\pi a)} \cdot \frac{\pi^2}{\sin^2(\pi z)}. \]

Example 10: Let $z \in \mathbb{C} \setminus \mathbb{Z}$. Then we have:
\[ \sum_{j=-\infty}^{\infty} \frac{1}{(j + z)^2} = \frac{\pi^2}{\sin^2(\pi z)}. \]
This follows by integrating
\[ q(\zeta) = \frac{\cot(\pi \zeta)}{(\zeta + z)^2} \]
along a closed rectangle $\gamma_n$ for $n \to \infty$.

Example 11: For $z \in \mathbb{C} \setminus \mathbb{Z}$ we have:
\[ \frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2} = \pi \cot(\pi z). \]

This partial fraction decomposition can be obtained by integrating the partial fraction decomposition of the previous example.

### 11.6 Practice Problems

**Problem 1:** Show

\[ \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} \, dx = \int_{-\infty}^{\infty} \frac{\sin x}{1 + x^2} \, dx = \pi e. \]

**Problem 2:** Show

\[ \int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} \, dx = \frac{\pi}{\sqrt{2}}. \]

**Problem 3:** Show

\[ \int_{0}^{\infty} \frac{1 - \cos x}{x^2} \, dx = \frac{\pi}{2}. \]
The Bernoulli Numbers, the Values $\zeta(2m)$, and Sums of Powers

12.1 The Bernoulli Numbers

The function $g(z)$ defined by

$$g(z) = \frac{z}{(e^z - 1)} \quad \text{for} \quad 0 < |z| < 2\pi, \quad g(0) = 1,$$

is holomorphic in $D(0, 2\pi)$. We write its Taylor series as

$$g(z) = \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} z^\nu, \quad |z| < 2\pi,$$

where the numbers $B_\nu$ are, by definition, the Bernoulli numbers. Since

$$g(z) = \frac{1}{1 + \frac{1}{2} z + \frac{1}{6} z^2 + \ldots}$$

$$= 1 - \frac{1}{2} z + \ldots$$

it follows that

$$B_0 = 1, \quad B_1 = -\frac{1}{2}.$$

Lemma 12.1 The function

$$h(z) = g(z) + \frac{z}{2}$$

is even. Consequently,

$$B_\nu = 0 \quad \text{for} \quad \nu \geq 3, \quad \nu \text{ odd}.$$

Proof: We must show that

$$g(-z) - \frac{z}{2} = g(z) + \frac{z}{2},$$

i.e.,

$$g(-z) - g(z) = z.$$

With $a = e^z$ we have

$$g(-z) - g(z) = \frac{-z}{e^{-z} - 1} - \frac{z}{e^z - 1}$$

$$= z \left( \frac{-1}{a - 1} - \frac{1}{a - 1} \right)$$

$$= z \left( \frac{-a}{1-a} - \frac{1}{a-1} \right)$$

$$= z.$$
One can compute the Bernoulli numbers easily using a recursion. Recall the binomial coefficients
\[ \binom{n}{\nu} = \frac{n!}{\nu!(n-\nu)!} \]

We claim:

**Lemma 12.2** For \( n \geq 1 \) we have
\[ \sum_{\nu=0}^{n} \binom{n+1}{\nu} B_\nu = 0 . \]

**Proof:** We have, for \( 0 < |z| < 2\pi \):
\[
1 = \frac{e^z - 1}{z} \cdot \frac{z}{e^z - 1} = \left( \sum_{\mu=0}^{\infty} \frac{z^\mu}{(\mu+1)!} \right) \cdot \left( \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} z^\nu \right) = \sum_{\mu,\nu=0}^{\infty} \frac{B_\nu}{\nu!(\mu+1)!} z^{\mu+\nu} \quad \text{(with } \mu = n - \nu) \]
\[
= \sum_{n=0}^{\infty} \left( \sum_{\nu=0}^{n} \frac{B_\nu}{\nu!(n+1-\nu)!} \right) z^n
\]

Since
\[ \binom{n+1}{\nu} = \frac{(n+1)!}{\nu!(n+1-\nu)!} \]
the lemma is proved. \( \diamond \)

Using Pascal’s triangle, we can compute the binomial coefficients. Then, using the previous lemma and \( B_0 = 1 \) we obtain:

For \( n = 1 \):
\[ B_0 + 2B_1 = 0, \quad \text{thus } B_1 = -\frac{1}{2} . \]

For \( n = 2 \):
\[ B_0 + 3B_1 + 3B_2 = 0, \quad \text{thus } B_2 = \frac{1}{6} . \]

For \( n = 3 \):
\[ B_0 + 4B_1 + 6B_2 + 4B_3 = 0, \quad \text{thus } B_3 = 0 . \]

For \( n = 4 \):
\[ B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 = 0, \quad \text{thus } B_4 = -\frac{1}{30} . \]
Continuing this process, one obtains the following non–zero Bernoulli numbers:

\[
\begin{align*}
B_6 &= \frac{1}{42} \\
B_8 &= -\frac{1}{30} \\
B_{10} &= \frac{5}{66} \\
B_{12} &= -\frac{691}{2730} \\
B_{14} &= \frac{7}{6}
\end{align*}
\]

etc.

**Remark:** The sequence \(|B_{2\nu}|\) is unbounded since otherwise the series (12.1) would have a finite radius of convergence. More precisely, by Hadamard’s formula,

\[
\limsup_{\nu \to \infty} \left( \frac{|B_{2\nu}|}{(2\nu)!} \right)^{1/(2\nu)} = \frac{1}{2\pi}.
\]

Also, we will see below that \((-1)^{\nu+1}B_{2\nu} > 0\). Thus, the sign pattern observed for \(B_2\) to \(B_{14}\) continuous.

### 12.2 The Taylor Series for \(z\cot z\) in Terms of Bernoulli Numbers

Recall that

\[
g(w) = \frac{w}{e^w - 1} = \sum_{\nu=0}^{\infty} \frac{B\nu}{\nu!} w^\nu.
\]

We now express the Taylor series for \(z\cot z\) about \(z = 0\) in terms of Bernoulli numbers. Note that

\[
\begin{align*}
\cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) \\
\sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) \\
\cot z &= i \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \\
&= i \frac{1 + e^{-2iz}}{1 - e^{-2iz}} \\
&= i \frac{1 - e^{-2iz} + 2e^{-2iz}}{1 - e^{-2iz}} \\
&= i \left(1 + \frac{2}{e^{2iz} - 1}\right) \quad \text{for} \quad 0 < |z| < \pi.
\end{align*}
\]

Therefore,
\[
\cot z = i + \frac{1}{z} \cdot \frac{2iz}{e^{2iz} - 1} \quad \text{for} \quad 0 < |z| < \pi ,
\]

thus, for \(|z| < \pi\):

\[
z \cot z = iz + g(2iz) = iz + 1 - \frac{1}{2} (2iz) + \sum_{\nu=2}^{\infty} \frac{B_{\nu}}{\nu!} (2iz)^{\nu} = 1 + \sum_{\nu=1}^{\infty} (-1)^{\nu} \frac{4^{\nu}}{(2\nu)!} B_{2\nu} z^{2\nu} .
\]

We substitute \(\pi z\) for \(z\) and summarize:

**Lemma 12.3** If \(B_{\nu}\) denotes the sequence of the Bernoulli numbers, then we have for \(|z| < 1\):

\[
\pi z \cot(\pi z) = 1 + \sum_{\nu=1}^{\infty} (-1)^{\nu} \frac{(2\pi)^{2\nu}}{(2\nu)!} B_{2\nu} z^{2\nu} . \tag{12.2}
\]

**12.3 The Mittag–Leffler Expansion of \(\pi z \cot(\pi z)\)**

We have shown the following partial fraction decomposition (also called Mittag–Leffler expansion):

\[
\pi \cot(\pi z) = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} , \quad z \in \mathbb{C} \setminus \mathbb{Z} .
\]

Therefore,

\[
\pi z \cot(\pi z) = 1 - 2 \sum_{n=1}^{\infty} \frac{z^2}{n^2 - z^2} , \quad z \in \mathbb{C} \setminus \mathbb{Z} .
\]

Here, for \(|z| < 1\):

\[
\frac{z^2}{n^2 - z^2} = \frac{(z/n)^2}{1 - (z/n)^2} = \sum_{m=1}^{\infty} \left( \frac{z}{n} \right)^{2m}
\]

Therefore,
\[ \pi z \cot(\pi z) = 1 - 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{z}{n}\right)^{2m} \]  \hspace{1cm} (12.3)

\[ = 1 - 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{z}{n}\right)^{2m} \]  \hspace{1cm} (12.4)

\[ = 1 - 2 \sum_{m=1}^{\infty} \zeta(2m) z^{2m} \]  \hspace{1cm} (12.5)

### 12.4 The Values of \( \zeta(2m) \)

Comparing the expressions (12.5) and (12.2), we obtain the following result about the values of the Riemann \( \zeta \)-function at even integers. (This result was already known to Euler in 1734.)

**Theorem 12.1** For \( m = 1, 2, \ldots \) the value of \( \zeta(2m) \) is

\[ \zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{1}{2} (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m} \]  \hspace{1cm} (12.6)

**Remark:** Since, clearly, \( \zeta(2m) > 0 \) we obtain that \( (-1)^{m+1} B_{2m} > 0 \).

**Examples:**

- For \( m = 1 \) we have \( B_2 = \frac{1}{6} \), thus
  \[ \zeta(2) = \frac{(2\pi)^2}{2 \cdot 2} \cdot \frac{1}{6} = \frac{\pi^2}{6} \cdot \]

- For \( m = 2 \) we have \( B_4 = -\frac{1}{30} \), thus
  \[ \zeta(4) = \frac{(2\pi)^4}{2 \cdot 4!} \cdot \frac{1}{30} = \frac{\pi^4}{90} \cdot \]

- For \( m = 3 \) we have \( B_6 = \frac{1}{42} \), thus
  \[ \zeta(6) = \frac{(2\pi)^6}{2 \cdot 6!} \cdot \frac{1}{42} = \frac{\pi^6}{945} \cdot \]

- For \( m = 4 \) one obtains
  \[ \zeta(8) = \frac{\pi^8}{9450} \cdot \]

**Remark:** No formula for \( \zeta(2m + 1) \) seems to be known if \( m \) is an integer.
12.5 Sums of Powers and Bernoulli Numbers

It is not difficult to show the following formulae by induction in $n$:

\[
S_1(n - 1) \equiv \sum_{j=1}^{n-1} j = \frac{1}{2} n^2 - \frac{1}{2} n
\]

\[
S_2(n - 1) \equiv \sum_{j=1}^{n-1} j^2 = \frac{1}{3} n^3 - \frac{1}{2} n^2 + \frac{1}{6} n
\]

\[
S_3(n - 1) \equiv \sum_{j=1}^{n-1} j^3 = \frac{1}{4} n^4 - \frac{1}{2} n^3 + \frac{1}{4} n^2 + 0 n
\]

Recalling that

\[
B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0
\]

we notice that the three formulae have the pattern:

\[
\sum_{j=1}^{n-1} j^k = \frac{1}{k+1} n^{k+1} - \frac{1}{2} n^k + \ldots + B_k n
\]

but it is not obvious how the general formula should read.

Define the sum

\[
S_k(n - 1) = \sum_{j=0}^{n-1} j^k
\]

where $k = 0, 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$. We claim that, for every fixed integer $k \geq 0$, the sum $S_k(n - 1)$ is a polynomial

\[
\Phi_k(n)
\]

of degree $k + 1$ in the variable $n$ and that the coefficients of $\Phi_k(n)$ can be obtained in terms of Bernoulli numbers. Precisely:

**Theorem 12.2** For every integer $k \geq 0$, let $\Phi_k(n)$ denote the polynomial of degree $k + 1$ given by

\[
\Phi_k(n) = \frac{1}{k+1} \sum_{\mu=0}^{k} \binom{k+1}{\mu} B_\mu n^{k+1-\mu}.
\]

Then we have

\[
S_k(n - 1) = \Phi_k(n) \quad \text{for all} \quad n = 1, 2, \cdots.
\]
Remark: Writing out a few terms of $\Phi_k(n)$, the theorem says that

$$S_k(n - 1) = \frac{1}{k + 1} n^{k+1} - \frac{1}{2} n^k + \frac{1}{k + 1} \left( \frac{k + 1}{2} \right) B_2 n^{k-1} + \cdots + B_k n.$$ 

Proof of Theorem: The trick is to write the finite geometric sum

$$E_n(w) = 1 + e^w + e^{2w} + \cdots + e^{(n-1)w}$$

in two ways and then to compare coefficients. We have

$$E_n(w) = \sum_{j=0}^{n-1} e^{jw} = \sum_{j=0}^{n-1} \sum_{k=0}^{\infty} \frac{j^k}{k!} w^k = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{n-1} \frac{1}{k!} w^k \right) = \sum_{k=0}^{\infty} \frac{1}{k!} S_k(n - 1) w^k$$

(Here we have used the convention $0^0 = 1$.)

On the other hand, we have

$$E_n(w) = \frac{e^{nw} - 1}{e^w - 1} = \frac{w}{e^w - 1} \cdot \frac{e^{nw} - 1}{w} = \left( \sum_{\mu=0}^{\infty} \frac{B_{\mu}}{\mu!} w^{\mu} \right) \cdot \left( \sum_{\lambda=0}^{\infty} \frac{n^{\lambda+1}}{\lambda+1)!} w^{\lambda} \right) = \sum_{k=0}^{\infty} \left( \sum_{\mu=0}^{\infty} \frac{B_{\mu}}{\mu!(\lambda+1)!} n^{\lambda+1} \right) w^k$$

Comparison yields that

$$S_k(n - 1) = \sum_{\mu=k}^{\infty} \frac{k!}{\mu!(\lambda+1)!} B_{\mu} n^{\lambda+1} \quad \text{(with } \lambda = k - \mu)$$

$$= \frac{1}{k + 1} \sum_{\mu=0}^{k} \frac{(k+1)!}{\mu!(k+1-\mu)!} B_{\mu} n^{k+1-\mu}$$

This proves the claim since

$$\binom{k+1}{\mu} = \frac{(k+1)!}{\mu!(k+1-\mu)!}.$$ 

\(\Box\)
13 Some Properties of the $\Gamma$–Function

13.1 The Reflection Formula

Let $0 < a < 1$. We have shown that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^{x}} \, dx = \frac{\pi}{\sin(\pi a)} .$$

We want to use this to prove the so–called reflection formula for the $\Gamma$–function:

**Theorem 13.1** For $0 < s < 1$ we have

$$\Gamma(s)\Gamma(1 - s) = \frac{\pi}{\sin(\pi s)} .$$

**Proof:** We have

$$\Gamma(s) = \int_{0}^{\infty} t^{s-1} e^{-t} \, dt$$

and

$$\Gamma(1 - s) = \int_{0}^{\infty} t^{-s} e^{-t} \, dt \quad \text{(rename $t = u$)}$$

$$= \int_{0}^{\infty} u^{-s} e^{-u} \, du \quad \text{(substitute $u = tv$)}$$

$$= t \int_{0}^{\infty} (tv)^{-s} e^{-tv} \, dv \quad \text{for} \quad t > 0 .$$

Obtain that

$$\Gamma(s)\Gamma(1 - s) = \int_{0}^{\infty} t^{s-1} e^{-t} \Gamma(1 - s) \, dt$$

$$= \int_{0}^{\infty} t^{s-1} e^{-t} \left( t \int_{0}^{\infty} (tv)^{-s} e^{-tv} \, dv \right) \, dt$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} v^{-s} e^{-(1+v)t} \, dv \, dt$$

$$= \int_{0}^{\infty} v^{-s} \int_{0}^{\infty} e^{-(1+v)t} \, dt \, dv$$

$$= \int_{0}^{\infty} v^{-s} \frac{1}{1 + v} \, dv \quad \text{(with $s = 1 - \sigma$)}$$

$$= \int_{0}^{\infty} \frac{v^{\sigma-1}}{1 + v} \, dv$$

$$= \int_{0}^{\infty} \frac{v^{\sigma}}{1 + v} \, dv$$

In the last integral, use the substitution
\[ v = e^x, \quad \frac{dv}{v} = dx, \]
to obtain
\[
\int_0^\infty \frac{e^{\sigma x}}{1 + x} \frac{dv}{v} = \int_{-\infty}^{\infty} \frac{e^x}{1 + e^x} \frac{dx}{\pi} \frac{\sin(\pi \sigma)}{\pi} = \frac{\sin(\pi (1 - s))}{\sin(\pi s)}
\]
This proves the reflection formula for \( 0 < s < 1. \)

13.2 Extension of the Domain of Definition of \( \Gamma \) Using the Functional Equation

**Notations:** Let
\[ Z_+ = \{0, 1, 2, \ldots\} \quad \text{and} \quad Z_- = \{0, -1, -2, \ldots\}. \]
We set
\[ U = \mathbb{C} \setminus Z_- . \]
Also, for \( k = 1, 2, \ldots \) let
\[ U_k = \{z : \Re z > -k\} \setminus Z_- \]
and note that
\[ U = \bigcup_{k=1}^{\infty} U_k . \]
To define the \( \Gamma \)-function, we have used the formula
\[ \Gamma(z) = \int_0^\infty t^z e^{-t} \frac{dt}{t}, \quad \Re z > 0 . \]
We have shown that \( \Gamma(z) \) is holomorphic for \( \Re z > 0 \) and satisfies the functional equation
\[ \Gamma(z + 1) = z\Gamma(z), \quad \Re z > 0 . \]
Fix any (large) \( k \in \{1, 2, \ldots\} \). We have, for \( \Re z > 0 \):
\[
\Gamma(z + k) = (z + k - 1)\Gamma(z + k - 1) = (z + k - 1)(z + k - 2)\Gamma(z + k - 2) = (z + k - 1)(z + k - 2) \cdots z\Gamma(z)
\]
Thus,
\[ \Gamma(z) = \frac{\Gamma(z + k)}{z(z + 1) \cdots (z + k - 1)}, \quad \text{Re } z > 0. \tag{13.1} \]
We note that the right-hand side is well-defined and holomorphic in \( U_k \). We set
\[ r_k(z) = \frac{\Gamma(z + k)}{z(z + 1) \cdots (z + k - 1)}, \quad z \in U_k. \]
Then, by (13.1), we have \( r_k(z) = \Gamma(z) \) for \( \text{Re } z > 0 \). The function \( r_k(z) \) is the unique holomorphic continuation of the \( \Gamma \)-function defined in \( U_k \). Since we can do this for every \( k = 1, 2, \ldots \) we obtain the holomorphic continuation of \( \Gamma \) in
\[ U = \bigcup_{k=1}^{\infty} U_k. \]

13.3 Extension of the Domain of Definition of \( \Gamma \) Using Series Expansion

One can extend the definition of \( \Gamma \) also as follows. First, assume again that \( \text{Re } z > 0 \) and write
\[ \Gamma(z) = \int_0^1 t^{z-1} e^{-t} dt + \int_1^{\infty} t^{z-1} e^{-t} dt =: g(z) + h(z) \]
It is easy to see that the formula
\[ h(z) = \int_1^{\infty} t^{z-1} e^{-t} dt, \quad z \in \mathbb{C}, \]
defines an entire function. (Use that \( t^{z-1} = e^{(\ln t)(z-1)} \) and apply Cauchy’s theorem and Morera’s theorem.)

In the formula defining \( g(z) \) we write out the exponential series and interchange summation and integration. Thus, for \( \text{Re } z > 0 \):
\[ g(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_0^1 t^{z-1} t^j dt = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{1}{z+j} \]
The infinite series converges for every
\[ z \in U := \mathbb{C} \setminus \{0, -1, -2, \ldots\} = \mathbb{C} \setminus \mathbb{Z}_- . \]
The convergence of
Figure 13.1: Gamma function on the real axis

\[ g_n(z) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \frac{1}{z+j} \]

to \( g(z) \) is normal in \( U \). Thus, \( g \in H(U) \).

To summarize, the formula

\[ \Gamma(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{1}{z+j} + \int_1^{\infty} t^{z-1} e^{-t} \, dt \]

defines \( \Gamma(z) \) as a holomorphic function in \( \mathbb{C} \setminus \mathbb{Z}_- \).

