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**RESEARCH ARTICLE**

# The Equivalence Between Likelihood Ratio Test and F Test for Testing Variance Component in a Balanced One Way Random Effects Model

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In mixed linear models, it is frequently of interest to test hypotheses on the variance components. F test and likelihood ratio test are commonly used for such purposes. Current likelihood ratio tests available in literature are based on limiting distribution theory. With the development of finite sample distribution theory, it becomes possible to derive the exact test for likelihood ratio statistic. In this paper, we consider the problem of testing null hypotheses on the variance component in a one way balanced random effects model. We use the exact test for the likelihood ratio statistic and compare the performance of F test and likelihood ratio test. Simulations provide strong support of the equivalence between these two tests. Furthermore, we prove the equivalence between these two tests mathematically.

**Keywords:** Finite sample distribution, Hypothesis, Mixed model, Simulations, Variance component

## 1. Introduction

This paper assumes a one way random effects model

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \tag{1}$$

where $i = 1, \ldots, m; j = 1, \ldots, k$. $\mu$ is the fixed unknown intercept, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)'$ is the random effect and $\boldsymbol{\varepsilon} = (\varepsilon_{11}, \varepsilon_{12}, \ldots, \varepsilon_{mk})'$ is the error term. Assume $\boldsymbol{\alpha}$ and $\boldsymbol{\varepsilon}$ are normally and independently distributed with mean $\mathbf{0}$ and variances $\sigma_\alpha^2 \mathbf{I}_m$, $\sigma^2 \mathbf{I}_{mk}$. A standard test of the variance component $\sigma_\alpha^2$ is as the following:

$$\mathbf{H}_0 : \sigma_\alpha^2 = 0 \quad \text{v.s} \quad \mathbf{H}_a : \sigma_\alpha^2 > 0. \tag{2}$$

F test is commonly used for this situation because, in this case, F test is a uniformly most powerful unbiased test. On the other hand, likelihood ratio test is a well known and widely used statistical test. One problem of test (2) is that zero is at the boundary of the parameter space, so the limiting distribution of the likelihood ratio statistic is not $\chi^2$. Hartly and Rao [1] stated without giving a proof that the asymptotic distribution of $-2log\mathbf{L}$ is a central $\chi^2$. Other papers related to asymptotic distribution of likelihood ratio statistic include Stram and

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Lee [2], Shephard and Harvey [3] and Stern and Welsh [4]. $\chi^2$ mixture [2] is another way to approximate the distribution of likelihood ratio statistic and it works well when the number of independent groups is large. There was another test called locally optimal test proposed by Westfall [5, 6], which followed papers by Harville and Fenech [7] and Seely and El-Bassinouni [8]. Westfall [5] compared the locally optimal test to the F test for unbalanced designs. However, "the literature on likelihood ratio tests in the context of linear mixed models is much less extensive" (Jiang [9, pg. 55]). Recently, Crainiceanu and Ruppert [10] derived finite sample distribution of likelihood ratio statistics in linear mixed models, which makes it possible to derive the exact test for likelihood ratio statistics. In this article, we consider the standard test (2) in a one way balanced random effects model. We discover that F test and likelihood ratio test are equivalent by simulation studies. Furthermore, we prove the equivalence between the two tests in theory.

This paper is organized as follows. In section 2, we review F test and likelihood ratio test. In section 3, we report our simulation results. Finally, a proof of the equivalence between the two tests is given in section 4.

## 2. Background

We first introduce some notation. Define $\bar{y}_{..} = \sum_{i=1}^m \sum_{j=1}^k y_{ij}/mk$, $\bar{y}_{i.} = \sum_{j=1}^k y_{ij}/k$ for each $i$, $SSE = \sum_{i=1}^m \sum_{j=1}^k (y_{ij} - \bar{y}_{i.})^2/(k-1)m$, $SSB = k\sum_{i=1}^m (\bar{y}_{i.} - \bar{y}_{..})^2$, $MSB = SSB/(m-1)$ and $MSE = SSE/(m(k-1))$.

