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Simultaneous confidence intervals for several inverse Gaussian populations

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ABSTRACT

In this research, we propose simultaneous confidence intervals for all pairwise comparisons of means from inverse Gaussian distribution. Our method is based on fiducial generalized pivotal quantities for vector parameters. We prove that the constructed confidence intervals have asymptotically correct coverage probabilities. Simulation results show that the simulated Type-I errors are close to the nominal level even for small samples. The proposed approach is illustrated by an example.

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1. Introduction

Inverse Gaussian (IG) distribution is widely used to describe and analyze positively right-skewed data. For example, it is useful to model lifetime distribution and wind energy distribution. Chhikara and Folks (1989, Ch.10) and Seshadri (1993, 1999) discussed some examples of IG distribution in fields such as cardiology, hydrology, demography and finance. For many observational and experimental data arising from several IG populations, the problem of testing equality (see Shi and Lv, 2012; Krishnamoorthy and Tian, 2008 etc.) and simultaneous pairwise comparisons (SPC) of the group means are usually of interest.

In standard analysis of variance, Scheffé's method (Scheffé, 1959), the Bonferroni inequality-based method, and the Tukey method (Tukey, 1953) are widely used for SPC. When variances are heteroscedastic and group sizes are unequal, exact frequentist tests are unavailable. In such situations, parametric bootstrap and generalized p-value (Tsui and Weerahandi, 1989) procedures are commonly used. Zhang (in press, 2014b) proposed parametric bootstrap simultaneous confidence intervals (SCI) for one-way and two-way ANOVA under heteroscedasticity. Weerahandi (1993) introduced the concept of a generalized pivotal quantity. Later, Hannig et al. (2006) introduced a subclass of Weerahandi's generalized pivotal quantity, called fiducial generalized pivotal quantities (FGPQs). Using the idea of FGPQ, Hannig et al. (2006) proposed simultaneous fiducial generalized confidence intervals for ratios of means of log-normal distributions. Xiong and Mu (2009) proposed two kinds of SCI based on FGPQ in a one-way layout under heteroscedasticity. Recently, Ye et al. (2014) discussed the reliability issue in one-way random models based on generalized pivotal quantities and FGPQ. Zhang and Falk (2014) proposed FGPQ-based SCI for several log-normal distributions. To our knowledge, there is no work on SCI for all-pairwise comparisons of IG distributions. In this research, we propose to construct FGPQ-based SCI for IG distributions to fill the gap.

This paper is organized as follows. In Section 2, we review IG distribution and several generalized pivotal quantities. In Section 3, we propose FGPQ-based SCI for means from several IG distributions and prove the asymptotic properties.

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In Section 4, we present simulation studies from the perspective of Type-I error and power of the tests. In Section 5, we give an example to illustrate the proposed approach. Section 6 gives conclusions.

2. Background

The density function of the two-parameter IG distribution $IG(\mu, \lambda)$ is defined as

$$f(x, \mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\}, \quad x > 0,\ \mu, \lambda > 0 \tag{1}$$

where μ is the mean parameter and λ is the scale parameter. Let $X_{ij} \sim IG(\mu_i, \lambda_i), i = 1, \ldots, k, j = 1, \ldots, n_i$ be an independent random sample from k IG populations. Let $n = \sum_{i=1}^{k} n_i$ be the total sample size, and $\bar{X}_i = \sum_{j=1}^{n_i} X_{ij}/n_i$ be the group mean. The maximum likelihood estimators of μ_i and λ_i can be found as

$$\hat{\mu}_i = \bar{X}_i, \qquad 1/\hat{\lambda}_i = \frac{1}{n_i}\sum_{j=1}^{n_i}(1/X_{ij} - 1/\bar{X}_i).$$

To simplify notation, let $V_i = 1/\hat{\lambda}_i$. It is well known that

$$\bar{X}_i \sim IG(\mu_i, n_i\lambda_i), \qquad n_i\lambda_i V_i \sim \chi^2_{n_i-1}, \quad i = 1, 2, \ldots, k, \tag{2}$$

$\bar{X}_i$ and V_i are complete sufficient statistics for (μ_i, λ_i) and are mutually independent.