**Poles of \( \Gamma \).** For every \( k \in \mathbb{Z}_+ \) we have

\[ g(z) = \frac{(-1)^k}{k!} \frac{1}{z+k} + \sum_{j=0, j \neq k}^{\infty} \frac{(-1)^j}{j!} \frac{1}{z+j} . \]

Here the infinite sum is holomorphic near \( z = k \). It follows that \( \Gamma(z) \) has a simple pole at every number \( z_k = -k \) with \( k \in \mathbb{Z}_+ \). Also,

\[ \text{Res}(\Gamma,-k) = \frac{(-1)^k}{k!}, \quad k \in \mathbb{Z}_+ . \]

Since \( \Gamma(x) \) is real for every \( x \in \mathbb{R} \setminus \mathbb{Z}_- \) and since \( \Gamma(x) > 0 \) for every \( x > 0 \), it follows that

\[ \Gamma(x) < 0 \quad \text{for} \quad -1 < x < 0 , \]
\[ \Gamma(x) > 0 \quad \text{for} \quad -2 < x < -1 , \]

etc.

### 13.4 Extension of the Reflection Formula

The function
\[
\phi(z) = \Gamma(z)\Gamma(1-z) - \frac{\pi}{\sin(\pi z)}
\]
is holomorphic in \( \mathbb{C} \setminus \mathbb{Z}_- \). Since \( \phi(s) = 0 \) for \( 0 < s < 1 \) it follows from the identity theorem (Theorem 9.1) that \( \phi(z) = 0 \) for all \( z \in \mathbb{C} \setminus \mathbb{Z}_- \). Therefore,
\[
\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \in \mathbb{C} \setminus \mathbb{Z}_- ,
\]
i.e., the reflection formula holds in \( \mathbb{C} \setminus \mathbb{Z} \). (Note that both sides in the reflection formula have a simple pole at every integer.)

### 13.5 Extension of the Functional Equation

Using the definition,
\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t} \, dt, \quad \text{Re} \, z > 0 ,
\]
it easily follows through integration by parts that
\[
\Gamma(z + 1) = z\Gamma(z), \quad \text{Re} \, z > 0 .
\]
The function
\[
\phi(z) = \Gamma(z + 1) - z\Gamma(z)
\]
is holomorphic in \( \mathbb{C} \setminus \mathbb{Z}_- \). Since \( \phi(s) = 0 \) for \( \text{Re} \, z > 0 \) it follows from the identity theorem that \( \phi(z) = 0 \) for all \( z \in \mathbb{C} \setminus \mathbb{Z}_- \). Therefore,
\[
\Gamma(z + 1) = z\Gamma(z), \quad z \in \mathbb{C} \setminus \mathbb{Z}_- ,
\]
i.e., the functional equation holds in the whole region where \( \Gamma \) is holomorphic, namely in the whole region where \( \Gamma \) is holomorphic, namely in \( \mathbb{C} \setminus \{0, -1, -2, \ldots \} \).

### 13.6 Some Special Values of \( \Gamma(z) \)

We have
\[
\Gamma(1) = \int_0^\infty e^{-t} \, dt = 1 .
\]
Using the functional equation:
\[
\begin{align*}
\Gamma(1+1) &= 1 \cdot \Gamma(1) = 1 \\
\Gamma(2+1) &= 2 \cdot \Gamma(2) = 2 \\
\Gamma(3+1) &= 3 \cdot \Gamma(3) = 2 \cdot 3
\end{align*}
\]
etc. In general,
\[
\Gamma(n+1) = n!, \quad n \in \mathbb{Z}_+ .
\]
From the reflection formula one obtains that
\[ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} . \]
Then one can use the functional equation to compute \( \Gamma\left(n + \frac{1}{2}\right) \) for every \( n \in \mathbb{N} \):
\[
\begin{align*}
\Gamma\left(\frac{1}{2} + 1\right) &= \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \sqrt{\pi} \\
\Gamma\left(\frac{3}{2} + 1\right) &= \frac{3}{2} \cdot \Gamma\left(\frac{3}{2}\right) = \frac{1 \cdot 3 \cdot \sqrt{\pi}}{2} \\
\Gamma\left(\frac{5}{2} + 1\right) &= \frac{5}{2} \cdot \Gamma\left(\frac{5}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdot \sqrt{\pi}}{2} \\
\end{align*}
\]
In general, for all \( n \in \mathbb{Z}_+ \):
\[
\Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot \ldots \cdot (2n - 1)}{2^n} \cdot \sqrt{\pi} = \frac{(2n)!}{4^n n! \sqrt{\pi}}
\]

### 13.7 Some Simple Applications

The \( \Gamma \)-function appears in many formulas.

**Example 1:** Using the substitution \( x^2 = t \), one obtains:
\[
\int_0^\infty x^{2n} e^{-x^2} dx = \frac{1}{2} \int_0^\infty t^{n+\frac{1}{2}-1} e^{-t} dt = \frac{1}{2} \Gamma\left(n + \frac{1}{2}\right)
\]

**Example 2:** Using the substitution
\[
t = -\ln x \quad \text{for} \quad 0 < x \leq 1, \quad e^{-t} = x, \quad dx = -e^{-t} dt,
\]
on one obtains for \( \text{Re} \ z > -1 \):
\[
\Gamma(z + 1) = \int_0^\infty t^z e^{-t} dt
\]
\[
= -\int_0^\infty t^z (-e^{-t}) dt
\]
\[
= \int_1^0 (\ln x)^z dx
\]
In particular, for \( n \in \mathbb{Z}_+ \),
\[
\int_0^1 (\ln x)^n dx = \Gamma(n + 1) = n! .
\]
13.8 The Function $\Delta(z) = 1/\Gamma(z)$

We know that $\Gamma(n) > 0$ for all $n \in \mathbb{N}$. Also, the reflection formula implies that $\Gamma(z) \neq 0$ for all $z \in \mathbb{C} \setminus \mathbb{Z}$. Therefore,

$$\Gamma(z) \neq 0 \quad \text{for all} \quad z \in \mathbb{C} \setminus \mathbb{Z}.$$ 

Since $\Gamma$ has a (simple) pole at every point $k \in \mathbb{Z}$, one can use Riemann’s removability theorem to show that the function $\Delta(z)$ defined by

$$\begin{align*}
\Delta(z) &= 1/\Gamma(z) \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{Z} \\
\Delta(z) &= 0 \quad \text{for} \quad z \in \mathbb{Z}
\end{align*}$$

is entire. Weierstrass based his theory of the $\Gamma$–function on the investigation of $\Delta(z)$.

13.9 Log–Convexity of $\Gamma(x)$

We know that

$$\Gamma : (0, \infty) \to (0, \infty)$$

is a $C^\infty$–function.

**Theorem 13.2** For all $x \in \mathbb{R}$:

$$\frac{d^2}{dx^2} \ln \Gamma(x) > 0.$$ 

**Proof:** If $\phi(x) = \ln \Gamma(x)$ then

$$\phi' = \frac{\Gamma'}{\Gamma}, \quad \phi'' = \frac{\Gamma'' \Gamma - \Gamma'^2}{\Gamma^2}.$$ 

We must show that

$$\Gamma''(x)\Gamma(x) > \Gamma'^2(x), \quad x \in \mathbb{R}.$$ (13.2)

We have

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt$$

with

$$t^x = e^{x \ln t} \quad \text{for} \quad t > 0.$$ 

Since

$$\begin{align*}
\frac{d}{dx} t^x &= (\ln t)e^{x \ln t} \\
\frac{d^2}{dx^2} t^x &= (\ln t)^2 e^{x \ln t}
\end{align*}$$
we obtain:

\[
\Gamma'(x) = \int_0^\infty (\ln t) t^{x-1} e^{-t} \, dt \\
\Gamma''(x) = \int_0^\infty (\ln t)^2 t^{x-1} e^{-t} \, dt
\]

For fixed \(0 < x < \infty\) define the quadratic

\[
g(u) = u^2 \Gamma(x) + 2u \Gamma'(x) + \Gamma''(x), \quad u \in \mathbb{R}.
\]

The above expressions for \(\Gamma(x)\) and its derivatives yields:

\[
g(u) = \int_0^\infty \left\{ u^2 + 2u \ln t + (\ln t)^2 \right\} t^{x-1} e^{-t} \, dt
\]

Here

\[
u^2 + 2u \ln t + (\ln t)^2 = (u + \ln t)^2.
\]

This implies that

\[
g(u) > 0 \quad \text{for all} \quad u \in \mathbb{R}.
\]

Since

\[
g'(u) = 2u \Gamma(x) + 2 \Gamma'(x)
\]

the function \(g(u)\) attains its minimum at

\[
u_0 = -\frac{\Gamma'(x)}{\Gamma(x)}.
\]

Evaluating \(g(u)\) at \(u = u_0\) one obtains:

\[
\min_u g(u) = g(u_0) = -\frac{\Gamma'^2(x)}{\Gamma(x)} + \Gamma''(x)
\]

Since

\[
\min g(u) > 0
\]

we have shown (13.2), and the theorem is proved. ☐
13.10 Summary

The formula

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t} \, dt$$

defines $\Gamma(s)$ for $\text{Re}\, s > 0$ as an analytic function. We have $s\Gamma(s) = \Gamma(s + 1)$
and $\Gamma(n + 1) = n!$ for $n = 0, 1, 2, \ldots$ Using the formula

$$\Gamma(s) = \sum_{j=0}^\infty \frac{(-1)^j}{j!} \frac{1}{s+j} + \int_1^\infty t^{s-1}e^{-t} \, dt$$

one obtains the analytic continuation of $\Gamma(s)$ in

$$U = \mathbb{C} \setminus \{0, -1, -2, \ldots \}.$$  

The function $\Gamma \in H(U)$ has a simple pole at $-k$ for $k = 0, 1, 2, \ldots$ and

$$\text{Res}(\Gamma, -k) = \frac{(-1)^k}{k!}.$$  

The reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \quad s \in \mathbb{C} \setminus \mathbb{Z},$$

holds. It implies that $\Gamma(1/2) = \sqrt{\pi}$ and that $\Gamma(s) \neq 0$ for all $s \in U$. The function $\Delta(s) = 1/\Gamma(s)$ is entire.

For real $s$, $s \in \mathbb{R} \setminus \{0, -1, -2, \ldots \}$, the value $\Gamma(s)$ is real. We have

$$(d/ds)^2 \ln \Gamma(s) > 0 \quad \text{for} \quad s > 0$$

and

$$\Gamma(s) = \left(\frac{s}{e}\right)^s \sqrt{2\pi s} \left(1 + O(s^{-1})\right) \quad \text{as} \quad s \to \infty,$$

which is Stirling’s formula.
14 Log Functions

Let \( U = \mathbb{C} \setminus (-\infty,0] \). The main branch of the complex logarithm can be introduced, as a function defined on \( U \), as follows: Take any \( z \in U \) and write

\[ z = re^{i\theta}, \quad r = |z| > 0, \quad -\pi < \theta < \pi. \]

The real numbers \( r \) and \( \theta \) are uniquely determined. Then we have

\[ \log z = \ln r + i\theta. \]

If \( z = x + iy \) then \( r = (x^2 + y^2)^{1/2} \) and

\[ \theta = \arctan(y/x). \]

Here one must choose the correct branch of the arctan–function and must be careful when \( x = 0 \). One then obtains

\[ \log(x + iy) = \ln \left((x^2 + y^2)^{1/2}\right) + i\arctan(y/x). \]

With some effort (in particular for \( x = 0 \)) one can use the Cauchy–Riemann equations to prove that the function \( \log(x + iy) \) is holomorphic on \( U \).

From the point of view of complex variables, there is a better way to introduce \( \log z, z \in U \), namely as the inverse of \( e^w \). We will do this below. To construct \( \log z \) we will use

\[ \log z = \int_{\Gamma_z} \frac{dw}{w}, \quad z \in U, \]

where \( \Gamma_z \) is a curve in \( U \) from \( z_0 = 1 \) to \( z \).

14.1 Auxiliary Results

We first recall the following:

**Theorem 14.1** Let \( U \subset \mathbb{C} \) be simply connected and let \( g \in H(U) \). Fix \( z_0 \in U \) and, for every \( z \in U \) choose a curve \( \Gamma_z \) in \( U \) from \( z_0 \) to \( z \). Then:

1. The function

\[ f(z) = \int_{\Gamma_z} g(w) \, dw, \quad z \in U, \]

is well–defined, i.e., it does not depend on the particular choice of \( \Gamma_z \).

2. We have \( f \in H(U) \) and \( f'(z) = g(z), z \in U. \)

3. \( f(z_0) = 0. \)

**Lemma 14.1** Let \( U \subset \mathbb{C} \) be connected and let \( g \in H(U) \). Assume that \( g'(z) = 0 \) for all \( z \in U \). Then \( g(z) \) is constant in \( U \).
**Proof:** Fix \( z_0 \in U \) and let \( z \in U \) be arbitrary. Choose a curve \( \Gamma_z \) in \( U \) from \( z_0 \) to \( z \). We have

\[
g(z) - g(z_0) = \int_{\Gamma_z} g'(w) \, dw = 0,
\]
thus \( g(z) = g(z_0) \).

\[ \diamond \]

### 14.2 The Main Branch of the Complex Logarithm

**Theorem 14.2** Let \( U = \mathbb{C} \setminus (-\infty, 0] \). There is a unique function \( L \in H(U) \) with

1. \( L(1) = 0 \);
2. \( e^{L(z)} = z \) for all \( z \in U \).

This function \( L(z) \) is denoted by

\[
L(z) = \log z, \quad z \in U,
\]
and is called the main branch of the complex logarithm. The function \( L(z) = \log z \) satisfies \( L'(z) = 1/z, z \in U \), and we have

\[
L(x) = \ln x \quad \text{for} \quad 0 < x < \infty.
\]

**Proof:** Let \( \Gamma_z \) denote a curve in \( U \) from \( z_0 = 1 \) to \( z \in U \).

**Uniqueness of \( L \).** Suppose \( L \in H(U) \) satisfies the conditions 1. and 2.

We have

\[
L'(z)e^{L(z)} = 1,
\]
thus

\[
L'(z) = e^{-L(z)} = \frac{1}{z}
\]
in \( U \). Therefore,

\[
L(z) = L(z) - L(1) = \int_{\Gamma_z} \frac{dw}{w}
\]

**Existence of \( L \).** Define

\[
L(z) = \int_{\Gamma_z} \frac{dw}{w}, \quad z \in U.
\]

We then have \( L(1) = 0 \) and \( L'(z) = \frac{1}{z}, z \in U \). Therefore,

\[
\frac{d}{dz} \left(ze^{-L(z)} \right) = e^{-L(z)} - zL'(z)e^{-L(z)} = 0.
\]
This shows that

\[ ze^{-L(z)} = \text{const} \]

At \( z_0 = 1 \) we obtain

\[ \text{const} = 1e^0 = 1 \]

thus \( e^{L(z)} = z \).

For \( 0 < x < \infty \) we have

\[ L(x) = \int_1^x \frac{dw}{w} = \ln x \]

Lemma 14.2 We have

\[ \log(1 + z) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} z^j}{j} \quad \text{for } |z| < 1 \]

Proof: The derivative of the left–hand side is

\[ l'(z) = \frac{1}{1 + z}, \quad |z| < 1 \]

The derivative of the right–hand side is

\[ r'(z) = \sum_{j=1}^{\infty} (-1)^{j-1} z^{j-1} = \sum_{k=0}^{\infty} (-z)^k = \frac{1}{1 + z}, \quad |z| < 1 \]

It follows that \( l(z) - r(z) \) is constant. Also,

\[ l(0) - r(0) = \log(1) = 0 \]

thus \( r(z) \equiv l(z) \) for \( |z| < 1 \). □

14.3 Complex Logarithms in Other Simply Connected Regions

Theorem 14.3 Let \( V \subseteq \mathbb{C} \) be open and simply connected. Assume that \( 0 \notin V \). Fix \( z_0 \in V \) and write

\[ z_0 = r_0 e^{i\theta_0}, \quad r_0 > 0 \]

Then there is a unique function \( L \in H(V) \) with

1. \( L(z_0) = \ln(r_0) + i\theta_0 \);
2. \( e^{L(z)} = z \) for all \( z \in V \).

This function \( L(z) \) satisfies \( L'(z) = 1/z, z \in V \).
**Proof:** Let $\Gamma_z$ denote a curve in $V$ from $z_0$ to $z \in V$.

**Uniqueness of $L$.** Suppose $L \in H(U)$ satisfies the conditions 1. and 2. We have

$$L'(z)e^{L(z)} = 1,$$

thus

$$L'(z) = e^{-L(z)} = \frac{1}{z}$$

in $V$. Therefore,

$$L(z) - L(z_0) = \int_{\Gamma_z} \frac{dw}{w}.$$ 

This shows that $L(z)$ is uniquely determined.

**Existence of $L$.** Define

$$L(z) = \ln(r_0) + i\theta_0 + \int_{\Gamma_z} \frac{dw}{w}, \quad z \in V.$$ 

We then have $L(z_0) = \ln(r_0) + i\theta_0$ and $L'(z) = \frac{1}{z}, z \in V$. Therefore,

$$\frac{d}{dz} \left( ze^{-L(z)} \right) = e^{-L(z)} - zL'(z)e^{-L(z)} = 0.$$ 

This shows that

$$ze^{-L(z)} = \text{const}$$

At $z = z_0$ we have

$$e^{L(z_0)} = r_0^e^{i\theta} = z_0,$$

thus

$$\text{const} = z_0 e^{-L(z_0)} = 1.$$ 

This proves that

$$e^{L(z)} = z, \quad z \in V.$$ 

\diamond

We call the function $L(z)$ the logarithm in $V$ with normalization $L(z_0) = \ln(r_0) + i\theta_0$. If we drop the dependency on the normalization in our notation, we write

$$L(z) = \log_V(z), \quad z \in V.$$ 

In particular, we have shown the existence statement of the following theorem:
Theorem 14.4 Let $V \subset \mathbb{C}$ be open and simply connected. Assume that $0 \notin V$. Then there is a function $L \in H(V)$ with $e^{L(z)} = z$ for all $z \in V$. Any such function satisfies $L'(z) = 1/z$ in $V$.

If $L_1, L_2 \in H(V)$ satisfy $e^{L_1(z)} = e^{L_2(z)}$ for all $z \in V$, then there exists $n \in \mathbb{Z}$ with

$$L_1(z) = L_2(z) + 2\pi i n, \quad z \in V . \tag{14.1}$$

Proof: We only have to show (14.1). We know that $e^w = 1$ holds if and only if $w = 2\pi i n$ for some $n \in \mathbb{Z}$. Therefore,

$$e^{L_1(z) - L_2(z)} = 1$$

implies that $L_1(z) - L_2(z) = 2\pi i n(z), n(z) \in \mathbb{Z}$. However, since $n(z) \in H(V)$, the function $n(z)$ is constant. \hfill \diamond

Definition: If $V \subset \mathbb{C}$ is an open set and if $L \in H(V)$ then we call $L$ a logarithm on $V$ if $e^{L(z)} = z$ for all $z \in V$.

Using the above terminology, Theorem 14.4 says that a logarithm exists on $V$ if $V$ is simply connected and $0 \notin V$. Furthermore, any two logarithms on $V$ differ by an integer multiple of $2\pi i$.

14.4 Argument Functions

Let $V \subset \mathbb{C}$ be open and simply connected and assume that $0 \notin V$. Let $L \in H(V)$ denote a logarithm on $V$ and write

$$L(z) = L_R(z) + iL_I(z)$$

with real functions $L_R$ and $L_I$. We have

$$z = e^{L_R(z)} e^{iL_I(z)},$$

thus

$$|z| = e^{L_R(z)}, \quad L_R(z) = \ln |z| .$$

Definition: Let $V \subset \mathbb{C}$ denote an open set. A $C^\infty$–function

$$\text{arg} : V \to \mathbb{R}$$

is called an argument function on $V$ if

$$z = e^{\ln |z| + i\text{arg}(z)} \quad \text{for all} \quad z \in V .$$

Our results say that an argument function exists on $V$ if $V$ is simply connected and $0 \notin V$. In fact, $\text{arg}(z) = \text{Im } L(z)$ is an argument function on $V$ if $L$ is a logarithm on $V$. Furthermore, any two argument functions on $V$ differ by an integer multiple of $2\pi i$. 