For model (1), the ratio of $MSB/(\sigma^2 + k\sigma_\alpha^2)$ to $MSE/\sigma^2$ has an F distribution with degrees of freedom (m-1,m(k-1)). Under $\mathbf{H}_0$ in (2), $MSB/MSE$ has an F distribution with degrees of freedom (m-1,m(k-1)).

Before we discuss the likelihood ratio test, we first write model (1) in matrix form,

$$\mathbf{Y} = \mathbf{X}\mu + \mathbf{Z}\boldsymbol{\alpha} + \boldsymbol{\varepsilon}, \tag{3}$$

where $\mathbf{X}$ is simply an $mk \times 1$ vector of 1s, $\mathbf{Z}$ is an $mk \times m$ matrix with every column containing only 0s with exception of a $k$ dimensional vector of 1s corresponding to the level parameter, $\mathbf{Y}$ is the response vector and $\boldsymbol{\varepsilon}$ is the random error vector.

Twice the log-likelihood function of (3), we have

$$2\log\{\mathbf{L}(\mu, \sigma_\alpha^2, \sigma^2)\} = -\log(\sigma^2) - \log|\mathbf{V}| - \frac{(\mathbf{Y} - \mathbf{X}\mu)^T \mathbf{V}^{-1} (\mathbf{Y} - \mathbf{X}\mu)}{\sigma^2},$$

where $\mathbf{V} = \mathbf{I}_{mk} + \lambda \mathbf{Z}\mathbf{Z}^T$, $\lambda = \sigma_\alpha^2/\sigma^2$ and the likelihood ratio statistic is defined as

$$LRT = 2\sup_{H_a}\{\mathbf{L}(\mu, \sigma_\alpha^2, \sigma^2)\} - 2\sup_{H_0}\{\mathbf{L}(\mu, \sigma_\alpha^2, \sigma^2)\}. \tag{4}$$

The standard maximum likelihood estimators are as follows

$$\hat{\mu} = (\mathbf{X}^T \mathbf{V}^{-1} \mathbf{X})^{-1} \mathbf{X}^T \mathbf{V}^{-1} \mathbf{Y},$$

and

$$\hat{\sigma}^2 = \frac{(\mathbf{Y} - \mathbf{X}\hat{\mu})^T \mathbf{V}^{-1} (\mathbf{Y} - \mathbf{X}\hat{\mu})}{mk}.$$

Under the null hypothesis, we obtain the likelihood estimators as follows

$$\hat{\mu} = \bar{y}_{..},$$

and

$$\hat{\sigma}_0^2 = \frac{1}{km} \sum_{i=1}^{m} \sum_{j=1}^{k} (y_{ij} - \bar{y}_{..})^2.$$

Under the alternative hypothesis, we obtain the likelihood estimators as follows

$$\hat{\mu} = \bar{y}_{..},$$

and

$$\hat{\sigma}^2 = \frac{1}{(k-1)m} \sum_{i=1}^{m} \sum_{j=1}^{k} (y_{ij} - \bar{y}_{i.})^2.$$

$$\hat{\sigma}_\alpha^2 = \begin{cases} \frac{1}{k}\left(\frac{k\sum_{i=1}^{m}(\bar{y}_{i.}-\bar{y}_{..})^2}{m} - \hat{\sigma}^2\right), & \text{if } \hat{\sigma}^2 \le \frac{k\sum_{i=1}^{m}(\bar{y}_{i.}-\bar{y}_{..})^2}{m}, \\ 0, & \text{otherwise} . \end{cases} \tag{5}$$

## 3. Simulations

We first introduce the result from Crainiceanu and Ruppert [10], which we use in our simulation study. Crainiceanu and Ruppert [10] developed a method to get finite sample distributions of likelihood ratio statistics, which follows that

$$\begin{aligned} LRT &\overset{D}{=} km\log(X_{m-1} + X_{(k-1)m}) \\ &- \inf_{d\ge 0} \left\{ km\log\left(\frac{X_{m-1}}{1+d} + X_{(k-1)m}\right) + m\log(1+d) \right\}, \end{aligned} \tag{6}$$

where notation $\overset{D}{=}$ denotes equivalence in distribution and $X_{m-1}$ and $X_{(k-1)m}$ are independent random variables with distribution $\chi^2_{m-1}$ and $\chi^2_{(k-1)m}$.