Ye et al. (2010) proposed the generalized pivotal quantities for λ_i and μ_i as follows

$$R_i = \frac{n_i\lambda_i V_i}{n_i v_i} \sim \frac{\chi^2_{n_i-1}}{n_i v_i}, \quad i = 1, \ldots, k, \tag{3}$$

and

$$T_{\mu_i} = \frac{\bar{x}_i}{\left|1 + \frac{\sqrt{n_i\lambda_i}(\bar{X}_i-\mu_i)}{\mu_i\sqrt{\bar{X}_i}}\sqrt{\frac{\bar{x}_i}{n_i R_i}}\right|} \overset{d}{\sim} \frac{\bar{x}_i}{\left|1 + Z_i\sqrt{\frac{\bar{x}_i}{n_i R_i}}\right|}, \tag{4}$$

where $\overset{d}{\sim}$ denotes "approximately distributed", $Z_i \sim N(0, 1)$, and $\bar{x}_i$ and v_i are the observed values of $\bar{X}_i$ and V_i. The approximate distribution comes from the moment matching method, that $\sqrt{n_i\lambda_i}(\bar{X}_i-\mu_i)/(\mu_i\sqrt{\bar{X}_i})$ has a limiting distribution of $N(0, 1)$. By substituting R_i in (4) with (3), Krishnamoorthy and Tian (2008) proposed an approximate generalized pivotal quantity for μ_i as follows

$$T^*_{\mu_i} = \frac{\bar{x}_i}{\max\left\{0, 1 + t_{n_i-1}\sqrt{\frac{\bar{x}_i v_i}{n_i-1}}\right\}}. \tag{5}$$

One problem of the pivotal quantity $T^*_{\mu_i}$ is that the denominator may be zero when t_{n_i-1} takes a negative value. To overcome this problem, Shi and Lv (2012) proposed a generalized pivotal quantity for the reciprocal of μ_i, say $\theta_i = 1/\mu_i$ by

$$T_{\theta_i} = \frac{1}{T_{\mu_i}} \overset{d}{\sim} \left|1 + Z_i\sqrt{\frac{\bar{x}_i}{n_i R_i}}\right| / \bar{x}_i. \tag{6}$$

Note that the observed value of T_{θ_i} is θ_i and the distribution of T_{θ_i} is free of any unknown parameter. Therefore, T_{θ_i} is a generalized pivotal quantity for θ_i. Based on the pivotal quantity T_{θ_i}, Shi and Lv (2012) proposed a new generalized p-value procedure for testing equality of inverse Gaussian means under heterogeneity.

3. Simultaneous confidence intervals for *k* inverse Gaussian populations

In this section, we propose FGPQ-based SCI for all-pairwise comparisons of means from k IG populations. The testing problem is as follows

$$H_0 : \theta_i = \theta_j \quad \text{for all } i \neq j \text{ vs } H_\alpha : \theta_i s \text{ are not all equal.} \tag{7}$$

Inspired by Shi and Lv (2012), we define FGPQ's for λ_i and θ_i as

$$R_{\lambda_i} = \frac{U_i^2}{n_i V_i}$$

and

$$R_{\theta_i} = \left|1 + Z_i\sqrt{\frac{\bar{X}_i}{n_i R_{\lambda_i}}}\right| / \bar{X}_i = \left|1 + Z_i\sqrt{\frac{\bar{X}_i V_i}{U_i^2}}\right| / \bar{X}_i,$$

where $U_i^2 \sim \chi^2_{n_i-1}$, $i = 1, \ldots, k$. Define $\theta_{ij} = \theta_i - \theta_j$. The FGPQ for θ_{ij} follows immediately

$$R_{\theta_{ij}} = R_{\theta_i} - R_{\theta_j} = \left|1 + Z_i\sqrt{\frac{\bar{X}_i V_i}{U_i^2}}\right| / \bar{X}_i - \left|1 + Z_j\sqrt{\frac{\bar{X}_j V_j}{U_j^2}}\right| / \bar{X}_j. \tag{8}$$

Let $\bar{\mathbf{X}} = (\bar{X}_1, \ldots, \bar{X}_k)'$ and $\mathbf{V} = (V_1, V_2, \ldots, V_k)'$. The conditional expectation and variance of $R_{\theta_{ij}}$ can be derived as follows

$$\eta_{ij} = E(R_{\theta_{ij}}|\bar{\mathbf{X}}, \mathbf{V}) = 1/\bar{X}_i - 1/\bar{X}_j, \tag{9}$$

$$V_{ij} = \mathrm{Var}(R_{\theta_{ij}}|\bar{\mathbf{X}}, \mathbf{V}) = \frac{V_i}{(n_i - 3)\bar{X}_i} + \frac{V_j}{(n_j - 3)\bar{X}_j}. \tag{10}$$