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15 The General Residue Theorem and the Argument Principle

15.1 Remarks on Solutions of Equations under Perturbations

A general question of mathematics, vaguely formulated, is the following: Suppose $u_0$ is the solution of an equation and the equations gets perturbed by $\varepsilon$. Will the perturbed equation have a solution $u(\varepsilon)$ near $u_0$? A precise result of this nature is formalized in the implicit function theorem, which is itself based on completeness of the underlying solution space (all Cauchy sequences converge) and contraction. To formalize ideas, assume that

$$F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

is a smooth map and consider the equation

$$F(u, \lambda) = 0 .$$

(15.1)

Here we consider $\lambda$ as a parameter in the parameter space $\mathbb{R}^m$. The solutions $u$ lie in the state space $\mathbb{R}^n$. The space $\mathbb{R}^n$ is also the space of right-hand sides so that, for fixed $\lambda \in \mathbb{R}^m$, the system $F(u, \lambda) = 0$ has $n$ unknowns and $n$ equations. Suppose that

$$F(u_0, \lambda_0) = 0 ,$$

i.e., for $\lambda = \lambda_0$ the equation (15.1) has the solution $u_0$. Let $\lambda = \lambda_0 + \varepsilon$ where $\varepsilon \in \mathbb{R}^m$ is small. We ask if the equation

$$F(u, \lambda_0 + \varepsilon)$$

(15.2)

has a solution $u = u(\varepsilon) \sim u_0$. To ensure that this is true, we assume that the Jacobian

$$A := F_u(u_0, \lambda_0) \in \mathbb{R}^{n \times n}$$

is nonsingular. Then, proceeding formally, we try to find a solution $u$ of (15.2) in the form

$$u = u_0 + \delta, \quad \delta \in \mathbb{R}^n ,$$

where $\delta$ is small. We have, formally,

$$0 = F(u_0 + \delta, \lambda_0 + \varepsilon) = F(u_0, \lambda_0) + A\delta + F_\lambda(u_0, \lambda_0)\varepsilon + Q(\delta, \varepsilon)$$

where

$$|Q(\delta, \varepsilon)| \leq C(|\delta|^2 + |\varepsilon|^2) .$$

Since $F(u_0, \lambda_0) = 0$ we obtain
\[ \delta = -A^{-1} F_\lambda(u_0, \lambda_0) \varepsilon - A^{-1} Q(\delta, \varepsilon). \]

This is a fixed point equation for \( \delta \), suggesting the iteration

\[ \delta^{j+1} = -A^{-1} F_\lambda(u_0, \lambda_0) \varepsilon - A^{-1} Q(\delta^j, \varepsilon), \quad \delta^0 = -A^{-1} F_\lambda(u_0, \lambda_0) \varepsilon. \]

If \( \varepsilon \) is small enough, one can use a contraction argument to show that the equation has a unique small solution \( \delta \in \mathbb{R}^n \). This is made precise in the proof of the implicit function theorem.

Complex variables offers another tool, different from contraction, to study the solutions of an equation under perturbation. The tool is, ultimately, Cauchy’s integral theorem, which allows us to count the number of zeros of a function in terms of an integral. The idea is as follows: If the function is perturbed slightly, the integral only changes slightly. Since the integral is an integer, it does not change at all and, consequently, the number of zeros of the perturbed function equals the number of zeros of the unperturbed function.

In a more general form, this tool is developed further in degree theory, an advanced topic of analysis and topology.

### 15.2 The Winding Number or Index

Let \( \gamma(t), a \leq t \leq b \), be a closed curve in \( \mathbb{C} \) and let \( P \in \mathbb{C} \setminus \tilde{\gamma} \), i.e., \( P \) is a point in the complex plane that does not lie on \( \tilde{\gamma} \).

The number

\[
\text{Ind}_\gamma(P) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - P} = \frac{1}{2\pi i} \int_{a}^{b} \frac{\gamma'(s)}{\gamma(s) - P} ds
\]

is called the index of \( \gamma \) w.r.t. \( P \) or the winding number of \( \gamma \) w.r.t. \( P \). Intuitively, \( \text{Ind}_\gamma(P) \) counts how many times \( \gamma \) winds around \( P \) in the positive sense. If \( \text{Ind}_\gamma(P) \) is negative, then \( \gamma \) winds around \( P \) clockwise.

It is surprisingly difficult to prove that the index is always an integer.

**Lemma 15.1** Under the above assumptions, the number \( \text{Ind}_\gamma(P) \) is an integer.

**Proof:** Let

\[
g(t) = \int_{a}^{t} \frac{\gamma'(s)}{\gamma(s) - P} ds, \quad a \leq t \leq b.
\]

We have \( g(a) = 0 \) and

\[
\frac{1}{2\pi i} g(b) = \text{Ind}_\gamma(P).
\]

Define
\[ \phi(t) = e^{-g(t)(\gamma(t) - P)}, \quad a \leq t \leq b. \]

We will prove that \( \phi(t) \) is constant. We have

\[
\begin{align*}
\phi(a) &= e^{-g(a)(\gamma(a) - P)} = \gamma(a) - P \\
\phi(b) &= e^{-g(b)(\gamma(b) - P)} = e^{-g(b)(\gamma(a) - P)}.
\end{align*}
\]

In the last equation we have used that \( \gamma(a) = \gamma(b) \), which holds since \( \gamma \) is assumed to be closed.

Note that the definition of \( g(t) \) yields

\[ g'(t) = \gamma'(t)(\gamma(t) - P)^{-1}. \]

We use this to prove that \( \phi(t) \) is constant:

\[
\begin{align*}
\phi'(t) &= e^{-g(t)(-g'(t))(\gamma(t) - P) + e^{-g(t)}\gamma'(t)} \\
&= e^{-g(t)(-\gamma'(t)) + e^{-g(t)}\gamma'(t)} \\
&= 0.
\end{align*}
\]

We obtain that

\[ \phi(a) = \phi(b). \]

Since \( \phi(a) - P \neq 0 \) one obtains from the expressions \( \phi(a) = \gamma(a) - P \) and \( \phi(b) = e^{-g(b)(\gamma(a) - P)} \) that

\[ e^{-g(b)} = 1. \]

This yields

\[ g(b) = 2\pi i n \quad \text{for some} \quad n \in \mathbb{Z}. \]

Finally,

\[ \text{Ind}_\gamma(P) = \frac{g(b)}{2\pi i} = n. \]

\( \diamond \)
15.3 The General Residue Theorem

Recall that an open connected set $U \subset \mathbb{C}$ is called a region. Also, recall that a curve $\Gamma$ in $U$ is called null–homotopic if one can deform $\Gamma$ continuously to a point in $U$ without leaving $U$.

**Theorem 15.1** Let $U$ be a region in $\mathbb{C}$. Let $P_1, \ldots, P_J \in U$ be $J$ distinct points in $U$ and let $f \in H \left( U \setminus \{P_1, \ldots, P_J\} \right)$. Let $\gamma$ be a closed curve in $U$ which is null–homotopic in $U$ and avoids the points $P_j$, i.e.,

$$ P_j \notin \gamma, \quad j = 1, \ldots, J. $$

Under these assumptions:

$$ \int_{\gamma} f(z) \, dz = 2\pi i \sum_{j=1}^{J} \text{Res}_f(P_j) \text{Ind}_{\gamma}(P_j). $$

**Proof:** For $0 < |z - P_j| < \varepsilon$:

$$ f(z) = \sum_{k=-\infty}^{-1} a_{-1}^{(j)} (z - P_j)^k + g_j(z) $$

where $g_j \in H(D(P_j, \varepsilon))$ and

$$ a_{-1}^{(j)} = \text{Res}_f(P_j). $$

The singular part of the Laurent expansion of $f$ near $P_j$ is:

$$ s_j(z) = \sum_{k=-\infty}^{-1} a_{k}^{(j)} (z - P_j)^k; $$

this function is holomorphic in $\mathbb{C} \setminus \{P_j\}$. (See the results on Laurent expansion.) Therefore,

$$ g(z) := f(z) - \sum_{j=1}^{J} s_j(z) $$

can be extended to a holomorphic function in $U$, i.e., the singularities of $g$ at every point $P_j$ is removable.

One obtains:
\[ \int_\gamma f(z) \, dz = \int_\gamma g(z) \, dz + \sum_{j=1}^{J} \int_\gamma s_j(z) \, dz \]
\[ = \sum_{j=1}^{J} a_j(1) \int_\gamma \frac{dz}{z - P_j} \]
\[ = \sum_{j=1}^{J} \text{Res}_f(P_j) 2\pi i \text{Ind}_\gamma(P_j) \]

\[ \diamond \]

15.4 Zero–Counting of Holomorphic Maps

We show here that the zeros of a holomorphic function \( f(z) \) can be counted (according to their multiplicity) by an integral. This is very useful if one perturbs the map \( f(z) \) or if one counts the solutions \( z_j \) of the perturbed equation

\[ f(z) - w = 0 \]

for small \( w \) instead of the zeros of \( f \).

15.4.1 The Multiplicity of a Zero

Let \( U \) be a region in \( \mathbb{C} \) and let \( f \in H(U) \). We assume that \( f \) is not identically zero. If \( z_0 \in U \) and \( f(z_0) = 0 \) then \( z_0 \) is called a zero of \( f \). For \( |z - z_0| < \varepsilon \) we can write:

\[ f(z) = \sum_{j=M}^{\infty} a_j(z - z_0)^j \]
\[ = (z - z_0)^M h(z) \]

where \( M \geq 1 \) and \( a_M \neq 0 \). The function \( h(z) \) is holomorphic in

\[ D(z_0, \varepsilon) \]

and we have, for sufficiently small \( \varepsilon \):

\[ h(z) \neq 0 \quad \text{for} \quad |z - z_0| \leq \varepsilon . \]

The number \( M \) is called the multiplicity of the zero \( z_0 \) of \( f \). We write

\[ M = \text{mult}_f(z_0) \]

and note that

\[ f^{(j)}(z_0) = 0 \quad \text{for} \quad j = 0, \ldots, M - 1, \quad f^{(M)}(z_0) = a_M M! \neq 0 . \]
15.4.2 The Zeros of a Holomorphic Function in a Disk

Let \( U \) be a region in \( \mathbb{C} \) and let \( f \in H(U) \). We assume that \( f \) is not identically zero.

Let \( \bar{D} = \bar{D}(P, r) \subset U \) be a closed disk in \( U \). We assume that
\[
f(z) \neq 0 \quad \text{for all} \quad z \in \partial \bar{D},
\]
i.e., \( f \) has no zero on the boundary of the disk \( \bar{D} \). Let
\[
\gamma(t) = P + re^{it}, \quad 0 \leq t \leq 2\pi,
\]
denote the positively oriented boundary curve of \( \bar{D} \).

Let \( z_1, \ldots, z_J \) denote the distinct zeros of \( f \) in the open disk \( D = D(P, r) \) with multiplicities
\[
M_j = \text{mult}_f(z_j).
\]

**Theorem 15.2** Under the above assumptions:
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{j=1}^{J} M_j,
\]
i.e., the integral can be used to count the zeros of \( f \) encircled by \( \gamma \) according to their multiplicities.

**Proof:** If \( \varepsilon > 0 \) is sufficiently small, then the curve
\[
\gamma_{\varepsilon}(t) = z_j + \varepsilon e^{it}, \quad 0 \leq t \leq 2\pi,
\]
encircles the zero \( z_j \) but no other zero of \( f \). We have
\[
\int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{j=1}^{J} \int_{\gamma_{\varepsilon}} \frac{f'(z)}{f(z)} \, dz.
\]
Fix \( j \) and set \( M = M_j \). From
\[
f(z) = (z - z_j)^M h(z) \quad \text{for} \quad z \in D_j := D(z_j, \varepsilon)
\]
with
\[
h \in H(D_j), \quad h(z) \neq 0 \quad \text{for} \quad z \in D_j,
\]
we obtain:
\[
f'(z) = M(z - z_j)^{M-1} h(z) + (z - z_j)^M h'(z)
\]
and
\[
\frac{f'(z)}{f(z)} = \frac{M}{z - z_j} + g(z), \quad g \in H(D_j).
\]
Therefore,
\[
\int_{\gamma_{\varepsilon}} \frac{f'(z)}{f(z)} \, dz = 2\pi i M = 2\pi i M_j.
\]
This proves the claim.

**Interpretation in Terms of \( f(\gamma(t)) \):** Let

\[
\mu(t) = f(\gamma(t)) = f(P + re^{it}), \quad 0 \leq t \leq 2\pi,
\]
denote the image of the curve \( \gamma(t) \) under the map \( f \). Then \( \mu(t) \neq 0 \) for \( 0 \leq t \leq 2\pi \) since, by assumption, \( f \) has no zero on \( \partial D \). The winding number of the curve \( \mu(t), 0 \leq t \leq 2\pi \), w.r.t. the point 0 is

\[
\text{Ind}_{\mu}(0) = \frac{1}{2\pi i} \int_{\mu} \frac{dw}{w} = \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{\mu'(t)}{\mu(t)} \, dt = \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))} \, dt = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz
\]

In other words, the left–hand side of (15.3) is the number of times by which the point \( f(\gamma(t)) \) moves around 0 when \( t \) changes from 0 to \( 2\pi \).

We obtain the following reformulation of Theorem 15.2:

**Theorem 15.3** Let \( \bar{D}(P, r) \subset U \) and let \( f \in H(U) \). Assume that \( f \) has no zero on \( \partial D(P, r) \). Then the number of zeros of \( f \) in \( D(P, r) \) (counting multiplicities) equals the number of times by which the curve

\[
\mu(t) = f(P + re^{it}), \quad 0 \leq t \leq 2\pi,
\]

winds around \( w = 0 \).

**Example:** Let \( f(z) = z^3 \) and let \( D = D(0, 1) \) denote the unit circle. The boundary curve of \( D \) is

\[
\gamma(t) = e^{it}, \quad 0 \leq t \leq 2\pi.
\]

The \( f \)–image of this curve is

\[
\mu(t) = e^{3it}, \quad 0 \leq t \leq 2\pi.
\]

Then \( \mu(t) \) winds three times around \( w = 0 \); accordingly, \( f \) has three zeros (counting multiplicities) in \( D \).

**15.4.3 The Argument Principle and Log–Functions**

The result of Theorem 15.2 is often called the **argument principle**. To explain this, we first make some remarks on log–functions.

If \( r > 0 \) we denote by \( \ln r \) the usual real natural logarithm of \( r \). We know that \( \frac{d}{dr} \ln r = \frac{1}{r}, r > 0 \).
The function
\[ r \to \ln r, \quad r > 0, \]
cannot be extended as a holomorphic function \( \log w \) defined for all \( w \in \mathbb{C} \setminus \{0\} \).
Otherwise, by the identity theorem,
\[ \frac{d}{dw} \log w = \frac{1}{w}, \quad w \neq 0. \]
However, we know that
\[ \int_{\gamma} \frac{dw}{w} = 2\pi i \neq 0, \]
where \( \gamma(t) = e^{it}, 0 \leq t \leq 2\pi \).

**The log–Function in a simply connected region.** If \( W \subset \mathbb{C} \) is a simply connected region with \( 0 \notin W \), then we can make a continuous choice for \( \arg(w), w \in W \), and write
\[ w = re^{i\arg(w)}, \quad r = |w| > 0, \quad w \in W. \]
We define
\[ \log_W(w) = \ln r + i\arg(w), \quad w \in W, \]
and obtain
\[ e^{\log_W(w)} = w, \quad w \in W. \]

Now let us make the same assumptions as in the previous subsection: Let \( U \) be a region in \( \mathbb{C} \) and let \( f \in H(U) \). We assume that \( f \) is not identically zero. Let \( \bar{D} = \bar{D}(P, r) \subset U \) be a closed disk in \( U \). We assume that \( f(z) \neq 0 \) for all \( z \in \partial \bar{D} \). Let
\[ \gamma(t) = P + re^{it}, \quad 0 \leq t \leq 2\pi, \]
declare the positively oriented boundary curve of \( \bar{D} \).

Let us assume that \( f \) has at least one zero in \( D \). Then the curve
\[ w(t) = f(\gamma(t)), \quad 0 \leq t \leq 2\pi, \]
winds around zero, and we cannot define \( \log w(t) \) consistently for \( 0 \leq t \leq 2\pi \).

Make a subdivision of the interval \( 0 \leq t \leq 2\pi \) into
\[ t_0 = 0 < t_1 < \ldots < t_K = 2\pi \]
and let
\[ \gamma_k(t) = \gamma(t), \quad t_{k-1} \leq t \leq t_k. \]
We have
\[ \gamma = \gamma_1 + \ldots + \gamma_K. \]

We make the subdivision fine enough so that each curve \( \gamma_k(t) \) lies in a simply connected region \( W_k \) with \( 0 \not\in W_k \). On \( W_k \) we have a log–function, which we call \( \log_k(w) \). We have

\[
\frac{d}{dw} \log_k(w) = \frac{1}{w}, \quad w \in W_k.
\]

Then, if \( z \) is chosen so that \( f(z) \in W_k \):

\[
\frac{d}{dz} \log_k(f(z)) = \frac{f'(z)}{f(z)}.
\]

Therefore,

\[
\int_{\gamma_k} \frac{f'(z)}{f(z)} \, dz = \log_k f(\gamma(t_k)) - \log_k f(\gamma(t_{k-1})).
\]

Define

\[
w_k = f(\gamma(t_k)), \quad 0 \leq k \leq K,
\]

and write

\[w_k = r_k e^{i \arg_k(w_k)}.\]

We then have

\[
\int_{\gamma_k} \frac{f'(z)}{f(z)} \, dz = \ln(r_k/r_{k-1}) + i \left( \arg_k(w_k) - \arg_k(w_{k-1}) \right).
\]

Summation over \( k \) from 1 to \( K \) yields

\[
\int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{k=1}^{K} \ln(r_k/r_{k-1}) + i \sum_{k=1}^{K} \arg_k(w_k) - \arg_k(w_{k-1}).
\]

The first part involves logs of real numbers:

\[
\sum_{k=1}^{K} \ln(r_k/r_{k-1}) = \ln \left( \frac{r_1}{r_0} \cdot \frac{r_2}{r_1} \cdots \frac{r_K}{r_{K-1}} \right)
= \ln 1
= 0
\]

Here we have used that \( w_0 = w_K \), thus \( r_0 = r_K \).

The real number

\[
\sum_{k=1}^{K} \arg_k(w_k) - \arg_k(w_{k-1})
\]
is the total change of argument of the function \( w(t) = f(\gamma(t)) \) as \( t \) goes from 0 to \( 2\pi \). In other words,

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi i} \sum_{k=1}^{K} \text{arg} g_k(w_k) - \text{arg} g_k(w_{k-1})
\]

is the number of times by which \( w(t) \) moves around zero when \( t \) goes from 0 to \( 2\pi \). This confirms our earlier interpretation of the left–hand side of the above equation.

**Example:** Let

\[
f(z) = z^3(z - 1)^2, \quad z \in \mathbb{C}.
\]

Let \( D = D(0, 2) \) and let \( \gamma(t) = 2e^{it}, 0 \leq t \leq 2\pi \), denote the boundary curve of \( D \). By Theorem 15.2 we have

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = 5
\]

since \( f \) has five zeros in \( D \). This is easily confirmed by the residue theorem: Since

\[
f'(z) = 3z^2(z - 1)^2 + 2z^3(z - 1)
\]

we have

\[
\frac{f'(z)}{f(z)} = 3 \cdot \frac{1}{z} + 2 \cdot \frac{1}{z - 1}.
\]

The curve

\[
\mu(t) = 8e^{3it}(2e^{it} - 1)^2, \quad 0 \leq t \leq 2\pi,
\]

is the image of \( \gamma(t) \) under \( f \). By Theorem 15.3 the curve \( \mu(t) \) winds five times around zero.

### 15.5 The Change of Argument and Zeros of Polynomials

Let \( U \subset \mathbb{C} \) denote an open set and let \( f \in H(U) \). Let \( \Gamma \) denote a curve in \( U \) and assume that \( f(z) \neq 0 \) for all \( z \in \Gamma \). The real number

\[
\text{Im} \int_{\Gamma} \frac{f'(z)}{f(z)} \, dz = \Delta_\Gamma \text{arg} f
\]

is called the change of argument of \( f \) along \( \Gamma \).