In this section, we perform simulation study to investigate the two tests. First, we compare the percentages of samples for which the test statistics exceed the critical value to the nominal level. Then we calculate power of the two tests. The following are the simulation details:

(1) Calculate critical values for both tests. We use (6) to generate finite sample distribution of likelihood ratio statistic and to find corresponding critical values. For each setting, we use a different seed and generate 100000 samples from (6). Critical value is the $100(1-\gamma)\%$ percentile, where $\gamma$ is the significant level of test. Critical value for F test is $F_\gamma[m-1, m(k-1)]$. Results are listed in Table (1).

(2) Compare to the nominal levels. The following model is used to generate samples:

$$y_{ij} = 0.5 + \alpha_i + \varepsilon_{ij}, \tag{7}$$

Table 1. Critical values for the two tests

| significance level ($\gamma$) | k | tests | m = 2 | m = 10 | m = 50 |
|---|---|---|---|---|---|
| 0.01 | 5 | F test | 11.25862 | 2.887560 | 1.634977 |
| | | LRT test | 3.062403 | 4.361615 | 4.91984 |
| | 10 | F test | 8.28542 | 2.610879 | 1.576229 |
| | | LRT test | 2.590879 | 4.137277 | 4.707984 |
| | 20 | F test | 7.352545 | 2.501878 | 1.551395 |
| | | LRT test | 2.444884 | 4.024391 | 4.670615 |
| 0.05 | 5 | F test | 5.317655 | 2.124029 | 1.418051 |
| | | LRT test | 0.9052984 | 1.877785 | 2.335454 |
| | 10 | F test | 4.413873 | 1.985595 | 1.382671 |
| | | LRT test | 0.6955273 | 1.764155 | 2.228158 |
| | 20 | F test | 4.098172 | 1.929425 | 1.367567 |
| | | LRT test | 0.6283923 | 1.705018 | 2.225503 |
| 0.1 | 5 | F test | 3.457919 | 1.792902 | 1.312488 |
| | | LRT test | 0.2665952 | 0.990356 | 1.362770 |
| | 10 | F test | 3.006977 | 1.702053 | 1.286975 |
| | | LRT test | 0.1779033 | 0.9079695 | 1.27152 |
| | 20 | F test | 2.842442 | 1.664704 | 1.276034 |
| | | LRT test | 0.1358003 | 0.8746464 | 1.274283 |

where $i = 1, \ldots, m; j = 1, \ldots, k$. $\boldsymbol{\alpha}$ and $\boldsymbol{\varepsilon}$ are normally and independently distributed with mean $\mathbf{0}$ and variance $\sigma_\alpha^2 \mathbf{I}_m$, $\sigma^2 \mathbf{I}_{mk}$ respectively.

For each setting, we use a different seed and generate 100000 samples from model (7) with $\sigma_\alpha^2 = 0$ and $\sigma^2 = 1$. For each sample, we apply F test and likelihood ratio test (LRT) and the percentages of samples for which the test statistics exceed the critical value are reported in Table (2). Both tests give almost the same results and work very well as we can see that the percentages of samples for which the test statistics exceed the critical value are very close to the nominal level.

(3) Calculate power of the tests. We generate 100000 samples from model (7) with $\sigma_\alpha^2 = 0.09, 1, 9$ and $\sigma^2 = 1$ for each setting using a different seed. The results are reported in Table (3), Table (4) and Table (5). We can see that the two tests almost have the same power. Equation (8) can also be used to calculate power.

$$P\left(\frac{MSB}{MSE} > F_\gamma | \sigma_\alpha^2 > 0\right) = P\left(\frac{MSB/(\sigma^2 + k\sigma_\alpha^2)}{MSE/\sigma^2} > \frac{\sigma^2}{\sigma^2 + k\sigma_\alpha^2} F_\gamma\right), \quad (8)$$

where $\gamma$ is the significance level, $F_\gamma[m-1, m(k-1)]$ is the critical value.