As pointed by Xiong and Mu (2009), FGPQs can be used to provide effective approximations to distributions. The distribution of

$$\max_{i<j}\left|\frac{\theta_{ij} - E(R_{\theta_{ij}}|\bar{\mathbf{X}}, \mathbf{V})}{\sqrt{\mathrm{Var}(R_{\theta_{ij}}|\bar{\mathbf{X}}, \mathbf{V})}}\right|$$

can be approximated by the conditional distribution of

$$Q = \max_{i<j}\left|\frac{R_{\theta_{ij}} - E(R_{\theta_{ij}}|\bar{\mathbf{X}}, \mathbf{V})}{\sqrt{V_{ij}}}\right|. \tag{11}$$

Let q_α be the conditional upper αth quantile of Q. We propose the $(1 - \alpha)100\%$ simultaneous confidence intervals for θ_{ij} as

$$\theta_{ij} \in \frac{1}{\bar{X}_i} - \frac{1}{\bar{X}_j} \pm q_\alpha\sqrt{V_{ij}} \quad \text{for all } i < j. \tag{12}$$

Algorithm 1 is proposed to calculate q_α.

Algorithm 1.

For given observations x_{ij}, $i = 1, \ldots, k$, $j = 1, \ldots, n_i$,
Compute $\bar{x}_i$ and v_i, $i = 1, \ldots, k$
For $l = 1, 2, \ldots, L$
Generate $Z_i \sim N(0, 1)$ and $U_i^2 \sim \chi^2_{n_i-1}$, $i = 1, \ldots, k$
Compute $R_{\theta_{ij}}$, η_{ij}, V_{ij}, and Q_l using Eqs. (8)–(11) respectively
End l loop.
Compute $q(\alpha)$, the $(1 - \alpha)100\%$ percentile of Q.

We now examine the properties of the SCIs in (12). The following theorems show that the proposed SCIs have asymptotically correct coverage probabilities. Proof of Theorem 1 is included in the Appendix.

Theorem 1. *Let $X_{i1}, \ldots, X_{in_i}$, $i = 1, \ldots, k$ be random samples from k IG populations and be mutually independent. Assume that $0 < \sigma_i^2 = Var(X_{i1}) < \infty$, $\mu_i = E(X_{i1})$, $N = \sum_{i=1}^k n_i$ and $\frac{n_i}{N} \to \tau_i \in (0, 1)$ as $N \to \infty$ for all i, we have*

$$P\left(\frac{1}{\mu_i} - \frac{1}{\mu_j} \in \frac{1}{\bar{X}_i} - \frac{1}{\bar{X}_j} \pm q(\alpha)\sqrt{V_{ij}}\, \textit{for all } i < j\right) \xrightarrow{p} 1 - \alpha.$$

Theorem 2. *With the same assumptions set out in Theorem 1, we have*

$$P(L_n < \mu_i - \mu_j < U_n \textit{ for all } i < j) \xrightarrow{p} 1 - \alpha,$$

where

$$L_n = K_n\left(\frac{1}{\bar{X}_i} - \frac{1}{\bar{X}_j} + q_\alpha\sqrt{V_{ij}}\right)$$

$$U_n = K_n\left(\frac{1}{\bar{X}_i} - \frac{1}{\bar{X}_j} - q_\alpha\sqrt{V_{ij}}\right)$$

$$K_n = -\hat{\mu}_i\hat{\mu}_j = -\bar{X}_i\bar{X}_j.$$

Table 1
Simulated Type-I errors of the proposed multiple comparison procedure for three groups: numbers in table are simulated Type-I errors.