**Interpretation:** Let \( z(t), a \leq t \leq b \), denote a parametrization of \( \Gamma \). The curve \( \Gamma \) goes from \( A = z(a) \) to \( B = z(b) \). First assume that \( f(\Gamma) \subset W \) and that \( L(w) = \log w \) is a logarithm on \( W \). We have, for \( z \) near \( \Gamma \),

\[
\frac{d}{dz} \log f(z) = \frac{f'(z)}{f(z)},
\]

thus
\[ \int_{\Gamma} \frac{f'(z)}{f(z)} \, dz = \log f(B) - \log f(A) \]
\[ = \ln \left| \frac{f(B)}{f(A)} \right| + i(\arg f(B) - \arg f(A)), \]

thus

\[ \Delta_{\Gamma} \arg f = \text{Im} \int_{\Gamma} \frac{f'(z)}{f(z)} \, dz \]
\[ = \arg f(B) - \arg f(A). \]

Note that the difference in argument does not depend on the specific argument function on \( W \), but does depend on \( W \), i.e., does depend on the assumption that \( f(\Gamma) \subset W \).

In the general case, choose a subdivision

\[ t_0 = a < t_1 < \ldots < t_K = b \]

and obtain

\[ \Delta_{\Gamma} \arg f = \text{Im} \int_{\Gamma} \frac{f'(z)}{f(z)} \, dz \]
\[ = \text{Im} \sum_k \int_{\Gamma_k} \frac{f'(z)}{f(z)} \, dz \]
\[ = \sum_k \left( \arg_k f(w_k) - \arg f(w_{k-1}) \right). \]

Here \( w_k = f(z(t_k)) \) and \( \Gamma_k \) is the curve with parametrization \( z(t), t_{k-1} \leq t \leq t_k \).
Also, \( f(\Gamma_k) \subset W_k \) and \( \arg_k \) is an argument function on \( W_k \).

**Example 1:** Let \( \Gamma_R \) denote the curve, along the imaginary axis, parametrized by \( z(t) = it, -R \leq t \leq R \), and let \( \Gamma \) denote the whole imaginary axis with parametrization \( z(t) = it, -\infty < t < \infty \).

Consider the polynomial \( f(z) = z + 1 \) with the simple zero \( z_1 = -1 \) to the left of \( \Gamma \). We note that \( f(\Gamma) \) lies in the right half-plane and we can work with the main branch, \( \log w \).

We have

\[ \int_{\Gamma_R} \frac{dz}{z + 1} = \log(iR + 1) - \log(-iR + 1) \]
\[ = \ln(R^2 + 1)^{1/2} + i\theta_R - (\ln(R^2 + 1)^{1/2} - i\theta_R) \]

with
\[
\theta_R = \frac{\pi}{2} - \alpha_R, \quad \alpha_R = \arctan(1/R) = O(1/R).
\]
Therefore,
\[
\Delta \Gamma_R \arg (z + 1) = \pi + O(1/R)
\]
and
\[
\Delta \Gamma \arg (z + 1) = \pi.
\]
Similarly, if \(z_1\) is any point to the left of \(\Gamma\), one finds that
\[
\Delta \Gamma \arg (z - z_1) = \pi.
\]

**Example 2:** Let \(\Gamma_R\) and \(\Gamma\) denote the same curves as in Example 1 and consider the polynomial \(f(z) = z - 1\) with zero \(z_2 = 1\) to the right of \(\Gamma\). In this case, \(f(\Gamma)\) lies to the left of the imaginary axis, and the main branch, \(\log w\), is not defined along \(f(\Gamma)\). One obtains
\[
\Delta \Gamma_R \arg (z - 1) = \left(\frac{\pi}{2} + \alpha_R\right) - \left(\frac{3\pi}{2} - \alpha_R\right) = -\pi + O(1/R)
\]
and
\[
\Delta \Gamma \arg (z - 1) = -\pi.
\]
Similarly, if \(z_2\) is any point to the right of \(\Gamma\), one finds that
\[
\Delta \Gamma \arg (z - z_2) = -\pi.
\]

Let
\[
f(z) = \Pi_{j=1}^n (z - z_j)
\]
denote a polynomial and assume that none of the zeros \(z_j\) of \(f\) lies on \(\Gamma\). Since
\[
\frac{f'(z)}{f(z)} = \sum \frac{1}{z - z_j}
\]
one obtains that
\[
\Delta \Gamma \arg f = \pi(p - q)
\]
if \(p\) of the zeros of \(f\) lie to the left and \(q = n - p\) of the zeros of \(f\) lie to the right of \(\Gamma\). With a change of variables, one obtains the following result.
Theorem 15.4  Let \( \Gamma \) denote a straight line with parametrization

\[
z(t) = A + Bt, \quad -\infty < t < \infty,
\]

where \( A, B \) are complex numbers and \( B \neq 0 \). If \( f(z) \) is any polynomial without a zero on \( \Gamma \) then we have

\[
\Delta_\Gamma \arg f = \pi(p - q)
\]

if \( f \) has \( p \) zeros to the left and \( q \) zeros to the right of \( \Gamma \).
16 Applications and Extensions of the Argument Principle

The argument principle can be used to study how zeros of a holomorphic function are perturbed if the function is perturbed. This, in turn, leads to the **Open Mapping Theorem**.

In Section 16.4 we give a useful generalization of the argument principle, which has similarity to the spectral theorem for matrices and linear operators. We will use it to show that the local inverse of a holomorphic function is holomorphic if the inverse exists.

**16.1 Example**

We first consider a simple example. Let

\[ f(z) = z^3, \quad z \in \mathbb{C}. \]

Let \( D = D(0,1) \) denote the unit disk with boundary curve \( \gamma \). In this case, \( f \) has a zero

\[ z_0 = 0 \]

of multiplicity \( M = 3 \). We have

\[ \frac{f'(z)}{f(z)} = \frac{3}{z} \]

and Theorem 15.2 says that

\[ \frac{1}{2\pi i} \int_{\gamma} \frac{3z^2}{z^3} \, dz = 3. \]

Now consider the perturbed equation

\[ z^3 = w \]

where \( w \in \mathbb{C} \) is small in absolute value, \( w = re^{i\theta}, r > 0 \). The solutions are

\[ z_1(w) = r^{1/3} e^{i\theta/3} \]
\[ z_2(w) = r^{1/3} e^{i\theta/3} e^{2\pi i/3} \]
\[ z_3(w) = r^{1/3} e^{i\theta/3} e^{4\pi i/3} \]

These are simple zeros of the function

\[ g(z) = z^3 - w. \]

Theorem 15.2 applied to \( g(z) \) says that

\[ \frac{1}{2\pi i} \int_{\gamma} \frac{3z^2}{z^3 - w} \, dz = 3. \]
In this example, the triple zero $z_0 = 0$ of $f(z) = z^3$ splits into three simple zeros for the perturbed function $g(z) = z^3 - w$ if $w \neq 0$. We want to generalize this result.

### 16.2 Perturbation of a Multiple Zero

We make the same assumptions as in 15.4.2: $U$ is a region in $\mathbb{C}$; $f \in H(U)$ is not identically zero.

Let $z_0 \in U$ be a zero of $f$ of multiplicity $M$. We can choose $r > 0$ with

a) $\bar{D} = \bar{D}(z_0, r) \subset U$;

b) $f(z) \neq 0$ for $0 < |z - z_0| \leq r$;

c) $f'(z) \neq 0$ for $0 < |z - z_0| \leq r$.

We let $\gamma(t) = z_0 + re^{it}, 0 \leq t \leq 2\pi$.

Let $\eta := \min\{|f(z)| : |z - z_0| = r\}$, thus $\eta > 0$. We consider the equation

$$f(z) = w, \quad z \in D(z_0, r),$$

for $|w| < \eta$.

Let us make a plausibility consideration first: Let $w \sim 0$ be given. We have, for $z$ close to $z_0$:

$$f(z) \sim a_M(z - z_0)^M, \quad a_M \neq 0.$$

We must solve

$$a_M(z - z_0)^M \sim w,$$

i.e.,

$$(z - z_0)^M \sim \frac{w}{a_M} =: \rho e^{i\theta}.$$

If $\rho > 0$ then the equation

$$q^M = \frac{w}{a_M} = \rho e^{i\theta}$$

has $M$ distinct solutions $q_j$:

$$
q_1 = \rho^{1/M} e^{i\theta/M} \\
q_2 = q_1 e^{2\pi i/M} \\
q_3 = q_1 e^{4\pi i/M} \\
\ldots = \ldots \\
q_M = q_1 e^{(M-1)2\pi i/M}
$$

We expect that the equation

$$f(z) = w$$

has $M$ solutions.
\[ z_j(w) \sim z_0 + q_j, \quad j = 1, \ldots, M. \]

**Theorem 16.1** Let \( U \) denote a region in \( \mathbb{C} \) and let \( f \in H(U) \). Assume that \( f \) is not identically zero. Let \( z_0 \in U \) denote a zero of \( f \) of multiplicity \( M \). Choose \( r > 0 \) so that the disk \( D = D(z_0,r) \) satisfies the conditions \( \bar{D}(z_0,r) \subset U; f(z) \neq 0 \) for \( 0 < |z - z_0| \leq r \); \( f'(z) \neq 0 \) for \( 0 < |z - z_0| \leq r \). Define \( \eta := \min\{|f(z)| : |z - z_0| = r\} \) and let \( 0 < |w| < \eta \). Then the equation \( f(z) = w \) has \( M \) distinct solutions \( z_1, \ldots, z_M \) in \( D \). Every \( z_j \) is a simple zero of the function \( g(z) = f(z) - w \), i.e., \( g'(z_j) = f'(z_j) \neq 0 \).

**Proof:** Let \( g(z) = f(z) - w \). If \( |w| < \eta \) and \( |z - z_0| = r \) then

\[ |g(z)| \geq |f(z)| - |w| \geq \eta - |w| > 0. \]

Therefore, the function

\[ F(w) := \int_{\gamma} \frac{f'(z)(f(z) - w)}{f(z) - w} \, dz, \quad |w| < \eta, \]

is integer valued. We know that \( F(w) \) is the number of zeros of \( g(z) \) in \( D \), where the zeros are counted according to their multiplicity.

We claim that \( F(w) \) is holomorphic for \( |w| < \eta \). In fact, for \( |z - z_0| = r \):

\[
\frac{1}{f(z) - w} = \frac{1}{f(z)} \cdot \frac{1}{1 - w/f(z)} = \sum_{j=0}^{\infty} \frac{w^j}{(f(z))^{j+1}}
\]

For every fixed \( w \) with \( |w| < \eta \) the convergence is uniform for \( z \in \tilde{\gamma} \). This yields:

\[ F(w) = \int_{\gamma} \frac{f'(z)}{f(z)} dz \left( \sum_{j=0}^{\infty} b_j w^j \right) \]

with

\[ b_j = \int_{\gamma} \frac{f'(z)}{(f(z))^{j+1}} dz. \]

A holomorphic function that is integer–valued is constant. One obtains that

\[ F(w) \equiv M. \]

It follows that the number of zeros of \( g(z) \) in \( D \) is \( M \) if zeros are counted according to their multiplicity.

Now let \( 0 < |w| < \eta \) and let \( z_1 \in D \) be a zero of \( g \). Then \( f(z_1) = w \), thus \( z_1 \neq z_0 \). It follows that \( f'(z_1) \neq 0 \); thus all zeros of \( g(z) \) are simple. \( \diamond \)
16.3 The Open Mapping Theorem

If $U$ and $V$ are metric spaces (or, more generally, topological spaces) and $f : U \to V$ is a map, then $f$ is called open if the set $f(\Omega)$ is an open subset of $V$ whenever $\Omega$ is an open subset of $U$. (This notion is different from continuity.

A map $f : U \to V$ can be shown to be continuous on $U$ if and only if $f^{-1}(W)$ is an open subset of $U$ whenever $W$ is an open subset of $V$. Here $f^{-1}(W) = \{u \in U : f(u) \in W\}$.

The following result is know as the Open Mapping Theorem of complex analysis. (There is another Open Mapping Theorem of functional analysis, which is different.)

Theorem 16.2 Let $U$ be a region in $\mathbb{C}$ and let $f \in H(U)$ be a non–constant function. Then the mapping $f : U \to \mathbb{C}$ is open.

Remark: Such a result is not true in $\mathbb{R}$. For example, if $f(x) = x^2$, then $f(\mathbb{R}) = [0, \infty)$. The set $[0, \infty)$ is not open in $\mathbb{R}$.

Proof: Let $\Omega \subset U$ be an open non–empty set. We must show that $f(\Omega)$ is open. To this end, let $Q \in f(\Omega)$ be an arbitrary point. We must show that there is $\varepsilon > 0$ with $D(Q, \varepsilon) \subset f(\Omega)$.

Since $Q \in f(\Omega)$ there is $P \in \Omega$ with $f(P) = Q$. We will apply Theorem 16.1 to the function

$$h(z) = f(z) - Q, \quad z \in \Omega.$$  

Note that $h(P) = f(P) - Q = 0$.

Let $M$ denote the multiplicity of the zero $P$ of the function $h(z)$. There is $r > 0$ with:

a) $D(P, r) \subset \Omega$;
b) $f(z) \neq Q$ for $0 < |z - P| \leq r$;
c) $f'(z) \neq 0$ for $0 < |z - P| \leq r$.

Let $\eta := \min\{|f(z) - Q| : |z - P| = r\}$, thus $\eta > 0$. If $|w| < \eta$ then the equation

$$f(z) = Q + w$$
has $M$ solutions $z_j \in D(P, r) \subset \Omega$. In particular, if $Q + w \in D(Q, \eta)$ then $Q + w$ lies in $f(\Omega)$. This says that $D(Q, \eta) \subset f(\Omega)$, proving the theorem. ⋄

16.4 An Analogue to the Spectral Theorem

The following is a useful generalization of Theorem 15.2. The assumptions are similar to those in Theorem 15.2, but a general function $\phi \in H(U)$ appears in Theorem 16.3 which is identically one in Theorem 15.2.

Theorem 16.3 Let $U$ be a region and let $f, \phi \in H(U)$. Let $\bar{D} = \bar{D}(P, r) \subset U$. Assume that

$$f(z) \neq 0 \quad \text{for} \quad |z - P| = r$$

and let $\gamma(t) = P + re^{it}$, $0 \leq t \leq 2\pi$. Let $z_1, \ldots, z_J$ denote the distinct zeros of $f$ in $D$ with multiplicities $M_j = \text{mult}_f(z_j)$. Then we have

$$\frac{1}{2\pi i} \int_{\gamma} \phi(z) \frac{f'(z)}{f(z)} \, dz = \sum_{j=1}^{J} M_j \phi(z_j).$$

Proof: Let $\varepsilon > 0$ be small enough and let

$$\gamma_{j\varepsilon}(t) = z_j + \varepsilon e^{it}, \quad 0 \leq t \leq 2\pi.$$ 

Fix $j$ and let $M = M_j$. In the following, the functions $h_k(z)$ are holomorphic for $|z - z_j| < \varepsilon$. We have

$$f(z) = (z - z_j)^M h_1(z), \quad h_1(z_j) \neq 0,$$

$$f'(z)/f(z) = M(z - z_j)^{-1} + h_2(z),$$

$$\phi(z) = \phi(z_j) + h_3(z), \quad h_3(z_j) = 0,$$

$$\phi(z)f'(z)/f(z) = M\phi(z_j)(z - z_j)^{-1} + h_4(z).$$

This implies that

$$\int_{\gamma_{j\varepsilon}} \phi(z) \frac{f'(z)}{f(z)} \, dz = 2\pi i M_j \phi(z_j).$$

The theorem follows by summing over $j$. ⋄

16.5 Local Inverses of Holomorphic Functions

Let $U$ be a region in $\mathbb{C}$ and let $f \in H(U)$. We ask for conditions under which the mapping $f : U \to \mathbb{C}$ is $1 - 1$. First assume that $f'(z_0) = 0$ for some $z_0 \in U$. If $Q := f(z_0)$ then $z_0$ is a zero of multiplicity $M \geq 2$ of the function

$$h(z) = f(z) - Q, \quad z \in U.$$
Choose any \( w \in \mathbb{C}, w \neq 0 \), with \(|w|\) small. By Theorem 16.1 there are \( M \) distinct points \( z_1, \ldots, z_M \in D(z_0, r) \) which have the same image under \( f \), thus
\[
f(z_j) = Q + w, \quad j = 1, \ldots, M.
\]
One obtains that \( f \) cannot be \( 1 \)-1 if \( f'(z_0) = 0 \) for some \( z_0 \).

Now assume that \( f'(z) \neq 0 \) for all \( z \in U \). The example \( f(z) = e^z \) shows that \( f \) may still fail to be \textit{globally} \( 1 \)-1 since
\[
e^0 = e^{2\pi i} = 1.
\]
However, as we will prove below, if \( f'(z_0) \neq 0 \), then \( f \) is \textit{locally} \( 1 \)-1 near \( z_0 \).

**Theorem 16.4** (local inversion of holomorphic functions) Let \( f \in H(U) \). Let \( P \in U \) with \( f'(P) \neq 0 \). Set \( Q = f(P) \). Then there is an open neighborhood \( U_0 \) of \( P \) with \( U_0 \subset U \) and there is an open disk \( D(Q, \eta) \) so that the following holds:

a) \( f : U_0 \rightarrow D(Q, \eta) \) is \( 1 \)-1 and onto;

b) there is a unique function \( g : D(Q, \eta) \rightarrow U_0 \) which is \( 1 \)-1 and onto satisfying
\[
f(g(w)) = w \quad \text{for all} \quad w \in D(Q, \eta)
\]
and
\[
g(f(z)) = z \quad \text{for all} \quad z \in U_0.
\]

This uniquely determined function \( g \) is holomorphic on \( D(Q, \eta) \).

**Proof:** 1) Choose \( r > 0 \) with

a) \( D(P, r) \subset U \);

b) \( f(z) \neq Q \) for \( 0 \leq |z - P| \leq r \);

c) \( f'(z) \neq 0 \) for \( 0 \leq |z - P| \leq r \).

If \( \gamma(t) = P + re^{it}, 0 \leq t \leq 2\pi \), then
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - Q} \, dz = 1.
\]

This holds since the equation \( f(z) - Q = 0 \) has precisely one solution \( z \) in \( D(P, r) \), namely \( z = P \), and the solution \( z = P \) is simple since \( f'(P) \neq 0 \).

Set
\[
\eta = \min \{|f(z) - Q| : |z - P| = r\} > 0.
\]

If \( |w - Q| < \eta \) then the equation
\[
f(z) = w
\]
has a unique solution \( z_1 \in D(P,r) \). We call this solution \( z_1 = g(w) \). In this way we have defined a function

\[ g : D(Q,\eta) \to D(P,r) \]

with

\[ f(g(w)) = w \quad \text{for all} \quad w \in D(Q,\eta) . \]

This equation implies that \( g \) is \( 1-1 \). We note that \( g(Q) = P \) since \( P \) is the unique solution of the equation \( f(z) = Q, z \in D(P,r) \).

2) Apply Theorem 16.3 with \( \phi(z) \equiv z \) to obtain

\[ \frac{1}{2\pi i} \int_\gamma z \cdot \frac{f'(z)}{f(z) - w} \, dz = g(w) \quad \text{for} \quad w \in D(Q,\eta) . \]  

(16.1)

(Note that \( z_1 = g(w) \) is the unique zero of the function \( f(z) - w \) and \( z_1 = g(w) \) is simple. Also, if \( \phi(z) \equiv z \), then \( \phi(g(w)) = g(w) \).

We use the representation (16.1) of the function \( g \) to prove that \( g \) is holomorphic on \( D(Q,\eta) \). To this end, note that for \( z \in \tilde{\gamma} \) and \( w \in D(Q,\eta) \):

\[ f(z) - w = (f(z) - Q) - (w - Q) \]

with

\[ |f(z) - Q| \geq \eta > |w - Q| . \]

Therefore,

\[ \frac{1}{f(z) - w} = \frac{1}{(f(z) - Q) - (w - Q)} \]
\[ = \frac{1}{f(z) - Q} \cdot \frac{1}{1 - \frac{w-Q}{f(z)-Q}} \]
\[ = \sum_{j=0}^{\infty} \frac{(w-Q)^j}{(f(z)-Q)^{j+1}} \]

The convergence is uniform for \( z \in \tilde{\gamma} \). Using the above series in (16.1) and exchanging summation and integration, we obtain the expansion

\[ g(w) = \frac{1}{2\pi i} \sum_{j=0}^{\infty} b_j (w - Q)^j \]

with

\[ b_j = \int_\gamma \frac{zf'(z)}{(f(z)-Q)^{j+1}} \, dz . \]

This proves that \( g \in H(D(Q,\eta)) \).
c) Define $U_0 := g(D(Q, \eta))$. Then, by the Open Mapping Theorem, $U_0$ is an open neighborhood of $P$ and $U_0 \subset D(P, r) \subset U$. The remaining claims of the theorem are now easily verified: The mapping

$$g : D(Q, \eta) \to U_0$$

is $1-1$ and onto. If $z \in U_0$ is given, then there exists a unique $w \in D(Q, \eta)$ with $g(w) = z$. We have $f(g(w)) = w$, thus

$$g(f(g(w))) = g(w).$$

Recalling that $z = g(w)$ this becomes:

$$g(f(z)) = z \quad \text{for all} \quad z \in U_0.$$

This equation implies that $f$ is $1-1$ on $U_0$.