For example, let $m = 2, k = 5, \gamma = 0.01, \lambda = 0.09$, we have $\sigma^2 F_\gamma/(\sigma^2 + k\sigma_\alpha^2) = 7.764566$ and power of the test from (8) is 0.02368, which is close to the empirical power reported in Table (3).

Table 2. Tests Comparison (the numbers in the table are the percentages of samples for which the test statistics exceed the critical value)

| significance level ($\gamma$) | k | m | | | |
|---|---|---|---|---|---|
| | | tests | 2 | 10 | 50 |
| 0.01 | 5 | F test | 0.01015 | 0.01026 | 0.01009 |
| | | LRT test | 0.01034 | 0.01006 | 0.00973 |
| | 10 | F test | 0.00958 | 0.00977 | 0.01013 |
| | | LRT test | 0.00978 | 0.0096 | 0.01061 |
| | 20 | F test | 0.00989 | 0.01041 | 0.00945 |
| | | LRT test | 0.00970 | 0.01041 | 0.00992 |
| 0.05 | 5 | F test | 0.05028 | 0.04888 | 0.05037 |
| | | LRT test | 0.05049 | 0.04959 | 0.04943 |
| | 10 | F test | 0.05022 | 0.05015 | 0.05001 |
| | | LRT test | 0.05025 | 0.05038 | 0.05107 |
| | 20 | F test | 0.04938 | 0.0496 | 0.05036 |
| | | LRT test | 0.04840 | 0.05003 | 0.05039 |
| 0.1 | 5 | F test | 0.09926 | 0.10016 | 0.10066 |
| | | LRT test | 0.09921 | 0.10123 | 0.09809 |
| | 10 | F test | 0.10153 | 0.09906 | 0.10078 |
| | | LRT test | 0.0996 | 0.10006 | 0.10221 |
| | 20 | F test | 0.10089 | 0.10019 | 0.10003 |
| | | LRT test | 0.09994 | 0.10062 | 0.0998 |

Table 3. Power of the tests $\lambda = 0.09$

| significance level ($\gamma$) | k | m | | | |
|---|---|---|---|---|---|
| | | tests | 2 | 10 | 50 |
| 0.01 | 5 | F test | 0.02286 | 0.06682 | 0.27836 |
| | | LRT test | 0.02332 | 0.06565 | 0.27426 |
| | 10 | F test | 0.05208 | 0.21231 | 0.78583 |
| | | LRT test | 0.05301 | 0.21117 | 0.78968 |
| | 20 | F test | 0.11299 | 0.53116 | 0.99473 |
| | | LRT test | 0.11186 | 0.53112 | 0.99490 |
| 0.05 | 5 | F test | 0.09213 | 0.19207 | 0.52209 |
| | | LRT test | 0.09244 | 0.19354 | 0.51860 |
| | 10 | F test | 0.14611 | 0.41039 | 0.91525 |
| | | LRT test | 0.14613 | 0.41108 | 0.91625 |
| | 20 | F test | 0.23492 | 0.72211 | 0.99865 |
| | | LRT test | 0.23280 | 0.72282 | 0.99865 |
| 0.1 | 5 | F test | 0.16075 | 0.30051 | 0.65041 |
| | | LRT test | 0.16056 | 0.30251 | 0.64618 |
| | 10 | F test | 0.22725 | 0.53221 | 0.95468 |
| | | LRT test | 0.22403 | 0.53373 | 0.95526 |
| | 20 | F test | 0.31735 | 0.79945 | 0.99968 |
| | | LRT test | 0.31589 | 0.79985 | 0.99966 |

## 4. Proof of the Equivalence

**Theorem 4.1:** *For model (1), the likelihood ratio test statistic (4) is a one to one function of $MSB/MSE$. Hence Likelihood ratio test is equivalent to F test.*

**Proof:** We first prove that the likelihood ratio statistic is a one to one function of F statistic by two cases: $\hat{\sigma}_\alpha^2 > 0$ and $\hat{\sigma}_\alpha^2 = 0$.