$(\boldsymbol{\mu}, \boldsymbol{\sigma})$	$\mathbf{n}_1^{(3)}$			$\mathbf{n}_2^{(3)}$			$\mathbf{n}_3^{(3)}$		
	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
$(\boldsymbol{\mu}_1^{(3)}, \boldsymbol{\lambda}_1^{(3)})$	0.0120	0.0515	0.1010	0.0075	0.0435	0.0775	0.0090	0.0430	0.0810
$(\boldsymbol{\mu}_1^{(3)}, \boldsymbol{\lambda}_2^{(3)})$	0.0080	0.0435	0.0845	0.0050	0.0390	0.0825	0.0065	0.0430	0.0935
$(\boldsymbol{\mu}_1^{(3)}, \boldsymbol{\lambda}_3^{(3)})$	0.0090	0.0470	0.0910	0.0090	0.0430	0.0910	0.0115	0.0565	0.0950
$(\boldsymbol{\mu}_2^{(3)}, \boldsymbol{\lambda}_1^{(3)})$	0.0130	0.0555	0.1140	0.0085	0.0580	0.1105	0.0100	0.0515	0.0980
$(\boldsymbol{\mu}_2^{(3)}, \boldsymbol{\lambda}_2^{(3)})$	0.0075	0.0440	0.0870	0.0115	0.0490	0.0940	0.0115	0.0415	0.0865
$(\boldsymbol{\mu}_2^{(3)}, \boldsymbol{\lambda}_3^{(3)})$	0.0080	0.0435	0.0955	0.0100	0.0510	0.1065	0.0135	0.0595	0.1155
$(\boldsymbol{\mu}_3^{(3)}, \boldsymbol{\lambda}_1^{(3)})$	0.0165	0.0675	0.1310	0.0125	0.0535	0.1135	0.0120	0.0485	0.0965
$(\boldsymbol{\mu}_3^{(3)}, \boldsymbol{\lambda}_2^{(3)})$	0.0100	0.0505	0.1015	0.0140	0.0530	0.1040	0.0115	0.0470	0.1000
$(\boldsymbol{\mu}_3^{(3)}, \boldsymbol{\lambda}_3^{(3)})$	0.0120	0.0560	0.1120	0.0095	0.0530	0.1095	0.0070	0.0495	0.1015

Proof. It follows from Theorem 1 and the facts that $\bar{X}_i \xrightarrow{p} \mu_i$ and $\bar{X}_j \xrightarrow{p} \mu_j$. □

4. Simulations

In this section, we use simulations to evaluate the proposed SCIs by Type-1 error and power of the tests under various settings. The simulation study was performed with factors: (1) number of groups k: $k = 3$, $k = 6$ and $k = 10$; (2) population scale parameter $\boldsymbol{\lambda}_h^{(k)} = (\lambda_1, \ldots, \lambda_k)$: various combinations, $h = 1, 2, 3$; (3) population mean $\boldsymbol{\mu}_h^{(k)} = (\mu_1, \ldots, \mu_k)$: various combinations; (4) significance level α: 0.01, 0.05 and 0.1; (5) group sizes $\mathbf{n}_h^{(k)} = (n_1, \ldots, n_k)$: various combinations. The specific combinations are given in the following paragraph.

For Tables 1–3: for three groups $k = 3$, group sizes are with $\mathbf{n}_1^{(3)} = (10, 16, 20)$, $\mathbf{n}_2^{(3)} = (10, 10, 10)$, $\mathbf{n}_3^{(3)} = (20, 16, 10)$, population means are with $\boldsymbol{\mu}_1^{(3)} = (1, 1, 1)$, $\boldsymbol{\mu}_2^{(3)} = (5, 5, 5)$, $\boldsymbol{\mu}_3^{(3)} = (10, 10, 10)$, scale parameters are with $\boldsymbol{\lambda}_1^{(3)} = (1, 5, 10)$, $\boldsymbol{\lambda}_2^{(3)} = (5, 5, 5)$, $\boldsymbol{\lambda}_3^{(3)} = (10, 5, 1)$; for six groups $k = 6$, group sizes are with $\mathbf{n}_1^{(6)} = (10, 10, 16, 16, 20, 20)$, $\mathbf{n}_2^{(6)} = (10, 10, 10, 10, 10, 10)$, $\mathbf{n}_3^{(6)} = (20, 20, 16, 16, 10, 10)$, population means are with $\boldsymbol{\mu}_1^{(6)} = (1, 1, 1, 1, 1, 1)$, $\boldsymbol{\mu}_2^{(6)} = (5, 5, 5, 5, 5, 5)$, $\boldsymbol{\mu}_3^{(6)} = (10, 10, 10, 10, 10, 10)$, population scale parameters are with $\boldsymbol{\lambda}_1^{(6)} = (1, 1, 5, 5, 10, 10)$, $\boldsymbol{\lambda}_2^{(6)} = (5, 5, 5, 5, 5, 5)$, $\boldsymbol{\lambda}_3^{(6)} = (1, 1, 10, 10, 5, 5)$; for ten groups $k = 10$, group sizes are with $\mathbf{n}_1^{(10)} = (10, 10, 10, 16, 16, 16, 20, 20, 20, 20)$, $\mathbf{n}_2^{(10)} = (10, 10, 10, 10, 10, 10, 10, 10, 10, 10)$, $\mathbf{n}_3^{(10)} = (20, 20, 20, 20, 16, 16, 16, 10, 10, 10)$, population means are with $\boldsymbol{\mu}_1^{(10)} = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$, $\boldsymbol{\mu}_2^{(10)} = (5, 5, 5, 5, 5, 5, 5, 5, 5, 5)$, $\boldsymbol{\mu}_3^{(10)} = (10, 10, 10, 10, 10, 10, 10, 10, 10, 10)$, population scale parameters are with $\boldsymbol{\lambda}_1^{(10)} = (0.5, 0.5, 0.5, 2.5, 2.5, 2.5, 5, 5, 5, 5)$, $\boldsymbol{\lambda}_2^{(10)} = (2.5, 2.5, 2.5, 2.5, 2.5, 2.5, 2.5, 2.5, 2.5, 2.5)$, and $\boldsymbol{\lambda}_3^{(10)} = (0.5, 0.5, 0.5, 5, 5, 5, 5, 2.5, 2.5, 2.5)$.