Let $w \in D(Q, \eta)$ be given. Then $z := g(w) \in U_0$ satisfies $f(z) = f(g(w)) = w$. Thus we have shown that $f : U_0 \to D(Q, \eta)$ is $1-1$ and onto, with inverse function $g$. The uniqueness of $g$ is trivial. ⊖

**Remark:** One can also prove the previous theorem by power series expansion. Assume $P = Q = 0$, for simplicity, and let

$$f(z) = \sum_{j=1}^{\infty} a_j z^j, \quad |z| < r, \quad a_1 \neq 0.$$

We try to determine a function

$$g(w) = \sum_{k=1}^{\infty} b_k w^k, \quad |w| < \eta,$$

with

$$|g(w)| < r \quad \text{and} \quad f(g(w)) = w \quad \text{for all} \quad |w| < \eta.$$

First proceeding formally, we write

$$f(g(w)) = a_1(b_1w + b_2w^2 + \ldots) + a_2(b_1w + b_2w^2 + \ldots)^2 + \ldots = a_1b_1w + w^2(a_1b_2 + a_2b_1^2) + \ldots$$

The condition $f(g(w)) = w$ yields

$$a_1b_1 = 1 \quad \text{thus} \quad b_1 = 1/a_1.$$

Further,

$$a_1b_2 + a_2b_1^2 = 0 \quad \text{thus} \quad b_2 = -a_2b_1^2/a_1.$$

This process can be continued. The $b_k$ are determined recursively. One then has to prove that the series...
\[ g(w) = \sum_{k=1}^{\infty} b_k w^k \]

has a positive radius of convergence.

### 16.6 The Argument Principle for Meromorphic Functions

Roughly speaking, a function which is holomorphic except for poles is called meromorphic. Let us be more precise.

**Definition:** A set \( S \subset \mathbb{C} \) is called discrete if for all \( z \in S \) there is \( r > 0 \) with \( S \cap D(z, r) = \{z\} \).

**Example:** Let \( S = \{\frac{1}{n} : n \in \mathbb{N}\} \). Then \( S \) is a discrete set. The set \( S \cup \{0\} \) is not discrete.

**Definition:** Let \( U \subset \mathbb{C} \) be open. Assume that \( S \subset U \) is a discrete subset of \( \mathbb{C} \) which is closed in \( U \), i.e., if \( z_n \in S \) and \( z_n \to z \in U \), then \( z \in S \). Let \( f \in H(U \setminus S) \). The function \( f \) is called meromorphic in \( U \) with singular set \( S \) if every \( z_j \in S \) is a pole of \( f \). Often one simply says that \( f \) is meromorphic in \( U \) and writes \( f \in M(U) \).

**Example:** Let \( S = \{\frac{1}{n} : n \in \mathbb{N}\} \). This set is not closed as a subset of \( \mathbb{C} \). However, if \( U = \{z = x + iy : x > 0\} \) denotes the right half–plane, then \( S \) is closed in \( U \).

**Example 1:** Let \( p(z) \) and \( q(z) \) be polynomials which have no common zero. The rational function

\[ f(z) = \frac{p(z)}{q(z)} \]

is meromorphic in \( \mathbb{C} \) with singular set

\[ S = \{z_j : q(z_j) = 0\} . \]

**Example 2:** The function

\[ f(z) = \frac{1}{\sin(\pi z)} \]

is meromorphic in \( \mathbb{C} \) with singular set \( S = \mathbb{Z} \).

**Example 3:** Let

\[ S = \{z_n = \frac{1}{n \pi} : n \in \mathbb{Z}, n \neq 0\} \]

and let

\[ S_0 = S \cup \{0\} . \]

Note that 0 is an accumulation point of \( S \) and of \( S_0 \). Consider the function

\[ f(z) = \frac{1}{\sin(1/z)}, \quad z \in \mathbb{C} \setminus S_0 . \]
Clearly, \( f \) is holomorphic on \( \mathbb{C} \setminus S_0 \). The function \( f \) is not meromorphic on \( \mathbb{C} \setminus S_0 \) since the singularity at \( z = 0 \) is not isolated. The singularity at \( z = 0 \) is neither a pole nor an essential singularity. If \( U = \mathbb{C} \setminus \{0\} \), then \( f \) is meromorphic on \( U \) with singular set \( S \).

**Theorem 16.5** Let \( U \subset \mathbb{C} \) be open and let \( f \in M(U) \). Let \( D = \bar{D}(P, r) \subset U \) be a closed disk in \( U \) and assume that \( f \) has no zero and no pole on the boundary \( \partial D \) of \( D \). Let \( z_1, \ldots, z_J \) denote the distinct zeros of \( f \) in \( D \) with multiplicities \( M_j = \text{mult}_f(z_j) \) and let \( p_1, \ldots, p_K \) denote the distinct poles of \( f \) in \( D \) with orders \( N_k = \text{ord}_f(p_k) \). Then we have

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{j=1}^{J} M_j - \sum_{k=1}^{K} N_k .
\]

In other words,

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} \, dz = \#(\text{zeros}) - \#(\text{poles}) .
\]

Here the zeros and poles of \( f \) in \( D \) are counted with their multiplicities.

**Proof:** The proof is similar to the proof of Theorem 15.2. We only note that if \( p_k \) is a pole of order \( N = N_k \) of \( f \), then we have for \( 0 < |z - p_k| < \varepsilon \):

\[
f(z) = a_{-N}(z - p_k)^{-N}(1 + h_1(z)), \quad a_{-N} \neq 0 ,
\]
and

\[
f'(z) = (-N)a_{-N}(z - p_k)^{-N-1}(1 + h_2(z)) ,
\]
thus

\[
\frac{f'(z)}{f(z)} = \frac{-N}{z - p_k} + h_3(z) .
\]

Here \( h_{1,2,3} \) are holomorphic near \( p_k \). The claim then follows as in the proof of Theorem 15.2. \( \diamond \)

**16.7 Rouché’s Theorem and Hurwitz’s Theorem**

**Theorem 16.6** (Rouché) Let \( U \) be open and let \( f, g \in H(U) \). Let \( D(P, r) \subset U \). Assume that \( f \) and \( g \) are close to each other in the sense that

\[
|f(z) - g(z)| < |f(z)| + |g(z)| \quad \text{for} \quad |z - P| = r . \quad (16.2)
\]

Then \( f \) and \( g \) have the same number of zeros in \( D(P, r) \) where zeros are counted with their multiplicities. In other words, if \( \gamma(t) = P + re^{it} \), then

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi i} \int_{\gamma} \frac{g'(z)}{g(z)} \, dz . \quad (16.3)
\]
**Proof:** Let \(|z - P| = r\). Then (16.2) implies that

\[ f(z) \neq 0 \neq g(z). \]

Therefore the integrals in (16.3) are defined. We claim that, for \(|z - P| = r\), the number

\[ \lambda = \frac{f(z)}{g(z)} \]

does not belong to \((-\infty, 0]\). Otherwise,

\[
\left| \frac{f(z)}{g(z)} - 1 \right| = |\lambda - 1| = -\lambda + 1 = \left| \frac{f(z)}{g(z)} \right| + 1
\]

Multiplying by \(|g(z)|\) one obtains that

\[ |f(z) - g(z)| = |f(z)| + |g(z)| \]

in contradiction to (16.2).

Consider the function

\[ f_t(z) = tf(z) + (1 - t)g(z), \quad z \in U, \]

for \(0 \leq t \leq 1\). If \(|z - P| = r\) then \(f_t(z) \neq 0\) since, otherwise, one obtains that \(f(z)/g(z)\) is negative. It follows that

\[ I(t) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_t'(z)}{f_t(z)} \, dz, \quad 0 \leq t \leq 1, \]

is integer valued and continuous. Therefore, \(I(0) = I(1)\), proving the theorem.

\[ \diamond \]

**Example:** Let \(f(z) = z^7 + 5z^3 - z - 2\) and \(g(z) = 5z^3\). For \(|z| = 1\) we have

\[ |f(z) - g(z)| = |z^7 - z - 2| \leq 4 \]

and

\[ |g(z)| = 5. \]

Therefore, (16.3) holds for \(|z| = 1\). Clearly, \(g(z)\) has a zero of multiplicity 3 in \(D(0, 1)\), and has no other zero. Therefore, by Rouché’s theorem, the polynomial \(f(z)\) has three zeros \(z_j\) with \(|z_j| < 1\). These three zeros are not necessarily distinct.

**Theorem 16.7 (Hurwitz)** Let \(U\) be a region and let \(f_n \in H(U)\) for \(n = 1, 2, \ldots\) Assume that \(f_n(z)\) converges normally to \(f(z)\). (Thus, \(f \in H(U)\).) If \(f_n(z) \neq 0\) for all \(z \in U\) and all \(n = 1, 2, \ldots\), then either \(f \equiv 0\) or \(f(z) \neq 0\) for all \(z \in U\).
**Proof:** Suppose that \( f \) is not identically zero, but \( f(P) = 0 \) for some \( P \in U \). Let \( M \) denote the multiplicity of the zero \( P \) of \( f \),

\[
M = \text{mult}_f(P) \geq 1.
\]

There exists \( r > 0 \) with \( \bar{D}(P, r) \subset U \) and \( f(z) \neq 0 \) for \( 0 < |z - P| \leq r \). Let \( \gamma(t) = P + re^{it} \). One obtains that

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = M, \quad \text{but} \quad \frac{1}{2\pi i} \int_{\gamma} \frac{f'_n(z)}{f_n(z)} \, dz = 0
\]

for all \( n \). As \( n \to \infty \), the quotient \( f'_n(z)/f_n(z) \) converges uniformly to \( f'(z)/f(z) \) on \( \gamma \), and we obtain a contradiction. \( \diamond \)
17 Matrix–Valued and Operator Valued Analytic Functions

17.1 Outline and Examples

Let $\gamma(t), a \leq t \leq b$, denote a simple closed positively oriented curve in $\mathbb{C}$ and let $\lambda$ be a complex number, $\lambda \notin \gamma$. By the residue theorem:

$$\frac{1}{2\pi i} \int_{\gamma} (z - \lambda)^{-1} \, dz = \begin{cases} 1, & \lambda \text{ inside } \gamma \\ 0, & \lambda \text{ outside } \gamma \end{cases} \quad (17.1)$$

It is interesting that one can generalize the formula to the case where $\lambda$ is replaced by a matrix $A \in \mathbb{C}^{n \times n}$ or a more general operator defined on a dense subspace of a Banach space. We will consider here only the case of a matrix $A$, but generalizations are possible.

Let $A \in \mathbb{C}^{n \times n}$. With $\sigma(A) = \{\lambda_1, \ldots, \lambda_s\}$ we denote the set of (distinct) eigenvalues of $A$. (More generally, $\sigma(A)$ denotes the spectrum of the operator $A$.) The matrix valued function

$$(zI - A)^{-1}, \quad z \in \mathbb{C} \setminus \sigma(A),$$

is the so-called resolvent of $A$. Suppose that $\gamma$ is a curve, as above, and $\lambda_j \notin \gamma$ for $j = 1, \ldots, s$. By Cramer’s rule, each matrix entry

$$((zI - A)^{-1})_{jk}$$

is a rational function of $z$ defined for $z \in \mathbb{C} \setminus \sigma(A)$. We let

$$P_A = \frac{1}{2\pi i} \int_{\gamma} (zI - A)^{-1} \, dz,$$

where the integral is defined elementwise, i.e.,

$$(P_A)_{jk} = \frac{1}{2\pi i} \int_{\gamma} ((zI - A)^{-1})_{jk} \, dz, \quad 1 \leq j, k \leq n.$$

**Example 1**: Let $A$ denote a $4 \times 4$ diagonal matrix,

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}.$$ 

Suppose that $\lambda_{1,2}$ are inside and $\lambda_{3,4}$ are outside $\gamma$. Using (17.1) it is clear that

$$P_A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Let $e^1, \ldots, e^4$ denote the standard bases of $\mathbb{C}^4$. For the diagonal matrix $A$, $U = \text{span}\{e^1, e^2\}$ is the sum of the eigenspaces of $\lambda_{1,2}$ and $V = \text{span}\{e^3, e^4\}$.
is the sum of the eigenspaces of $\lambda_{3,4}$. The matrix $P_A$ is the projector onto $U$ along $V$.

The result generalizes, but one must consider generalized eigenspaces instead of geometric eigenspaces.

**Example 2:** Let $A$ denote a $2 \times 2$ Jordan matrix:

$$A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}.$$  

The resolvent is

$$(zI - A)^{-1} = \frac{1}{(z - \lambda_1)^2} \begin{pmatrix} z - \lambda_1 & 1 \\ 0 & z - \lambda_1 \end{pmatrix}.$$  

One obtains that $P_A = I$ if $\lambda_1$ is inside and $P_A = 0$ if $\lambda_1$ is outside $\gamma$. This makes it clear that the generalized eigenspace is important, not the geometric eigenspace.

**Notation:** Let $A \in \mathbb{C}^{n \times n}$. If $\lambda_j$ is an eigenvalue of $A$, then

$$E(\lambda_j) = \{ u \in \mathbb{C}^n : (A - \lambda_j I)^m u = 0 \text{ for some } m = 1, 2, \ldots, n \}$$

denotes the generalized eigenspace to $\lambda_j$.

It is not difficult to show:

**Theorem 17.1** Let

$$U = E(\lambda_1) \oplus \ldots \oplus E(\lambda_k)$$

and

$$V = E(\lambda_{k+1}) \oplus \ldots \oplus E(\lambda_s)$$

where $\lambda_1, \ldots, \lambda_k$ are inside and $\lambda_{k+1}, \ldots, \lambda_s$ are outside $\gamma$. Then

$$P_A = \frac{1}{2\pi i} \int_{\gamma} (zI - A)^{-1} \, dz \quad (17.2)$$

is the projector onto $U$ along $V$.

This formula for $P_A$ is very useful if one wants to study perturbations of $A$. Assume, for example, that $A = A(w)$ depends analytically in a parameter $w \in \mathbb{C}$. The eigenvalues $\lambda_j(w)$ are continuous functions of $w$ (if this is properly defined), but they are generally not smooth. The formula (17.2) shows, however, that $P_{A(w)}$ depends analytically on $w$ as long as the eigenvalues $\lambda_j(w)$ do not cross $\tilde{\gamma}$. Thus, the projector $P_A$ behaves much better under perturbations of $A$ than the eigenvalues of $A$.

**Example 3:** Let $A(w)$ denote a $2 \times 2$ matrix

$$A(w) = \begin{pmatrix} 0 & 1 \\ w & 0 \end{pmatrix}.$$
The eigenvalues are
\[
\lambda_1 = \sqrt{w}, \quad \lambda_2 = -\sqrt{w}.
\]
These functions are not differentiable at \( w = 0 \) and are not analytic in \( D(0, r) \setminus \{0\} \).

17.2 Analyticity of the Resolvent

Lemma 17.1 Let \( A \in \mathbb{C}^{n \times n} \) and let \( \sigma(A) \) denote the set of eigenvalues of \( A \). Then each matrix entry of the resolvent \( (A - zI)^{-1} \),
\[
((A - zI)^{-1})_{jk}
\]
is a rational function on \( \mathbb{C} \setminus \sigma(A) \).

This result follows from Cramer’s rule for the inverse of a matrix.

Another way to prove analyticity of the functions (17.3) uses the Neumann series. This proves generalizes to operators in Banach spaces.

17.3 Complementary Subspaces and Projectors

We want to make the ideas of the previous section more precise.

Definition: Let \( W \) be a vector space. Two subspaces \( U \) and \( V \) of \( W \) are called complementary if for every \( w \in W \) there is a unique \( u \in U \) and a unique \( v \in V \) with
\[
w = u + v, \quad u \in U, \quad v \in V.
\]
If \( U, V \) are complementary subspaces of \( W \) one writes
\[
W = U \oplus V.
\]

Definition: Let \( W \) be a vector space. A linear map \( P : W \to W \) is called a projector if \( P^2 = P \).

There is a close relation between pairs of complementary subspaces of \( W \) and projectors \( P \) from \( W \) into itself. The following is rather easily shown:

Theorem 17.2 1. Let \( U, V \) be complementary subspaces of \( W \). The map \( P : W \to W \) defined by
\[
Pw = u \quad \text{where} \quad w = u + v, \quad u \in U, \quad v \in V,
\]
is a projector. We have
\[
U = R(P), \quad V = N(P).
\]
The projectors \( P \) is called the projector onto \( U \) along \( V \).

2. Let \( P : W \to W \) be any projector. Then the subspaces
are complementary and the projector onto $U$ along $V$ is $P$.

3. If $P : W \to W$ is a projector, then $Q = I - P$ is also a projector. We have

\[ R(P) = N(Q), \quad N(P) = R(Q). \]

### 17.3.1 The Matrix Representation if a Projector

Let

\[ \mathbb{C}^n = U \oplus V. \]

Let

\[ t^1, \ldots, t^r \]

be a basis of $U$ and let

\[ t^{r+1}, \ldots, t^n \]

be a basis of $V$. Then

\[ T = (t^1, \ldots, t^n) \in \mathbb{C}^{n \times n} \]

is a nonsingular matrix.

**Lemma 17.2** Under the above assumptions, the projector $P$ onto $U$ along $V$ has the matrix representation

\[ P = T \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} T^{-1}. \]

**Proof:** If $w \in \mathbb{C}^n$ is any given vector, we write

\[ w = x_1 t^1 + \ldots + x_n t^n = Tx, \quad x \in \mathbb{C}^n, \]

and obtain

\[ u = Pw = x_1 t^1 + \ldots + x_r t^r. \]

If we partition $x$ as

\[ x = \begin{pmatrix} x^I \\ x^{I'} \end{pmatrix}, \quad x^I \in \mathbb{C}^r, \quad x^{I'} \in \mathbb{C}^{n-r}, \]

then we have
\[ u = Pw \]
\[ = T \begin{pmatrix} x^I \\ 0 \end{pmatrix} \]
\[ = T \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} x \]
\[ = T \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} T^{-1} w. \]

\[ \diamond \]

### 17.4 The Dunford–Taylor Integral

We first recall some familiar facts.

Let \( \Gamma \) denote a positively oriented, simple closed curve in \( \mathbb{C} \). Then \( \mathbb{C} \setminus \Gamma \) has two connected components, the interior of \( \Gamma \) and the exterior of \( \Gamma \). These are denoted by

\[ \text{int} \, \Gamma \quad \text{and} \quad \text{ext} \, \Gamma, \]

respectively. Let \( a \in \mathbb{C} \setminus \Gamma \). We have

\[ \frac{1}{2\pi i} \int_{\Gamma} \frac{dz}{z - a} = 1 \quad \text{if} \quad a \in \text{int} \, \Gamma \]

and

\[ \frac{1}{2\pi i} \int_{\Gamma} \frac{dz}{z - a} = 0 \quad \text{if} \quad a \in \text{ext} \, \Gamma. \]

**Lemma 17.3** Let \( U \) be an open set containing \( \Gamma \) and \( \text{int} \, \Gamma \). Let \( \phi \in H(U) \). Assuming \( a \in \text{int} \, \Gamma \), we have

\[ \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(z)}{z - a} \, dz = \phi(a). \]

We want to generalize the formula to the case where the number \( a \) is replaced by a matrix \( A \in \mathbb{C}^{n \times n} \). Then \( 1/(z - a) \) will be replaced by

\[ (zI - A)^{-1}. \]

From our previous results, we have:

**Lemma 17.4** Let \( \Gamma \) be a curve as above and let \( A \in \mathbb{C}^{n \times n} \). Assuming that

\[ \sigma(A) \subseteq \text{int} \, \Gamma \]

we have

\[ \frac{1}{2\pi i} \int_{\Gamma} (zI - A)^{-1} \, dz = I. \]
We now introduce a function \( \phi(z) \) multiplying \((zI - A)^{-1}\) in the integral, i.e., we consider the so-called Dunford–Taylor integral

\[
\frac{1}{2\pi i} \int_{\Gamma} \phi(z)(zI - A)^{-1} \, dz.
\]

Under suitable assumptions, this formula can be used to define \( \phi(A) \) in a reasonable way.