Table 4. Power of the tests $\lambda = 1$

| significance level ($\gamma$) | k | m tests | 2 | 10 | 50 |
|---|---|---|---|---|---|
| 0.01 | 5 | F test | 0.20632 | 0.87851 | 1.00000 |
| | | LRT test | 0.20821 | 0.87713 | 1.00000 |
| | 10 | F test | 0.39500 | 0.98768 | 1.00000 |
| | | LRT test | 0.39680 | 0.98756 | 1.00000 |
| | 20 | F test | 0.55872 | 0.99917 | 1.00000 |
| | | LRT test | 0.55755 | 0.99917 | 1.00000 |
| 0.05 | 5 | F test | 0.37424 | 0.94984 | 1.00000 |
| | | LRT test | 0.37472 | 0.95021 | 1.00000 |
| | 10 | F test | 0.53460 | 0.99604 | 1.00000 |
| | | LRT test | 0.53461 | 0.99605 | 1.00000 |
| | 20 | F test | 0.65973 | 0.99965 | 1.00000 |
| | | LRT test | 0.65829 | 0.99965 | 1.00000 |
| 0.1 | 5 | F test | 0.47081 | 0.97072 | 1.00000 |
| | | LRT test | 0.47068 | 0.97096 | 1.00000 |
| | 10 | F test | 0.60843 | 0.99737 | 1.00000 |
| | | LRT test | 0.60605 | 0.99739 | 1.00000 |
| | 20 | F test | 0.71258 | 0.99984 | 1.00000 |
| | | LRT test | 0.71187 | 0.99984 | 1.00000 |

Table 5. Power of the tests $\lambda = 9$

| significance level ($\gamma$) | k | m tests | 2 | 10 | 50 |
|---|---|---|---|---|---|
| 0.01 | 5 | F test | 0.63433 | 0.99994 | 1.00000 |
| | | LRT test | 0.63573 | 0.99994 | 1.00000 |
| | 10 | F test | 0.76535 | 1.00000 | 1.00000 |
| | | LRT test2 | 0.76617 | 1.00000 | 1.00000 |
| | 20 | F test | 0.84207 | 1.00000 | 1.00000 |
| | | LRT test | 0.84153 | 1.00000 | 1.00000 |
| 0.05 | 5 | F test | 0.74275 | 0.99998 | 1.00000 |
| | | LRT test | 0.74298 | 0.99998 | 1.00000 |
| | 10 | F test | 0.82701 | 1.00000 | 1.00000 |
| | | LRT test | 0.82702 | 1.00000 | 1.00000 |
| | 20 | F test | 0.88116 | 1.00000 | 1.00000 |
| | | LRT test | 0.88054 | 1.00000 | 1.00000 |
| 0.1 | 5 | F test | 0.79241 | 0.99999 | 1.00000 |
| | | LRT test | 0.79236 | 0.99999 | 1.00000 |
| | 10 | F test | 0.85743 | 1.00000 | 1.00000 |
| | | LRT test | 0.85654 | 1.00000 | 1.00000 |
| | 20 | F test | 0.90209 | 1.00000 | 1.00000 |
| | | LRT test | 0.90180 | 1.00000 | 1.00000 |

Case 1: $\hat{\sigma}_\alpha^2 > 0$
Since $\hat{\sigma}^2 = MSE$, $\hat{\sigma}_\alpha^2 = \frac{1}{k}\left(\frac{m-1}{m}MSB - MSE\right)$ and $SSE + SSB = \sum_{i=1}^m \sum_{j=1}^k (y_{ij} - \bar{y}_{..})^2$,