For a given sample size and parameter configuration, we generated 2000 observed vectors $(\bar{x}_1, \ldots, \bar{x}_k, v_1, \ldots, v_k)$ by $\bar{x}_i \sim IG(\mu_i, n_i\lambda_i)$ and $v_i \sim \chi^2_{n_i-1}/(n_i\lambda_i)$. For each generated vector, we use 5000 runs to estimate the conditional upper αth quantile q_α by Algorithm 1. Finally, we report the simulated type I error probability for simultaneous tests in (7) and power of the tests. The following algorithm is used to find the simulated type-I error and power of the test:

Algorithm 2.

For $m = 1, 2, \ldots, M$
Generated vector $(\bar{x}_1^{(m)}, \ldots, \bar{x}_k^{(m)}, v_1^{(m)}, \ldots, v_k^{(m)})$:
Calculate $q_{ij}^{(m)} = |1/\bar{x}_i^{(m)} - 1/\bar{x}_j^{(m)}|/\sqrt{v_{ij}^{(m)}}$, $i, j = 1, \ldots, k$, $i < j$, let $q^{(m)} = \max(q_{ij}^{(m)})$
Use Algorithm 1 to find $q_\alpha^{(m)}$, the $1 - \alpha$ percentile of the simulated distribution of Q
end m loop.
The simulated Type-I error is the proportion of the M simulations when $q^{(m)} > q_\alpha^{(m)}$.

Tables 1–3 report the simulated Type-I error of the suggested simultaneous pairwise comparison procedure. From Tables 1–3, we can see that the simulated Type-I errors are close to the nominal level. Taking a closer look, we found that Type-1 error reported in Table 3 ($k = 10$) is slightly inflated compared to those for six groups and three groups. Notice that we have greater number of pairwise comparisons (45 pairs ($k = 10$) v.s. 15 pairs ($k = 6$)), and bigger variance (smaller $\boldsymbol{\lambda}$s) in Table 3 settings. We suggest increasing number of units within each group for experiments with larger number of groups and larger variance.

Table 4 reports the simulation results for power of the multiple comparisons. It is clear that power of the tests increased when sample size increased. Power of the tests also increased with decreased variances (increased $\boldsymbol{\lambda}$s), i.e., it is more easy to detect the difference in means when variance is small. Under most of the settings, we have 100% of the confidence to detect the difference of the group means.

Table 2
Simulated Type-I errors of the proposed multiple comparison procedure for six groups: numbers in table are simulated Type-I errors.