17.4.1 The Case of a Polynomial

We first prove:

**Lemma 17.5** Let \( \Gamma \) be a curve as above and let \( A \in \mathbb{C}^{n \times n} \) with \( \sigma(A) \subset \text{int}\, \Gamma \). For \( j = 0, 1, \ldots \) we have

\[
\frac{1}{2\pi i} \int_{\Gamma} z^j (zI - A)^{-1} \, dz = A^j.
\]

**Proof:** Write

\[
(zI)^j = (zI - A + A)^j = (zI - A)^j + \ldots + A^j.
\]

Consider a term

\[
(zI - A)^k A^j - k(zI - A)^{-1} = A^j - k(zI - A)^{k-1}.
\]

If \( k \geq 1 \) then the function is holomorphic in \( z \) and the corresponding integral is zero. Thus, a nontrivial contribution is obtained for \( k = 0 \) only. One obtains

\[
\int_{\Gamma} z^j (zI - A)^{-1} \, dz = A^j \int_{\Gamma} (zI - A)^{-1} \, dz = 2\pi i A^j.
\]

This proves the lemma. \( \diamond \)

If

\[
p(z) = \sum_{j=0}^{N} a_j z^j
\]

is a polynomial, then one defines

\[
p(A) = \sum_{j=0}^{N} a_j A^j.
\]

Using the previous lemma, it is clear that

\[
\frac{1}{2\pi i} \int_{\Gamma} p(z)(zI - A)^{-1} \, dz = p(A). 
\]

(17.5)
17.4.2 The Case of a Power Series

Next let
\[ \phi(z) = \sum_{j=0}^{\infty} a_j z^j, \ |z| < \rho , \]
denote a convergent power series with radius of convergence \( 0 < \rho \leq \infty \). We let
\[ \phi_N(z) = \sum_{j=0}^{N} a_j z^j, \ |z| < \rho \]
denote the partial sums of \( \phi(z) \).

Lemma 17.6 If \( \sigma(A) \subset D(0, \rho) \) then the sequence of matrices
\[ S_N := \phi_N(A) = \sum_{j=0}^{N} a_j A^j, \quad N = 1, 2, \ldots \]
converges in \( \mathbb{C}^{n\times n} \). The limit is denoted by
\[ \lim_{N \to \infty} S_N = \phi(A) = \sum_{j=0}^{\infty} a_j A^j . \]

Proof: Since the spectral radius of \( A \) is strictly less than \( \rho \), there exists a vector norm \( \| \cdot \| \) on \( \mathbb{C}^n \) so that the corresponding matrix norm of \( A \) satisfies
\[ r := \| A \| < \rho . \]
For \( N > M \geq N(\varepsilon) \) we have
\[ \| S_N - S_M \| = \| \sum_{j=M+1}^{N} a_j A^j \| \]
\[ \leq \sum_{j=M+1}^{N} |a_j| r^j \]
\[ \leq \varepsilon . \]

Thus, \( S_N \) is a Cauchy sequence in \( \mathbb{C}^{n\times n} \).

Let us connect this result with the Dunford–Taylor integral.

Theorem 17.3 We make the same assumptions on \( A \) and \( \phi(z) \) as in the previous lemma. Let \( \Gamma \) be a positively oriented, simple closed curve in \( D(0, \rho) \) with \( \sigma(A) \subset \text{int} \ \Gamma \). Then we have
\[ \frac{1}{2\pi i} \int_{\Gamma} \phi(z)(zI - A)^{-1} \, dz = \phi(A) . \quad (17.6) \]
Here $\phi(A)$ is defined as the limit of the matrix sequence $\phi_N(A)$ considered in Lemma 17.6.

**Proof:** By (17.5) we have

$$\frac{1}{2\pi i} \int_{\Gamma} \phi_N(z)(zI - A)^{-1} \, dz = \phi_N(A)$$

for every finite $N = 1, 2, \ldots$. Taking the limit as $N \to \infty$, we obtain (17.6). \(\diamondsuit\)

**Example:** Let $t \in \mathbb{R}$ be fixed and let $\phi(z) = e^{t z}$. If $A \in \mathbb{C}^{n \times n}$ is any matrix and if $\Gamma$ is a positively oriented, simple closed curve surrounding $\sigma(A)$, then

$$e^{tA} = \frac{1}{2\pi i} \int_{\Gamma} e^{zt}(zI - A)^{-1} \, dz .$$

Here, by definition, $e^{tA} = \sum_{j=0}^{\infty} \frac{1}{j!} (tA)^j$.

### 17.4.3 A General Holomorphic Function

The formula

$$\frac{1}{2\pi i} \int_{\Gamma} \phi(z)(zI - A)^{-1} \, dz =: \phi(A) \quad (17.7)$$

can be used to define $\phi(A)$ under more general assumptions than those of Theorem 17.3, where $\phi(z)$ was assumed to be a power series. All one needs is $\phi \in H(U)$ and a positively oriented, simple closed curve $\Gamma$ in $U$ with

$$\sigma(A) \subset \text{int} \, \Gamma \subset U .$$

**Example:** Let $A$ be any nonsingular matrix. Choose a simply connected region $U$ with

$$0 \notin U, \quad \sigma(A) \subset U .$$

We know that there is a logarithm function $\log_U \in H(U)$ with

$$\exp(\log_U(z)) = z \quad \text{for all} \quad z \in U .$$

If $\Gamma$ is a positively oriented, simple closed curve in $U$ with $\sigma(A) \subset \text{int} \, \Gamma$, then

$$\log_U(A) := \frac{1}{2\pi i} \int_{\Gamma} \log_U(z)(zI - A)^{-1} \, dz$$

is a well-defined matrix. One can prove that

$$\exp(\log_U(A)) = A .$$
17.4.4 Remarks on Unbounded Operators

An important point of the formula
\[ \frac{1}{2\pi i} \int_{\Gamma} \phi(z)(zI - A)^{-1} \, dz =: \phi(A) \quad (17.8) \]
is that one can use it for linear operators \( A \) more general than matrices, even for unbounded operators \( A \) that are densely defined in some Banach space. Such operators \( A \) appear when one formulates initial value problems for PDEs abstractly as
\[ u_t = Au, \quad u(0) = u^{(0)}, \quad t \geq 0. \]
The formal solution is
\[ u(t) = e^{tA}u^{(0)}, \quad t \geq 0, \]
but if \( A \) is unbounded, one cannot use the exponential series to define \( e^{tA} \). Instead, one considers the resolvent
\[ (A - zI)^{-1}, \quad z \in \mathbb{C} \setminus \sigma(A), \]
and (under suitable assumptions) defines
\[ e^{tA} := \frac{1}{2\pi i} \int_{\Gamma} e^{tz}(zI - A)^{-1} \, dz. \quad (17.9) \]

Typically, the spectrum \( \sigma(A) \) is unbounded and \( \Gamma \) cannot surround \( \sigma(A) \). Instead, \( \Gamma \) is chosen as an infinite line,
\[ \Gamma : z(\xi) = b + i\xi, \quad -\infty < \xi < \infty, \]
which must lie to the right of \( \sigma(A) \). Since \( dz = i\,d\xi \) one obtains
\[ e^{tA} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(b+i\xi)t}((b+i\xi)I - A)^{-1} \, d\xi. \quad (17.10) \]
(The Laplace transform of the scalar function \( e^{ta} \) is
\[ \mathcal{L}(e^{ta})(s) = \int_{0}^{\infty} e^{-(s-a)t} \, dt = \frac{1}{s-a}. \]
The formulas (17.9) and (17.10) are versions of the inverse Laplace transform of an exponential.)

Details of these ideas lead to so-called semi-group theory, a part of functional analysis. The name semi-group arises since the family of operators
\[ e^{tA}, \quad t \geq 0, \]
satisfies
\[ e^{sA}e^{tA} = e^{(s+t)A} \quad \text{for all} \quad s, t \geq 0, \quad e^{0A} = I. \]
In other words, one can multiply the operators $e^{tA}$, obtaining the rules of associativity and commutativity. However, in general, the operator $e^{tA}$ does not have an inverse since $e^{-tA}$ does not exist for $t > 0$. Thus, the family of operators $e^{tA}, t \geq 0$, does not have the structure of a group.
18 The Maximum Modulus Principle for Holomorphic Functions

**Definition:** Let $U \subset \mathbb{C}$ be an open set and let $\phi : U \rightarrow \mathbb{R}$ be a real–valued function. Then $z_0 \in U$ is called a local maximum of $\phi$ if there is $r > 0$ with $D(z_0, r) \subset U$ and

$$\phi(z) \leq \phi(z_0) \quad \text{for all} \quad z \in D(z_0, r).$$

We will apply this concept to functions $\phi(z) = |f(z)|$ where $f \in H(U)$.

**Theorem 18.1** (local maximum modulus principle for holomorphic functions)

Let $U$ be a region and let $f \in H(U)$. Assume that $f$ is not constant. Then the function $|f(z)|$ does not attain any local maximum in $U$.

**Proof:** 1) Suppose that $|f(z)|$ attains a local maximum at $z_0$, i.e.,

$$|f(z)| \leq |f(z_0)| \quad \text{for} \quad |z - z_0| \leq \varepsilon,$$

where $\varepsilon > 0$. The set $W := f(D(z_0, \varepsilon))$ is open and $w = f(z_0) \in W$. Therefore, there exists $\eta > 0$ so that $D(w, 2\eta) \subset W$. Let

$$w = re^{i\theta}, \quad w_1 = (r + \eta)e^{i\theta}.$$

Then we have

$$|w_1| > |w| = |f(z_0)|$$

and $w_1 \in W$, a contradiction.

2) We give a second proof, not using the open mapping theorem. Suppose that $|f(z)|$ attains a local maximum at $z_0$. We can write

$$f(z) = f(z_0) + \sum_{j=M}^{\infty} a_j(z - z_0)^j \quad \text{for} \quad |z - z_0| < 2\varepsilon,$$

with $M \geq 1$ and $a_M \neq 0$.

Obtain that

$$f(z) = f(z_0) + (z - z_0)^M(a_M + h(z))$$

with $h(z_0) = 0$, thus

$$|h(z)| \leq \frac{1}{2}|a_M| \quad \text{for} \quad |z - z_0| \leq \varepsilon.$$

Set $a_0 = f(z_0)$. We may assume $a_0 \neq 0$.

Let $a_M/a_0 = \rho e^{i\theta}$ and let

$$z - z_0 = \varepsilon e^{i\phi}.$$

Then we have
\[ f(z) = a_0 \left( 1 + (z - z_0)^M \left( \frac{a_M}{a_0} + \frac{h(z)}{a_0} \right) \right) \]

with

\[ (z - z_0)^M \frac{a_M}{a_0} = \varepsilon^M \rho e^{iM\phi + i\theta} . \]

Choosing

\[ \phi = -\frac{\theta}{M} \]

one obtains that

\[ (z - z_0)^M \frac{a_M}{a_0} = \varepsilon^M \rho > 0 . \]

Also,

\[ \left| (z - z_0)^M \frac{h(z)}{a_0} \right| \leq \varepsilon^M \frac{1}{2} \rho . \]

This yields that

\[ |f(z)| \geq |a_0|(1 + \varepsilon^M \rho) - |a_0|\frac{1}{2} \varepsilon^M \rho \]
\[ = |a_0|(1 + \frac{1}{2} \varepsilon^M \rho) \]
\[ > |a_0| \]
\[ = |f(z_0)| \]

This contradiction proves the theorem. \diamond

Another form of the maximum modulus theorem is the following.

**Theorem 18.2** Let \( U \) be a bounded region. Let \( f \in H(U) \cap C(\overline{U}) \) and set

\[ M_0 := |f|_{\partial U} = \max_{z \in \partial U} |f(z)| . \]

Then

\[ |f(z)| < |f|_{\partial U} \quad \text{for all} \quad z \in U \]

unless \( f \) is constant.

**Proof:** Let

\[ M_1 = \max_{z \in \overline{U}} |f(z)| = |f(z_1)| . \]

First assume that \( M_1 > M_0 \). In this case, \( |f(z)| \) attains a local maximum at a point \( z_1 \in U \). By the previous theorem, \( f \) is constant, a contradiction.
We therefore have that $M_1 = M_0$. Again, if there is $z_1 \in U$ with $|f(z_1)| = M_1$, then $f$ is constant, a contradiction. It follows that $|f(z)| < M_0$ for all $z \in U$. \hfill \diamond

For some applications (in particular to the Paley–Wiener theorems of Fourier analysis) it is important to extend the maximum modulus theorem to unbounded domains. A straightforward generalization is wrong, however.

**Example:** Let

$$U = \{ z = re^{i\theta} : r > 0, \ |\theta| < \frac{\pi}{4}\}$$

and consider

$$f(z) = e^{(z^2)}, \quad z \in \mathbb{C}.$$  

Clearly, $f \in H(U) \cap C(\bar{U})$. If $z \in \partial U$ then

$$z = x(1 + i) \quad \text{or} \quad z = x(1 - i), \quad x \geq 0.$$  

Therefore,

$$z^2 = \pm 2ix^2,$$

thus

$$|f(z)| = 1 \quad \text{for all} \quad z \in \partial U.$$  

However, $f(x) = e^{x^2}$ is unbounded for $x > 0$. Thus, the values of $|f(z)|$ for $z \in U$ are not bounded by the boundary values of $|f(z)|$.

The following is an example of a Phragmén–Lindelöf theorem.

**Theorem 18.3** Let $U$ denote the unbounded region of the above example and let $f \in H(U) \cap C(\bar{U})$. Assume that $|f(z)| \leq 1$ for all $z \in \partial U$ and assume that

$$|f(z)| \leq Ce^{c|z|} \quad \text{for all} \quad z \in U,$$  

where $C$ and $c$ are positive constants. Then the bound

$$|f(z)| \leq 1$$

holds for all $z \in U$.

**Proof:** For $z \in \bar{U}$ we can write

$$z = re^{i\theta} \quad \text{with} \quad r \geq 0 \quad \text{and} \quad |\theta| \leq \frac{\pi}{4};$$

we define

$$z^{3/2} = r^{3/2} e^{i3\theta/2}.$$  

Let $\varepsilon > 0$. With the above definition of $z^{3/2}$ we set

$$f_\varepsilon(z) = f(z)e^{-\varepsilon z^{3/2}}.$$  

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Figure 18.1: Region U in Theorem 17.3

Note that $|\theta| \leq \frac{\pi}{4}$ yields

$$3|\theta|/2 \leq 3\pi/8 ,$$

and, therefore,

$$|z^{3/2}| = r^{3/2} \cos(3\theta/2) \geq c_1 r^{3/2}, \quad c_1 > 0 .$$

This implies that

$$|f_\varepsilon(z)| \leq C e^{cr} e^{-\varepsilon c_1 r^{3/2}} .$$

The bound tends to zero as $r \to \infty$.

We may assume that $f$ is not identically zero and let

$$M_\varepsilon = \sup_{z \in U} |f_\varepsilon(z)| > 0 .$$

There is $z_0 = z_0(\varepsilon) \in \bar{U}$ with

$$M_\varepsilon = |f_\varepsilon(z_0)| .$$

Note that for $z \in \partial U$:

$$z = re^{\pm i\pi/4} ,$$

thus

$$z^{3/2} = r^{3/2} \left( \cos(3\pi/8) \pm i \sin(3\pi/8) \right) .$$

It follows that

$$|e^{-\varepsilon z^{3/2}}| \leq 1 ,$$

thus

$$|f_\varepsilon(z)| \leq 1 \quad \text{for} \quad z \in \partial U .$$
Suppose that $M_\varepsilon > 1$. Then the function $f_\varepsilon$ attains its maximum at an interior point, at $z_0 \in U$, say, and we obtain a contradiction to the open mapping theorem. We conclude that $M_\varepsilon \leq 1$, which yields that

$$|f(z)| \leq e^{\varepsilon z^3/2}, \quad z \in \bar{U}.$$  

Since $\varepsilon > 0$ was arbitrary, we have shown that $|f(z)| \leq 1$. ◆
19 Harmonic Functions

19.1 Basic Concepts

Let $U \subset \mathbb{R}^n$ be an open set. A function $u \in C^2(U)$ is called harmonic in $U$ if $\Delta u = 0$ in $U$. Here

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

is the Laplace operator.

**Applications:** Stationary states of the heat equation $u_t = \Delta u$ are given by harmonic functions. If $\rho$ is charge density and $u$ is the potential of the electric field generated by $\rho$, then (in suitable units) $-\Delta u = \rho$. This is Poisson's equation. In regions free of charge, the potential $u$ is a harmonic function.

**Theorem 19.1** Let $U \subset \mathbb{C}$ be open and let $f \in H(U)$. Write $f(z) = u(x, y) + iv(x, y)$ for $z = x + iy$. Then $\Delta u = \Delta v = 0$.

**Proof:** This follows directly from the Cauchy–Riemann equations and $v_{xy} = v_{yx}$, $u_{xy} = u_{yx}$. ♦

In the following, let $U$ be a region in $\mathbb{C}$. If $f = u + iv$ is holomorphic in $U$ then one calls $v$ a harmonic conjugate of $u$ in $U$. Harmonic conjugates, if they exist, are unique up to a constant. To see this, assume that $f_1 = u + iv_1$ and $f_2 = u + iv_2$ are holomorphic in $U$. Then $f_1 - f_2 = i(v_1 - v_2)$ is also holomorphic. By the open mapping theorem, $f_1 - f_2$ is constant in the region $U$.

If $U$ is simply connected and $\Delta u = 0$ in $U$, then $u$ has a harmonic conjugate $v$ in $U$; see Section 19.2. In Section 19.3 we consider the harmonic function $u(x, y) = \ln((x^2 + y^2)^{1/2})$ to show that a harmonic conjugate does not always exist if the domain $U$ is not simply connected.

An elementary observation is the following: Let $v$ be a harmonic conjugate of $u$ in $U$, i.e., $f = u + iv \in H(U)$. The Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

imply that

$$(u_x, u_y) \cdot (v_x, v_y) = u_xv_x + u_yv_y = 0.$$

In other words, $\nabla u$ is orthogonal to $\nabla v$ at every point of $U$. Therefore, the family of lines

$$u(x, y) = c_1$$

is orthogonal to the family of lines

$$v(x, y) = c_2.$$

In other words, every function $f = u + iv \in H(U)$ yields two families of mutually orthogonal coordinate lines in $U$. 173
Example: Let \( f(z) = z^2 \), thus
\[
f(x + iy) = (x + iy)^2 = x^2 - y^2 + 2ixy .
\]
The equations
\[
x^2 - y^2 = c_1
\]
and
\[
2xy = c_2
\]
determine two families of hyperbolas. Each hyperbola of the family
\[
y = \pm \sqrt{x^2 - c_1}
\]
is orthogonal to each hyperbola
\[
y = \frac{c_2}{2x}
\]
of the other family.

19.2 The Harmonic Conjugate in a Simply Connected Region

We begin with a simple lemma, showing uniqueness of harmonic conjugates up to a constant. The argument is elementary and does not use the open mapping theorem.

Lemma 19.1 Let \( U \subset \mathbb{C} \) be a region and let \( u \in C^2(U, \mathbb{R}) \) be harmonic. If \( v \) and \( w \) are harmonic conjugates of \( u \) in \( U \), then \( v(x,y) = w(x,y) + c \) in \( U \) for some constant \( c \).

Proof: Let \( b = v - w \). We have
\[
v_x = -u_y \quad \text{and} \quad w_x = -u_y ,
\]
thus \( b_x = 0 \). Similarly, \( b_y = 0 \) in \( U \). Let \( P, Q \in U \) be arbitrary points and let \( \Gamma \) be a curve in \( U \) from \( P \) to \( Q \). Let \( \gamma(t), 0 \leq t \leq 1 \), parametrize \( \Gamma \) and define the auxiliary function \( h(t) = b(\gamma(t)) \). We have
\[
b(Q) - b(P) = h(1) - h(0) = \int_0^1 h'(t) \, dt .
\]
Here, by the change rule,
\[
h'(t) = b_x(\gamma(t))\gamma'_1(t) + b_y(\gamma(t))\gamma'_2(t) \equiv 0 .
\]
Therefore, \( h(Q) = h(P) \). Fixing \( P \) and letting \( Q \in U \) vary, we find that \( b \) is constant. \( \diamond \)

Existence of a harmonic conjugate is assured if the region \( U \) is simply connected.
Theorem 19.2 Let $U \subset \mathbb{C}$ be a simply connected region and let $u \in C^2(U, \mathbb{R})$ be harmonic. Then there exists a function $v \in C^2(U, \mathbb{R})$ so that $f = u + iv$ is holomorphic in $U$.