$$
\begin{aligned}
LRT &= -(mk-m)\log(\hat{\sigma}^2) - \sum_{i=1}^m \log(\hat{\sigma}^2 + k\hat{\sigma}_\alpha^2) - \frac{1}{\hat{\sigma}^2}\sum_{i=1}^m\sum_{j=1}^k (y_{ij} - \hat{\mu})^2 \\
&= -(mk-m)\log(\hat{\sigma}^2) - m\log(\hat{\sigma}^2 + k\hat{\sigma}_\alpha^2) - \frac{mk\hat{\sigma}_0^2}{\hat{\sigma}^2} \\
&+ \frac{\hat{\sigma}_\alpha^2}{\hat{\sigma}^2}\left(\frac{k^2}{\hat{\sigma}^2 + k\hat{\sigma}_\alpha^2}\right)\sum_{i=1}^m (\bar{y}_{i.} - \bar{y}_{..})^2 + mk\log(\hat{\sigma}_0^2) + mk \\
&= -(mk-m)\log\left(\frac{SSE}{(k-1)m}\right) - m\log\left(\frac{SSB}{m}\right) - \frac{(k-1)m(SSE+SSB)}{SSE} \\
&+ \frac{m^2(k-1)\left(\frac{SSB}{m} - \frac{SSE}{(k-1)m}\right)}{SSE} + mk\log\left(\frac{1}{mk}(SSE+SSB)\right) + mk \\
&= c + mk\log(SSE+SSB) - mk\log(SSE) + m\log(SSE) - m\log(SSB) \\
&= c + m\left[\log\left(1 + c^*\frac{MSB}{MSE}\right)^k - \log\left(c^*\frac{MSB}{MSE}\right)\right] \\
&= c - m\log(c^*) + m\log\left(\frac{\left(1 + c^*\frac{MSB}{MSE}\right)^k}{\frac{MSB}{MSE}}\right) \qquad (9)
\end{aligned}
$$

where $c$ and $c^* = (m-1)/((k-1)m)$ are constants. Since $\hat{\sigma}_\alpha^2 > 0$, we have $MSB/MSE > m/(m-1)$. If $x > m/(m-1)$ and $k \geq 2$, derivative of function $f(x) = (1 + c^*x)^k/x$ is positive . So $LRT$ is a strictly increasing function.
Case 2: $\hat{\sigma}_\alpha^2 = 0$
In this case,

$$
\begin{aligned}
LRT &= -(mk-m)\log(\hat{\sigma}^2) - \sum_{i=1}^m \log(\hat{\sigma}^2) - \frac{1}{\hat{\sigma}^2}\sum_{i=1}^m\sum_{j=1}^k (y_{ij} - \hat{\mu})^2 \\
&+ mk\log(\hat{\sigma}_0^2) + \frac{1}{\hat{\sigma}_0^2}\sum_{i=1}^m\sum_{j=1}^k (y_{ij} - \hat{\mu})^2 \\
&= -(mk-m)\log\left(\frac{SSE}{(k-1)m}\right) - m\log\left(\frac{SSE}{m(k-1)}\right) - \frac{(k-1)m(SSE+SSB)}{SSE} \\
&+ mk\log\left(\frac{1}{mk}(SSE+SSB)\right) + mk \\
&= c' - (k-1)m\frac{SSB}{SSE} + mk\log\left(1 + \frac{SSB}{SSE}\right) \\
&= c' - (m-1)\frac{MSB}{MSE} + mk\log\left(1 + c^*\frac{MSB}{MSE}\right)
\end{aligned}
$$

where $c'$ and $c^* = (m-1)/((k-1)m)$ are constants. Since $\hat{\sigma}_\alpha^2 = 0$, we have $MSB/MSE \leq m/(m-1)$. If $x \leq m/(m-1)$ and $m, k \geq 2$, derivative of function

$f(x) = -(m-1)x + mk \log(1 + c^* x)$ is positive. So $LRT$ in case 2 is also a strictly increasing function.

From the above proof, we conclude that the likelihood ratio statistic is a one-to-one function of F statistic $MSB/MSE$ under both cases $\hat{\sigma}_\alpha^2 > 0$ and $\hat{\sigma}_\alpha^2 = 0$.

The proof of equivalence can be obtained by proving that the two tests have the same results of rejection or acceptance. Consider case 1: $\hat{\sigma}_\alpha^2 > 0$. Given an arbitrary significance level $\gamma$, let $F_\gamma$ be the critical value of F test and $L_\gamma$ be the critical value of the likelihood ratio test. Let $LRT = g(MSB/MSE)$, where $g$ represents the one to one continuous increasing function. Clearly $L_\gamma = g(F_\gamma)$ and the statement $MSB/MSE > F_\gamma$ is equivalent to the statement $g(MSB/MSE) > g(F_\gamma)$ i.e., $LRT > L_\gamma$. Proof of equivalence in case 2: $\hat{\sigma}_\alpha^2 = 0$ can be obtained similarly. □

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