$(\boldsymbol{\mu}, \boldsymbol{\sigma})$	$\mathbf{n}_1^{(6)}$			$\mathbf{n}_2^{(6)}$			$\mathbf{n}_3^{(6)}$		
	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
$(\boldsymbol{\mu}_1^{(6)}, \boldsymbol{\lambda}_1^{(6)})$	0.0135	0.0545	0.1005	0.0090	0.0480	0.0855	0.0045	0.0420	0.0880
$(\boldsymbol{\mu}_1^{(6)}, \boldsymbol{\lambda}_2^{(6)})$	0.0060	0.0490	0.0980	0.0065	0.0370	0.0770	0.0070	0.0460	0.0960
$(\boldsymbol{\mu}_1^{(6)}, \boldsymbol{\lambda}_3^{(6)})$	0.0105	0.0515	0.0925	0.0095	0.0420	0.0890	0.0070	0.0410	0.0875
$(\boldsymbol{\mu}_2^{(6)}, \boldsymbol{\lambda}_1^{(6)})$	0.0115	0.0610	0.1140	0.0105	0.0555	0.1025	0.0095	0.0460	0.0900
$(\boldsymbol{\mu}_2^{(6)}, \boldsymbol{\lambda}_2^{(6)})$	0.0095	0.0495	0.0950	0.0050	0.0425	0.0850	0.0090	0.0435	0.0870
$(\boldsymbol{\mu}_2^{(6)}, \boldsymbol{\lambda}_3^{(6)})$	0.0115	0.0545	0.1085	0.0135	0.0540	0.1045	0.0105	0.0500	0.1020
$(\boldsymbol{\mu}_3^{(6)}, \boldsymbol{\lambda}_1^{(6)})$	0.0140	0.0585	0.1170	0.0100	0.0590	0.1170	0.0115	0.0530	0.1020
$(\boldsymbol{\mu}_3^{(6)}, \boldsymbol{\lambda}_2^{(6)})$	0.0140	0.0615	0.1025	0.0135	0.0605	0.1115	0.0085	0.0635	0.1245
$(\boldsymbol{\mu}_3^{(6)}, \boldsymbol{\lambda}_3^{(6)})$	0.0120	0.0565	0.1040	0.0150	0.0680	0.1270	0.0120	0.0555	0.1195

Table 3
Simulated Type-I errors of the proposed multiple comparison procedure for ten groups: numbers in table are simulated Type-I errors.

$(\boldsymbol{\mu}, \boldsymbol{\sigma})$	$\mathbf{n}_1^{(10)}$			$\mathbf{n}_2^{(10)}$			$\mathbf{n}_3^{(10)}$		
	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
$(\boldsymbol{\mu}_1^{(10)}, \boldsymbol{\lambda}_1^{(10)})$	0.0105	0.0545	0.1125	0.0120	0.0560	0.1110	0.0055	0.0485	0.0920
$(\boldsymbol{\mu}_1^{(10)}, \boldsymbol{\lambda}_2^{(10)})$	0.0055	0.0390	0.0805	0.0090	0.0370	0.0765	0.0055	0.0385	0.0785
$(\boldsymbol{\mu}_1^{(10)}, \boldsymbol{\lambda}_3^{(10)})$	0.0155	0.0610	0.1100	0.0105	0.0445	0.1045	0.0110	0.0405	0.0805
$(\boldsymbol{\mu}_2^{(10)}, \boldsymbol{\lambda}_1^{(10)})$	0.0125	0.0585	0.1155	0.0120	0.0535	0.1130	0.0140	0.0645	0.1255
$(\boldsymbol{\mu}_2^{(10)}, \boldsymbol{\lambda}_2^{(10)})$	0.0115	0.0520	0.1040	0.0105	0.0570	0.1185	0.0125	0.0520	0.1070
$(\boldsymbol{\mu}_2^{(10)}, \boldsymbol{\lambda}_3^{(10)})$	0.0130	0.0620	0.1200	0.0150	0.0690	0.1240	0.0165	0.0660	0.1180
$(\boldsymbol{\mu}_3^{(10)}, \boldsymbol{\lambda}_1^{(10)})$	0.0165	0.0680	0.1245	0.0190	0.0725	0.1350	0.0135	0.0715	0.1320
$(\boldsymbol{\mu}_3^{(10)}, \boldsymbol{\lambda}_2^{(10)})$	0.0200	0.0670	0.1360	0.0160	0.0870	0.1510	0.0200	0.0820	0.1555
$(\boldsymbol{\mu}_3^{(10)}, \boldsymbol{\lambda}_3^{(10)})$	0.0145	0.0610	0.1190	0.0150	0.0755	0.1330	0.0135	0.0655	0.1250

Table 4
Simulation result for power of the multiple comparisons: numbers in table are power of the test. Group sizes are with $\mathbf{n}_{1*}^{(3)} = (10, 16, 20) = \mathbf{n}_1^{(3)}, \mathbf{n}_{2*}^{(3)} = (50, 50, 50)$ and $\mathbf{n}_{3*}^{(3)} = (50, 80, 100)$; scale parameters are with $\boldsymbol{\lambda}_{1*}^{(3)} = (1, 1, 1), \boldsymbol{\lambda}_{2*}^{(3)} = (5, 5, 5) = \boldsymbol{\lambda}_2^{(3)}, \boldsymbol{\lambda}_{3*}^{(3)} = (1, 5, 10)$.