Proof: 1. (real analysis proof of the existence of $v$) We must show existence of a function $v \in C^2$ satisfying the Cauchy–Riemann equations:

$$v_x = -u_y, \quad v_y = u_x .$$

In terms of real analysis, we try to find a potential $v$ of the vector field $F = (-u_y, u_x)$, because the Cauchy–Riemann equations require that

$$\nabla v = (-u_y, u_x) .$$

The Jacobian of $F$ is

$$J_F = \begin{pmatrix} -u_{yx} & -u_{yy} \\ u_{xx} & u_{xy} \end{pmatrix} .$$

The assumption $u_{xx} + u_{yy} = 0$ implies that the Jacobian $J_F$ is symmetric. Then, by a theorem of real analysis (see Theorem 19.3), the vector field $F = (-u_y, u_x)$ has a potential in $U$. Any potential $v$ of the vector field $F = (-u_y, u_x)$ is a harmonic conjugate of $u$.

2. (complex variables proof of the existence of $v$) Suppose first that $v$ is a harmonic conjugate of $u$ and set $f = u + iv$. Then we have

$$f' = u_x + iv_x = u_x - iu_y .$$

In other words, $f'$ can be determined in terms of $u$. This motivates to define

$$g = u_x - iu_y .$$

Let us prove that $g \in H(U)$: The Jacobian of $g$ is

$$J_g = \begin{pmatrix} u_{xx} & u_{xy} \\ -u_{yx} & -u_{yy} \end{pmatrix} .$$

We see that the Cauchy–Riemann equations are fulfilled for the real and imaginary parts of $g$ since

$$(\text{Re} \ g)_x = u_{xx}, \quad (\text{Im} \ g)_y = -u_{yy}$$

etc. Consequently, $g \in H(U)$. By a theorem proved earlier, there is $f \in H(U)$ with $f' = g$. Let $f = a(x, y) + ib(x, y)$. Then we have

$$f' = a_x + ib_x = a_x - ia_y$$

and the equation $f' = g = u_x - iu_y$ yields that

$$a_x = u_x, \quad a_y = u_y .$$

By the previous lemma, this implies $a(x, y) = u(x, y) + c$ where $c$ is a real constant. Since $b$ is a harmonic conjugate of $a = u + c$, the function $b$ is also
a harmonic conjugate of $u$. Just note that $u + ib = a - c + ib = f - c$ is holomorphic. Thus we have shown that $u$ has a harmonic conjugate in $U$. \hfill \Box

In real analysis, one shows the following:

**Theorem 19.3** Let $U \subset \mathbb{R}^n$ be open and simply connected. Let $F : U \to \mathbb{R}^n$ be a $C^1$ vector field and assume that the Jacobian

$$J_F(x) = \left( \frac{\partial F_j(x)}{\partial x_i} \right)_{1 \leq i,j \leq n}$$

is a symmetric matrix for all $x \in U$. Then $F$ has a potential in $U$, i.e., there is a scalar $C^1$ function $v : U \to \mathbb{R}$ with

$$\nabla v(x) = F(x), \quad x \in U.$$  

### 19.3 A Harmonic Function in $\mathbb{C} \setminus \{0\}$ Without Harmonic Conjugate

In this section, let

$$U = \mathbb{C} \setminus \{0\}, \quad U_1 = \mathbb{C} \setminus (-\infty, 0].$$

Both sets are open and connected. The set $U_1$ is simply connected, but $U$ is not simply connected.

We will show:

**Lemma 19.2** The function

$$u(x, y) = \ln \left( (x^2 + y^2)^{1/2} \right), \quad (x, y) \neq (0, 0),$$

is harmonic in $U$ but does not have a harmonic conjugate in $U$.

**Proof:** 1. Recall the main branch of the complex logarithm defined in $U_1$: If $z \in U_1$ then

$$z = re^{i\theta}, \quad r > 0, \quad -\pi < \theta < \pi,$$

and

$$f(z) := \log z = \ln r + i\theta.$$

If one writes

$$f(x + iy) = u(x, y) + iv(x, y), \quad x + iy \in U_1,$$

then

$$u(x, y) = \ln r = \ln \left( (x^2 + y^2)^{1/2} \right)$$

and
Here one must choose the correct branch of the arctan–function and the correct limiting values for \( x = 0 \).

Since \( f \in H(U_1) \) we have

\[
\Delta u = \Delta v = 0 \quad \text{in} \quad U_1.
\]

The function \( u \) is \( C^\infty \) in \( U \), and one obtains that

\[
\Delta u = 0 \quad \text{in} \quad U.
\]

Of course, this can also be verified calculus.

2. Suppose that \( u(x, y) \) has a complex conjugate \( w(x, y) \) in \( U \). Then the function

\[
g(x + iy) = u(x, y) + iw(x, y), \quad x + iy \in U,
\]

is holomorphic in \( U \). We have, for \( z \in U_1 \),

\[
f(z) - g(z) = i(v(x, y) - w(x, y)).
\]

By the open mapping theorem, one finds that \( f(z) - g(z) = \text{const} \) in \( U_1 \).

Therefore,

\[
f'(z) - g'(z) = 0 \quad \text{in} \quad U_1.
\]

Therefore,

\[
g'(z) = \frac{1}{z}, \quad z \in U_1.
\]

By assumption, \( g \in H(U) \), thus \( g' \in H(U) \). Also, the function \( \frac{1}{z} \) is in \( H(U) \).

By the identity theorem, we find that

\[
g'(z) = \frac{1}{z}, \quad z \in U.
\]

This would mean that the function \( \frac{1}{z} \) has an antiderivative in \( U \), namely \( g(z) \). Then, if \( \Gamma \) is any closed curve in \( U \), we would obtain that

\[
\int_{\Gamma} \frac{dz}{z} = 0.
\]

Since this is not true, we conclude that \( u(x, y) \) cannot have a harmonic conjugate in \( U \). \( \Diamond \)

**Real Analysis Arguments.** We want to show the above lemma using arguments of real analysis. In the following, let

\[
\arctan : \mathbb{R} \to (-\pi/2, \pi/2)
\]

denote the main branch of the inverse tangent. Set

\[
V = \mathbb{C} \setminus \{iy : y \in \mathbb{R}\}.
\]
Lemma 19.3 Define the functions
\[ u(x, y) = \ln\left(\left(x^2 + y^2\right)^{1/2}\right), \quad (x, y) \neq (0, 0), \]
and
\[ v(x, y) = \arctan(y/x), \quad x \neq 0. \]
We have \( \Delta u = 0 \) in \( U \), \( \Delta v = 0 \) in \( V \) and
\[ u_x = v_y, \quad u_y = -v_x \quad \text{in} \quad V. \]
Thus, \( v \) is a harmonic conjugate of \( u \) in \( V \).

Proof: Using calculus:
\[
\begin{align*}
  u_x &= x(x^2 + y^2)^{-1} \\
  u_{xx} &= (x^2 + y^2)^{-1} - 2x^2(x^2 + y^2)^{-2} \\
  u_y &= y(x^2 + y^2)^{-1} \\
  u_{yy} &= (x^2 + y^2)^{-1} - 2y^2(x^2 + y^2)^{-2}
\end{align*}
\]
It follows that \( \Delta u = 0 \).

Also,
\[
\begin{align*}
  v_x &= \frac{1}{1 + y^2/x^2} \cdot (-y x^{-2}) \\
  &= -y(x^2 + y^2)^{-1} \\
  v_{xx} &= 2xy(x^2 + y^2)^{-2} \\
  v_y &= \frac{1}{1 + y^2/x^2} \cdot x^{-1} \\
  &= x(x^2 + y^2)^{-1} \\
  v_{yy} &= -2xy(x^2 + y^2)^{-2}
\end{align*}
\]
It follows that \( \Delta v = 0 \) in \( V \). We also see that
\[ u_x = v_y, \quad u_y = -v_x \quad \text{in} \quad V. \diamondsuit \]
Let us prove that \( u \) does not have a harmonic conjugate in \( U \). Suppose that \( w(x, y) \) is a harmonic conjugate of \( u \) in \( U \). By Lemma 19.1 there are constants, \( c_1 \) and \( c_2 \), with
\[ w(x, y) = \arctan(y/x) + c_1 \quad \text{for} \quad x > 0 \]
and
\[ w(x, y) = \arctan(y/x) + c_2 \quad \text{for} \quad x < 0. \]

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Fix $y = 1$, for example, and consider the limit as $x \to 0$. We obtain, for $x > 0$ and $x \to 0$:

$$w(0, 1) = \frac{\pi}{2} + c_1 .$$

For $x < 0$ and $x \to 0$:

$$w(0, 1) = -\frac{\pi}{2} + c_2 .$$

Now fix $y = -1$, for example, and again consider the limit as $x \to 0$. For $x > 0$ and $x \to 0$:

$$w(0, -1) = -\frac{\pi}{2} + c_1 .$$

For $x < 0$ and $x \to 0$:

$$w(0, -1) = \frac{\pi}{2} + c_2 .$$

Therefore,

$$\frac{\pi}{2} + c_1 = -\frac{\pi}{2} + c_2$$

and

$$-\frac{\pi}{2} + c_1 = \frac{\pi}{2} + c_2 .$$

The first equation requires that

$$c_1 - c_2 = -\pi$$

and the second equation requires that

$$c_1 - c_2 = \pi .$$

This contradiction implies that $u$ cannot have a harmonic conjugate in $U$ though $u$ has a harmonic conjugate in the open left half-plane (namely $v(x, y) = \arctan(y/x), x < 0$) and another harmonic conjugate in the open right half-plane (namely $v(x, y) = \arctan(y/x), x > 0$).

### 19.4 Dirichlet’s Problem and the Poisson Kernel for the Unit Disk

Let $U \subset \mathbb{C}$ be a bounded region with boundary $\partial U$. Let $u_0 \in C(\partial U)$, i.e, $u_0$ is a continuous function on $\partial U$. We assume that $u_0$ is real. The Dirichlet problem for Laplace’s equation is: Determine $u \in \mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$ with

$$\Delta u = 0 \quad \text{in} \quad U, \quad u = u_0 \quad \text{on} \quad \partial U .$$  \hspace{1cm} (19.1)

If $U$ is unbounded, one must specify additional conditions about the behavior of $u(x, y)$ for large $(x, y)$. In this section we consider (19.1) for
\[ U = \mathbb{D} = D(0, 1) , \]

i.e., \( U \) is the unit disk \( \mathbb{D} \). We let \( \gamma(t) = e^{it}, 0 \leq t \leq 2\pi \), and denote the boundary curve of \( \mathbb{D} \) by \( \Gamma \).

Let \( f \in H(D(0, 1 + \varepsilon)) \). By Cauchy’s integral formula:

\[ f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} \, dw, \quad z \in \mathbb{D}. \quad (19.2) \]

For \( 0 < |z| < 1 \) let

\[ z_1 = \frac{1}{z} . \]

(The mapping

\[ z = re^{i\theta} \rightarrow z_1 = \frac{1}{r} e^{i\theta} \]

is a reflection w.r.t. \( \partial\mathbb{D} \), the boundary of the unit disk.) Since \( |z_1| > 1 \) we have

\[ 0 = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z_1} \, dw, \quad 0 < |z| < 1 . \quad (19.3) \]

Note that

\[ \frac{1}{w - z_1} = \frac{\bar{z}}{w \bar{z} - 1} . \]

Therefore,

\[ 0 = \frac{1}{2\pi i} \int_{\Gamma} \frac{\bar{z}f(w)}{\bar{z}w - 1} \, dw, \quad z \in \mathbb{D} . \quad (19.4) \]

From (19.2) and (19.4) obtain:

\[ f(z) = \int_{\Gamma} H(z, w)f(w) \, dw \]

with (for \( |z| < 1, |w| = 1 \)):

\[ H(z, w) = \frac{1}{2\pi i} \left( \frac{1}{w - z} + \frac{\bar{z}}{1 - w\bar{z}} \right) \quad (19.5) \]

\[ = \frac{1}{2\pi i} \frac{1 - |z|^2}{w - z - w^2\bar{z} + |z|^2} \quad (19.6) \]

\[ = \frac{1}{2\pi i w} \frac{1 - |z|^2}{1 - wz - w\bar{z} + |z|^2} \quad (19.7) \]

\[ = \frac{1}{2\pi i w} \frac{1 - |z|^2}{|w - z|^2} \quad (19.8) \]

With

\[ w = e^{it}, \quad dw = iw \, dt \]
one obtains
\[ f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} f(e^{it}) \, dt \]
or, with \( z = re^{i\theta} \):
\[ f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} f(e^{it}) \, dt . \]

One defines the Poisson kernel for the unit disk by
\[ P_r(\alpha) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r \cos \alpha + r^2}, \quad 0 \leq r < 1, \quad \alpha \in \mathbb{R} . \]
Our derivation shows:

**Lemma 19.4** Let \( f \in H(D(0, 1 + \varepsilon)) \) for some \( \varepsilon > 0 \). Then we have
\[ f(re^{i\theta}) = \int_0^{2\pi} P_r(\theta - t)f(e^{it}) \, dt \]
for \( re^{i\theta} \in U \).

**Properties of the Poisson Kernel:** 1. For \( 0 \leq r < 1 \) and all real \( \alpha \) we have
\[
1 - 2r \cos \alpha + r^2 = 1 - 2r + r^2 + 2r(1 - \cos \alpha) \\
\geq (1 - r)^2 \\
> 0
\]
thus
\[
0 < P_r(\alpha) \leq P_r(0) = \frac{1}{2\pi} \frac{1 + r}{1 - r}. \]
In particular, \( P_r(0) \to \infty \) as \( r \to 1 \).

2) Applying the previous lemma with \( f \equiv 1 \) yields
\[ \int_{-\pi}^{\pi} P_r(\alpha) \, d\alpha = 1, \quad 0 \leq r < 1 . \quad (19.9) \]

3) Despite the fact that \( P_r(0) \to \infty \) as \( r \to 1 \), we will show that \( P_r(\alpha) \to 0 \) as \( r \to 1 \) if \( \alpha \) is bounded away from zero. A precise statement is:

**Lemma 19.5** For any \( \delta_1 > 0, \varepsilon_1 > 0 \) there exists \( \eta > 0 \) with
\[ P_r(\alpha) \leq \varepsilon_1 \]
if
\[
0 < \delta_1 \leq |\alpha| \leq \pi \quad \text{and} \quad 1 - \eta \leq r < 1 .
\]
Proof: For $\frac{1}{2} \leq r < 1$ and $\delta_1 \leq |\alpha| \leq \pi$ we have:

\[
1 - 2r \cos \alpha + r^2 = 1 - 2r + r^2 + 2r(1 - \cos \alpha) \\
\geq 1 - \cos \alpha \\
\geq \delta_2 > 0
\]

where $\delta_2 = 1 - \cos \delta_1$, i.e., $\delta_2$ depends only on $\delta_1$. Therefore,

\[
P_r(\alpha) \leq \frac{1 - r^2}{2\pi \delta_2} \leq \varepsilon_1 \quad \text{for} \quad 1 - \eta \leq r < 1
\]

if $\eta > 0$ is small enough. ◇

We now use these properties of $P_r(\alpha)$ to prove the following result about the Poisson kernel.

**Theorem 19.4** Let $D = D(0,1)$ denote the unit disk and let $u_0 \in C(\partial D)$. The function $u(z)$ defined for $z \in \mathbb{D}$ by

\[
u(re^{i\theta}) = \int_0^{2\pi} P_r(\theta - t)u_0(e^{it}) \, dt \quad \text{for} \quad 0 \leq r < 1 \quad (19.10)
\]

\[
u(e^{i\theta}) = u_0(e^{i\theta}) \quad \text{for} \quad r = 1 \quad (19.11)
\]

solves the Dirichlet problem with boundary data $u_0$ on $\partial D$. In particular:

a) $u \in C^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$;

b) $\Delta u = 0$ in $\mathbb{D}$.

To show that $u$ is harmonic in $\mathbb{D}$, we use the following simple result:

**Lemma 19.6** Suppose that $g(z)$ is a holomorphic function in some open set $V$ and let

\[
G(z) = g(\bar{z}) \quad \text{for} \quad z \in V_1 = \{z : \bar{z} \in V\}.
\]

Then the real and imaginary parts of $G$ are harmonic in $V_1$.

Proof: If $g(x + iy) = u(x, y) + iv(x, y)$ then

\[
G(x + iy) = u(x, -y) + iv(x, -y).
\]

◇

To show that the function $u$ defined by (19.10) is harmonic in $\mathbb{D}$, recall that our derivation shows:

\[
u(z) = \int_\gamma H(z, w)u_0(w) \, dw, \quad z \in \mathbb{D}, \quad (19.12)
\]

where $H(z, w)$ is defined in (19.5). By the previous lemma, the real and imaginary parts of $z \to H(z, w)$ are harmonic in $\mathbb{D}$, for each fixed $w \in \gamma$. Since one
can differentiate (19.12) under the integral sign, it follows that \( \Delta u = 0 \) in \( \mathbb{D} \).
(For another argument, using series, see the next section.)

We now show that the function \( u(z) \) defined by (19.10) and (19.11) is continuous in every point \( z_0 = e^{it_0} \).

Because of (19.9) we have

\[
\begin{align*}
  u(re^{i\theta}) - u(e^{it_0}) &= \int_0^{2\pi} P_r(\theta - t) \left( u_0(e^{it}) - u_0(e^{it_0}) \right) dt . \\
  &\quad \text{(19.13)}
\end{align*}
\]

For given \( \varepsilon > 0 \) there is \( \delta > 0 \) with

\[
\begin{align*}
  |u_0(e^{it}) - u_0(e^{it_0})| &\leq \varepsilon \quad \text{for} \quad |t - t_0| \leq \delta . \\
  &\quad \text{(19.14)}
\end{align*}
\]

We split the integral in (19.13) into

\[
\begin{align*}
  \int_0^{2\pi} = \int_{|t-t_0|<\delta} + \int_{|t-t_0|>\delta} =: I_1 + I_2 .
\end{align*}
\]

Using (19.9) and (19.14) we have

\[
I_1 \leq \varepsilon .
\]

To estimate \( I_2 \) we assume that \( |\theta - t_0| < \delta/2 \). Then the assumption \( |t - t_0| > \delta \) yields

\[
|\theta - t| > \frac{\delta}{2} =: \delta_1 .
\]

It follows that

\[
I_2 \leq 2|u_0|_{\infty} \cdot 2\pi \max_{\delta_1 \leq |\alpha| \leq \pi} P_r(\alpha) .
\]

Using Lemma 19.5 we obtain that

\[
I_2 \leq \varepsilon \quad \text{for} \quad 1 - \eta \leq r < 1
\]

if \( \eta > 0 \) is sufficiently small. To summarize, if \( \varepsilon > 0 \) is given, then there is \( \delta > 0 \) and \( \eta > 0 \) with

\[
|u(re^{i\theta}) - u(e^{it_0})| \leq 2\varepsilon
\]

if \( |\theta - t_0| < \delta/2 \) and \( 1 - \eta \leq r < 1 \). Since \( u \) is continuous on \( \partial U \) this shows that \( u \) is continuous in \( z_0 = e^{it_0} \). This completes the proof of Theorem 19.4.

\[\Box\]

Remark: We have derived the Poisson kernel for the unit disk,

\[
P_r(\alpha) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r \cos \alpha + r^2}, \quad 0 \leq r < 1, \quad \alpha \in \mathbb{R} .
\]

For a disk of radius \( R > 0 \) the Poisson kernel is

\[
P_r^{(R)}(\alpha) = \frac{1}{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos \alpha + r^2}, \quad 0 \leq r < R, \quad \alpha \in \mathbb{R} .
\]

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The solution of the Dirichlet problem,

\[ \Delta u = 0 \text{ in } D(0, R), \quad u(Re^{i\theta}) = u_0(Re^{i\theta}) \text{ for } 0 \leq \theta \leq 2\pi, \]

is

\[ u(Re^{i\theta}) = \int_0^{2\pi} P_r^{(R)}(\theta - t)u_0(Re^{i\theta}) \, d\theta \text{ for } 0 \leq r < R, \]

and

\[ u(Re^{i\theta}) = u_0(Re^{i\theta}) \text{ for } 0 \leq \theta \leq 2\pi. \]

19.5 The Poisson Kernel and Fourier Expansion

We have derived the Poisson kernel for the unit disk from Cauchy’s integral formula. An alternative derivation proceeds via Fourier expansion.