$\boldsymbol{\mu}_{h*}^{(3)}$	$(\boldsymbol{\lambda}_{h*}^{(3)}, \mathbf{n}_{h*}^{(3)})$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
$\boldsymbol{\mu}_{1*}^{(3)} = (1, 2, 1)$	$(\boldsymbol{\lambda}_{1*}^{(3)}, \mathbf{n}_{1*}^{(3)})$	0.1040	0.2930	0.4190
	$(\boldsymbol{\lambda}_{1*}^{(3)}, \mathbf{n}_{2*}^{(3)})$	0.6030	0.8270	0.9025
	$(\boldsymbol{\lambda}_{1*}^{(3)}, \mathbf{n}_{3*}^{(3)})$	0.8630	0.9570	0.9795
	$(\boldsymbol{\lambda}_{2*}^{(3)}, \mathbf{n}_{1*}^{(3)})$	0.7640	0.9350	0.9655
	$(\boldsymbol{\lambda}_{3*}^{(3)}, \mathbf{n}_{1*}^{(3)})$	0.8905	0.9720	0.9900
$\boldsymbol{\mu}_{2*}^{(3)} = (1, 5, 1)$	$(\boldsymbol{\lambda}_{1*}^{(3)}, \mathbf{n}_{1*}^{(3)})$	0.5110	0.7975	0.8785
$\boldsymbol{\mu}_{3*}^{(3)} = (1, 5, 10)$	$(\boldsymbol{\lambda}_{1*}^{(3)}, \mathbf{n}_{1*}^{(3)})$	0.3240	0.6660	0.8240
	$(\boldsymbol{\lambda}_{3*}^{(3)}, \mathbf{n}_{1*}^{(3)})$	0.3920	0.7655	0.9115
All other combinations		1	1	1

5. Example

In this section, we use an example (follows from Ye et al., 2010) to illustrate the usage of the proposed FGPQ-based SCI in practice. This data set was provided by National Transportation Safety Administration. The experiments were given by crashing the stock automobiles into a wall at 35MPH with dummies in the driver and front passenger seat. The response variable “injury” describes the extent of head injuries, chest deceleration, and left and right femur load. We consider simultaneous comparisons (refer to (7)) of the left femur load injuries among three car makes: Dodge (group 1), Honda (group 2) and Hyundai (group 3). The summary statistics are as follows: $n_1 = 8, n_2 = 7, n_3 = 5$; $\bar{x}_1 = 8.578, \bar{x}_2 = 8.053, \bar{x}_3 = 15.968$; $v_1 = 0.0254$, $v_2 = 0.0214$, and $v_3 = 0.0164$. Using our proposed method, we found that $\mu_1 - \mu_2 \in (-5.3775, 6.4275)$, $\mu_1 - \mu_3 \in (-18.3691, 3.5891)$, $\mu_2 - \mu_3 \in (-18.5530, 2.7230)$. We see that the confidence intervals involving μ_3 are wider than the confidence interval of $\mu_1 - \mu_2$, as $\bar{x}_3$ is far different from $\bar{x}_1$ and $\bar{x}_2$. Since the simultaneous confidence intervals all include 0, we conclude that simultaneously, the left femur loads are not different among these three car makes at 5% nominal level. This result is consistent with the overall tests by Tian (2006) and Ye et al. (2010).

6. Conclusions

IG distribution is widely used to describe and analyze positively right-skewed data. To our knowledge, there is no previous work on simultaneous pairwise comparisons of IG distributions. In this article, we propose an FGPQ-based new method to construct simultaneous confidence intervals for means from several IG distributions. Simulation studies show that these intervals perform well from Type-I error and power perspective. We also prove that the constructed confidence intervals have correct asymptotic coverage probabilities. The proposed methods could be applied to group mean comparisons when data are arising from IG distributions. The approach in constructing the SCIs could be extended to all pairwise comparisons from multivariate IG distributions.

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Appendix

Proof of Theorem 1.