Let \( u_0 : \partial \mathbb{D} \to \mathbb{C} \) denote a continuous function. We want to find a function

\[ u \in C^2(\mathbb{D}) \cap C(\overline{\mathbb{D}}) \]

with

\[ \Delta u = 0 \text{ in } \mathbb{D}, \quad u(z) = u_0(z) \text{ for } |z| = 1. \]

Set

\[ g(t) = u_0(e^{it}), \quad t \in \mathbb{R}. \]

Then \( g(t) \) is a continuous, \( 2\pi \)-periodic function and

\[ g(t) = \sum_{k=-\infty}^{\infty} \hat{g}(k)e^{ikt}, \quad \hat{g}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt}g(t) \, dt, \]

is the Fourier expansion of \( g(t) \). We ignore questions of convergence. We obtain, formally,

\[ u_0(z) = \sum_{k=-\infty}^{\infty} \hat{g}(k)z^k, \quad |z| = 1. \]

Note that, for \( |z| = 1 \) we have \( z^{-1} = \bar{z} \), thus

\[ z^k = \bar{z}^{|k|} \text{ for } |z| = 1, \quad k < 0. \]

Therefore, formally,

\[ u_0(z) = \sum_{k=0}^{\infty} \hat{g}(k)z^k + \sum_{k=-\infty}^{-1} \hat{g}(k)\bar{z}^{|k|}, \quad |z| = 1. \]

This second representation has the advantage that every term
is a harmonic function in $D$. In contrast, the function $z^k$ has a pole at $z = 0$ if $k < 0$.

We claim that the solution of the Dirichlet problem is given by

$$u(z) = \sum_{k=0}^{\infty} \hat{g}(k) z^k + \sum_{k=-\infty}^{-1} \hat{g}(k) \bar{z}^{|k|} \quad \text{for} \quad |z| < 1$$

(19.15)

and

$$u(z) = u_0(z) \quad \text{for} \quad |z| = 1$$

First note that the sequence of Fourier coefficients $\hat{g}(k)$ is bounded. Therefore,

$$u_1(z) = \sum_{k=0}^{\infty} \hat{g}(k) z^k \quad \text{for} \quad |z| < 1$$

and

$$u_2(z) = \sum_{k=-\infty}^{-1} \hat{g}(k) \bar{z}^{|k|} \quad \text{for} \quad |z| < 1$$

are harmonic functions in $D$. (Note that $\bar{u}_2(x)$ is holomorphic in $D$.) Thus, $u \in C^\infty(D)$ and $\Delta u = 0$ in $D$.

It remains to prove that $u \in C(\bar{D})$. To show this, we derive an integral representation of $u(z)$, the Poisson integral formula.

Setting $z = re^{i\theta}$ for $0 \leq r < 1$ we have

$$u(re^{i\theta}) = \frac{1}{2\pi} \sum_{k=0}^{\infty} r^k \int_0^{2\pi} e^{ik(\theta-t)} g(t) dt + \frac{1}{2\pi} \sum_{k=-\infty}^{-1} r^{|k|} \int_0^{2\pi} e^{ik(\theta-t)} g(t) dt$$

$$= \int_0^{2\pi} P_r(\theta - t) g(t) dt$$

with

$$P_r(\alpha) = \frac{1}{2\pi} \sum_{k=0}^{\infty} r^k e^{ik\alpha} + \frac{1}{2\pi} \sum_{k=-\infty}^{-1} r^{|k|} e^{ik\alpha}.$$ 

We have used the integral formula for $\hat{g}(k)$ in (19.15) and have changed the order of summation and integration. This is allowed since the series converge uniformly in $t$ for fixed $r$ with $0 \leq r < 1$.

Set

$$w = re^{i\alpha}.$$ 

Then we have
\[ 2\pi P_r(\alpha) = \sum_{k=0}^{\infty} w^k + \sum_{k=1}^{\infty} \bar{w}^k \]
\[ = \frac{1}{1-w} + \frac{1}{1-w} \]
\[ = \frac{1 - |w|^2}{1 - w - \bar{w} + |w|^2} \]
\[ = \frac{1}{1 - 2r\cos \alpha + r^2} \]

We have obtained the Poisson kernel for the unit disk \( \mathbb{D} \).

**Remarks on Fourier Expansion:** Let \( X \) denote the space of all \( 2\pi \)-periodic continuous functions

\[ g : \mathbb{R} \to \mathbb{C} . \]

(More generally, one could take \( X = L_2(0, 2\pi) \).) On \( X \) one defines the \( L_2 \)-inner product and norm by

\[ (u, v)_{L_2} = \int_0^{2\pi} \bar{u}(t)v(t)dt, \quad \|u\|_{L_2}^2 = (u, u)_{L_2} . \]

The functions in the sequence

\[ e^{ikt}, \quad k \in \mathbb{Z} , \]

are \( L_2 \)-orthogonal to each other and

\[ (e^{ikt}, e^{ikt})_{L_2} = 2\pi \delta_{jk} . \]

If \( g \in X \) then its Fourier series is

\[ \sum_{k=-\infty}^{\infty} \hat{g}(k)e^{ikt} \]

where

\[ \hat{g}(k) = \frac{1}{2\pi} (e^{ikt}, u(t))_{L_2}, \quad k \in \mathbb{Z} , \]

is the \( k \)-th Fourier coefficient of \( g \). Let

\[ S_n(t) = \sum_{k=-n}^{n} \hat{g}(k)e^{ikt} \]

denote the \( n \)-th partial sum of the Fourier series of \( g \). Then it is known that

\[ \|g - S_n\|_{L_2} \to 0 \quad \text{as} \quad n \to \infty , \]

i.e., the Fourier series of \( g \) represents \( g \) in the \( L_2 \)-sense. Pointwise convergence and convergence in maximum norm hold if \( g \in C^1 \), for example.
19.6 The Mean Value Property of Harmonic Functions

Let $U$ be an open set and let $f \in H(U)$. If $\bar{D}(P, r) \subset U$ and $\gamma(t) = P + re^{it}$, then, by Cauchy’s integral formula:

$$f(P) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-P} \, dz = \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{it}) \, dt.$$ 

This says that $f(P)$ is the mean value of the values of $f$ along the circle $\partial D(P, r)$.

Let $u : U \to \mathbb{R}$ be harmonic in $U$ and let $\bar{D}(P, r) \subset U$, as above. In $D(P, r + \varepsilon)$ there is a harmonic conjugate $v$ of $u$. Applying the above equation to $f = u + iv$ and taking real parts, one obtains:

$$u(P) = \frac{1}{2\pi} \int_0^{2\pi} u(P + re^{it}) \, dt .$$

In other words, harmonic functions have the following mean value property: If $\bar{D}(P, r)$ lies in the region $U$ where $u$ is harmonic, then $u(P)$ equals the mean value of $u$ on the circle $\partial D((P, r)$.

**Example:** Let $f(z) = e^z$ and take $P = 0, r = 1$. Cauchy’s integral formula says that

$$1 = e^0 = \frac{1}{2\pi i} \int_{\gamma} \frac{e^z}{z} \, dz .$$

This also follows from the residue theorem, of course.

Using

$$z(t) = e^{it} = \cos t + i \sin t, \quad dz = iz \, dt ,$$

one obtains the mean value formula

$$2\pi = \int_0^{2\pi} e^{\cos t + i \sin t} \, dt = \int_0^{2\pi} e^{\cos t} (\cos(\sin t) + i \sin(\sin t)) \, dt .$$

This yields

$$I_1 = \int_0^{2\pi} e^{\cos t} \cos(\sin t) \, dt = 2\pi$$

and

$$I_2 = \int_0^{2\pi} e^{\cos t} \sin(\sin t) \, dt = 0 .$$

Let $h(t) = e^{\cos t} \sin(\sin t)$. Then $h(-t) = -h(t)$ and $h(t)$ has period $2\pi$. Therefore, $I_2 = \int_{-\pi}^0 h(t) \, dt = 0$ is directly obvious. The formula for $I_1$ does not seem to be obvious.
19.7 The Maximum Principle for Harmonic Functions

Let $U$ be a bounded region and let $u \in C^2(U) \cap C(\bar{U})$ be a real valued function. Assume that $\Delta u = 0$ in $U$ and that $u$ is not constant. Let

$$M_1 = \max\{u(z) : z \in \bar{U}\}.$$ 

We claim that

$$u(P) < M_1$$

for all $P \in U$. Suppose the strict inequality $u(P) < M_1$ does not hold for some $P \in U$. Then we have $u(P) = M_1$, and $P$ is a local maximum of $U$. Using the mean value property, one finds that for some $r > 0$:

$$u(z) = M_1 \text{ for } |z - P| \leq r.$$ 

Set

$$Z = \{z \in U : u(z) = M_1\}.$$ 

The above argument shows that $Z$ is open. Also, by continuity, $Z$ is closed in $U$. Since $U$ is assumed to be connected, one obtains that $Z = U$. Thus, $u$ is constant.

We can apply the same reasoning to $-u$ and obtain:

**Theorem 19.5** Let $U$ be a bounded region and let $u \in C^2(U) \cap C(\bar{U})$ be harmonic in $U$. Assume that $u$ is not constant. Then, for every $P \in U$:

$$\min_{z \in \partial U} u(z) < u(P) < \max_{z \in \partial U} u(z).$$

A simple implication is the following: If $U$ is a bounded region, then the solution of the Dirichlet problem

$$\Delta u = f \text{ in } U, \quad u = u_0 \text{ on } \partial U,$$

is unique (if the solution exists). (If $u_1$ and $u_2$ are two solutions, then $u = u_1 - u_2$ is harmonic in $U$ and has zero boundary values. By Theorem 19.5 it follows that $u \equiv 0$.)

19.8 The Dirichlet Problem in More General Regions

Let us first summarize our results for the Dirichlet problem in the unit disk, $\mathbb{D} = D(0, 1)$.

**Theorem 19.6** Let $u_0 : \partial \mathbb{D} \to \mathbb{R}$ be a continuous function. Then there is a unique function

$$u \in C^2(\mathbb{D}) \cap C(\bar{\mathbb{D}})$$

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with
\[
\Delta u = 0 \quad \text{in} \quad \mathbb{D}, \quad u = u_0 \quad \text{on} \quad \partial\mathbb{D}.
\]
For \( z = re^{i\theta} \in \mathbb{D} \) the solution \( u \) is given by
\[
u(\alpha) = \text{the Poisson kernel for } \mathbb{D}.
\]
\[
 u(re^{i\theta}) = \int_0^{2\pi} P_r(\theta - t)u_0(e^{it}) \, dt
\]
where \( P_r(\alpha) \) is the Poisson kernel for \( \mathbb{D} \).

Let \( V \subset \mathbb{C} \) be any bounded region and assume that there are holomorphic mappings
\[
f : \mathbb{D} \to V, \quad g : V \to \mathbb{D}
\]
which are 1 – 1, onto, and inverse to each other. We also assume that \( f \) and \( g \) can be continuously extended as bijective mappings to the closures of \( \mathbb{D} \) and \( V \), respectively. We denote the extensions again by \( f \) and \( g \). Thus we assume that
\[
f : \overline{\mathbb{D}} \to \overline{V}, \quad g : \overline{V} \to \overline{\mathbb{D}},
\]
are continuous, 1 – 1, onto and
\[
f(g(z)) = z \quad \text{for all} \quad z \in \overline{V}, \quad g(f(w)) = w \quad \text{for all} \quad w \in \overline{\mathbb{D}}.
\]
This implies that boundaries are mapped to boundaries:
\[
f(\partial\mathbb{D}) = \partial V, \quad g(\partial V) = \partial \mathbb{D}.
\]
We will discuss the existence and construction of such mappings \( f \) and \( g \) later in connection with the Riemann Mapping Theorem.

Now let \( v_0 : \partial V \to \mathbb{R} \) be a given continuous function and consider the Dirichlet problem: Find
\[
v \in C^2(V) \cap C(\overline{V})
\]
with
\[
\Delta v = 0 \quad \text{in} \quad V, \quad v = v_0 \quad \text{on} \quad \partial V.
\]
We can transform this problem to the Dirichlet problem on \( \mathbb{D} \) in the following way: Set
\[
u_0(w) = v_0(f(w)), \quad w \in \partial\mathbb{D}.
\]
This transforms the given boundary function \( v_0 \), defined on \( \partial V \), to a boundary function \( u_0 \) defined on \( \partial\mathbb{D} \).

Then let \( u \in C^2(\mathbb{D}) \cap C(\overline{D}) \) solve the Dirichlet problem in \( \mathbb{D} \) with boundary data \( u_0 \). We claim that
$v(z) = u(g(z))$, \( z \in \bar{V} \),

solves the Dirichlet problem in \( V \). Clearly, if \( z \in \partial V \), then \( g(z) \in \partial \mathbb{D} \) and

\[
\begin{align*}
v(z) &= u_0(g(z)) \\
      &= v_0(f(g(z))) \\
      &= v_0(z),
\end{align*}
\]

showing that \( v \) satisfies the boundary conditions. It remains to prove that \( v \) is harmonic in \( V \). This follows from the following result.

**Theorem 19.7** Let \( U, V \) be regions and let \( g : V \to U \) be holomorphic. Let \( u_1 : U \to \mathbb{R} \) be harmonic in \( U \). Then \( v_1(z) = u_1(g(z)) \) is harmonic in \( V \).

**Proof:** Fix \( z_0 \in V \). We must show that \( \Delta v_1(z_0) = 0 \). We have \( g(z_0) \in U \) and there is \( r > 0 \) with

\[
D = D(g(z_0), r) \subset U.
\]

Since \( u_1 \) is harmonic in \( D \) it has a harmonic conjugate \( u_2 \) in \( D \). Then the function \( u = u_1 + iu_2 \) is holomorphic in \( D \). It follows that the function \( v(z) = u(g(z)) \) is holomorphic in a neighborhood of \( z_0 \). Since \( v_1 \) is the real part of \( v \), we conclude that \( v_1 \) is harmonic in a neighborhood of \( z_0 \). \( \diamond \)
20 Extensions of Cauchy’s Theorem in a Disk

20.1 Homotopic Curves

In the following, let \( U \subset \mathbb{C} \) be a region, i.e., \( U \) is open and connected. Let \( \gamma_0(t), \gamma_1(t), a \leq t \leq b \), denote two curves in \( U \) with

\[
\gamma_0(a) = \gamma_1(a) = P, \quad \gamma_0(b) = \gamma_1(b) = Q.
\]

Thus, \( \gamma_0 \) and \( \gamma_1 \) have the same starting point, \( P \), and the same endpoint, \( Q \).

**Definition:** The curve \( \gamma_0 \) is homotopic to the curve \( \gamma_1 \) in \( U \) (with fixed end points), if there is a continuous function

\[
\gamma : [0, 1] \times [a, b] \to U
\]

with the following properties:

1) For \( a \leq t \leq b \):

\[
\gamma(0, t) = \gamma_0(t), \quad \gamma(1, t) = \gamma_1(t).
\]

2) For \( 0 \leq s \leq 1 \):

\[
\gamma(s, a) = P, \quad \gamma(s, b) = Q.
\]

3) For every parameter \( s \in [0, 1] \) the function

\[
t \to \gamma(s, t), \quad a \leq t \leq b,
\]

is piecewise \( C^1 \).

**Terminology:** The function \( \gamma(s, t) \) is called a homotopy (with fixed end points). The parameter \( s \) is called the homotopy parameter and \( t \) is called the curve parameter. Intuitively, \( \gamma \) describes a continuous deformation of the curve \( \gamma_0 \) into \( \gamma_1 \).

We will only consider homotopies with fixed end points. Therefore we will drop the term.

20.2 Cauchy’s Theorem

**Theorem 20.1** Let \( U \) be a region in \( \mathbb{C} \) and let \( f \in H(U) \). If \( \gamma_0 \) and \( \gamma_1 \) are two curves in \( U \) which are homotopic in \( U \) then

\[
\int_{\gamma_0} f(z) \, dz = \int_{\gamma_1} f(z) \, dz.
\]

**Proof:** a) The set

\[
K = \gamma([0, 1] \times [a, b])
\]

is a compact subset of \( U \). Assume that \( U^c = \mathbb{C} \setminus U \) is not empty. (Otherwise, the following will be trivial.) Let

\[
\varepsilon := \text{dist}(K, U^c) = \inf \{|k - z| : k \in K, z \in U\}
\]
denote the distance between $K$ and $U^c$. Since $K$ is compact and $U^c$ is closed and $K \cap U^c = \emptyset$, it follows that $\varepsilon > 0$. (Proof of this statement: If $\varepsilon = 0$ then, for every $n \in \mathbb{N}$ there is $k_n \in K$ and $z_n \in U$ with

$$|k_n - z_n| < \frac{1}{n}.$$  

For a subsequence, $k_n \to k$ and $z_n \to z$. Since $k \in K$ and $z \in U^c$ and $|k - z| = 0$, one obtains a contradiction to $K \cap U^c = \emptyset$.)

It follows that

$$D(\gamma(s, t), \varepsilon) \subset U$$

for all $(s, t) \in [0, 1] \times [a, b]$.

b) Since $\gamma$ is uniformly continuous, there is $\delta > 0$ with

$$|s - s'| + |t - t'| < \delta \quad \Rightarrow \quad |\gamma(s, t) - \gamma(s', t')| < \varepsilon .$$

c) Choose $N \in \mathbb{N}$ so large that

$$\frac{1}{N} + \frac{b - a}{N} < \delta .$$

Define a grid in

$$Q = [0, 1] \times [a, b]$$

by

$$s_j = \frac{j}{N}, \quad t_k = a + (b - a) \frac{k}{N}, \quad 0 \leq j, k \leq N .$$

The rectangle $Q$ is partitioned into the subrectangles

$$Q_{jk} = [s_j, s_{j+1}] \times [t_k, t_{k+1}] .$$

If $(s, t)$ and $(s', t')$ are two points in $Q_{jk}$, then

$$|s - s'| + |t - t'| < \delta .$$

Therefore,

$$\gamma(Q_{jk}) \subset D(\gamma(s_j, t_k), \varepsilon) \subset U .$$

d) Set

$$\gamma_{s_j}(t) = \gamma(s_j, t), \quad a \leq t \leq b .$$

We claim that

$$\int_{\gamma_{s_j}} f \, dz = \int_{\gamma_{s_{j+1}}} f \, dz .$$

To see this, we apply Cauchy’s integral theorem in the disks.
successively for \( k = 0, 1, \ldots, N - 1 \) to deform the curve \( \gamma_{s_j} \) into \( \gamma_{s_{j+1}} \). Since the deformation takes place in disks that lie in \( U \), the integral does not change.

**Definition:** Let \( U \) be a region in \( \mathbb{C} \) and let \( \gamma_0(t), a \leq t \leq b \), be a closed curve in \( U \). Then \( \gamma_0 \) is called null–homotopic in \( U \) if \( \gamma_0 \) is homotopic to the constant curve

\[
\gamma_1(t) = \gamma_0(a) = \gamma_0(b), \quad a \leq t \leq b.
\]

The following three theorems are different versions of Cauchy’s Theorem.

**Theorem 20.2** Let \( U \) be a region in \( \mathbb{C} \) and let \( \gamma \) be null–homotopic in \( U \). If \( f \in H(U) \) then

\[
\int_\gamma f(z) \, dz = 0.
\]

**Definition:** A region \( U \) in \( \mathbb{C} \) is called (topologically) simply connected if every closed curve in \( U \) is null–homotopic in \( U \).

**Theorem 20.3** Let \( U \) be a simply connected region in \( \mathbb{C} \). If \( \gamma \) is a closed curve in \( U \) and \( f \in H(U) \) then

\[
\int_\gamma f(z) \, dz = 0.
\]

**Theorem 20.4** Let \( U \) be a region in \( \mathbb{C} \). (It is not assumed that \( U \) is simply connected.) Let \( f \in H(U) \). There is a function \( F \in H(U) \) with \( F' = f \) in \( U \) if and only if

\[
\int_\gamma f(z) \, dz = 0
\]

for every closed curve \( \gamma \) in \( U \).