Proof. By the central limit theorem, we have

$$\sqrt{N}\left((\eta_{12}-\theta_{12}),(\eta_{13}-\theta_{13}),\ldots,(\eta_{k-1,k}-\theta_{k-1,k})\right)\xrightarrow{d}N(0,\mathbf{U}),$$

where $\mathbf{U}$ is an $k(k-1)/2\times k(k-1)/2$ positive definite matrix and u_{ab}, $a,b=1,2,\cdots,k(k-1)/2$ is the (a,b)th entry. Let ξ_{ij} be the variance of η_{ij}. It can be shown that

$$\xi_{ij}=\mathrm{Var}\left\{E(R_{\theta_{ij}}|\bar{\mathbf{X}},\mathbf{V})\right\}=\mathrm{Var}\left(\frac{1}{\bar{X}_i}-\frac{1}{\bar{X}_j}\right)=\frac{\theta_i}{n_i\lambda_i}+\frac{\theta_j}{n_j\lambda_j}+\frac{2}{(n_i\lambda_i)^2}+\frac{2}{(n_j\lambda_j)^2},$$

$$u_{aa}=\frac{\theta_i}{\tau_i\lambda_i}+\frac{\theta_j}{\tau_j\lambda_j}$$

and

$$NV_{ij}\to\frac{\theta_i}{\tau_i\lambda_i}+\frac{\theta_j}{\tau_i\lambda_j}$$

almost surely. Therefore,

$$\left(\frac{\eta_{12}-\theta_{12}}{\sqrt{V_{12}}},\frac{\eta_{13}-\theta_{13}}{\sqrt{V_{13}}},\ldots,\frac{\eta_{k-1,k}-\theta_{k-1,k}}{\sqrt{V_{k-1,k}}}\right)\xrightarrow{d}N(0,\mathbf{U}^*),$$

where the (a,b)th entry of $\mathbf{U}^*$ is $u_{ab}/\sqrt{u_{aa}u_{bb}}$. Take a random vector $(Z_1,Z_2,\ldots,Z_{k(k-1)/2})$ distributed according to $N(0,\mathbf{U}^*)$. By the continuous mapping theorem

$$\max_{i<j}\left|\frac{\theta_{ij}-\eta_{ij}}{\sqrt{V_{ij}}}\right|\xrightarrow{d}\max|Z_a|$$

for $1\le a\le k(k-1)/2$.

For $i=1,\ldots,k$, $U_i^2/n_i\xrightarrow{p}1$. For all $i\neq j$,

$$\begin{aligned}\sqrt{N}(R_{\theta_{ij}}-\eta_{ij})&=\frac{\left|1+Z_i\sqrt{\frac{\bar{X}_iV_i}{U_i^2}}\right|}{\bar{x}_i}-\frac{\left|1+Z_i\sqrt{\frac{\bar{X}_iV_i}{U_i^2}}\right|}{\bar{x}_i}-\frac{1}{\bar{X}_i/\sqrt{n}}+\frac{1}{\bar{X}_j/\sqrt{n}}\\&=Z_i\sqrt{\frac{\theta_i}{Z_i\lambda_i}}-Z_j\sqrt{\frac{\theta_j}{Z_j\lambda_j}}+o_p(1)\end{aligned}\qquad(13)$$

conditionally on $T=(\bar{\mathbf{X}},\mathbf{V})$ almost surely.

Recall that $NV_{ij}\to\frac{\theta_i}{\tau_i\lambda_i}+\frac{\theta_j}{\tau_i\lambda_j}$ almost surely and note that (13) implies

$$\max_{i<j}\left|\frac{R_{\theta_{ij}}-\eta_{ij}}{\sqrt{V_{ij}}}\right|\xrightarrow{d}\max_{1\le a\le k(k-1)/2}|Z_a|\qquad(14)$$

on T almost surely. Let F be the cumulative distribution function of $\max_{1\le a\le k(k-1)/2}|Z_a|$. By the continuity of F

$$\sup_x |F_n(x|T) - F(x)| \to 0$$

almost surely, where F_n is the conditional distribution function of the left side of (14). As a result,

$$\begin{aligned} P\left(\theta_{ij} \in \eta_{ij} \pm q(\alpha)\sqrt{V_{ij}} \text{ for all } i<j\right) &= P\left\{F_n\left(\max_{i<j}\left|\frac{\theta_{ij}-\eta_{ij}}{\sqrt{V_{ij}}}\right|\,\middle|\,T\right) \le 1-\alpha\right\} \\ &= P\left\{F\left(\max_{i<j}\left|\frac{\theta_{ij}-\eta_{ij}}{\sqrt{V_{ij}}}\right|\right) + o_p(1) \le 1-\alpha\right\} \\ &\xrightarrow{d} 1-\alpha. \quad \square \end{aligned}$$

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