ACF and PACF of an AR(p)

- We will only present the general ideas on how to obtain the ACF and PACF of an AR(p) model since the details follow closely the AR(1) and AR(2) cases presented before.

- Recall that AR(p) model is given by the equation

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + \omega_t$$

- For the ACF, first we multiply by X_{t-k} both side of the autoregressive model equation to obtain,

$$X_{t-k} X_t = \phi_1 X_{t-k} X_{t-1} + \ldots + \phi_p X_{t-k} X_{t-p} + \omega_t X_{t-k}$$

- By taking expectation at both sides this equation, we get

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \ldots + \phi_p \rho_{k-p}$$

  or written in lag-operator

$$\Phi(B)\rho_k = 0$$

  with $\Phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p$

- The implied set of equations for different values of $k = 1, 2, \ldots,$ are known as *Yule-Walker* equations.
- We try again a solution of the form $\rho_k = \lambda^k$. which leads to the equation

$$\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - \ldots - \phi_p = 0$$

- The solutions to this equation are the reciprocal roots

$\alpha_1, \alpha_2, \ldots, \alpha_p$ of the characteristic polynomial $\Phi(B)$

- The general solution for the difference equation is

$$\rho_k = \sum_{i=1}^{p} A_i(\alpha_i)^k = A_1(\alpha_1)^k + A_2(\alpha_2)^k + \ldots A_p(\alpha_p)^k$$

- ρ_k has an exponential behavior and cyclical patterns (damped sine wave) may appear if some of the $\alpha_j' s$ are complex numbers

- **Theorem.** Given a general difference equation of the form $C(B)Z_t = 0$ where $C(B) = 1 + C_1 B + C_2 B^2 + \ldots C_n B^n$ and $C(B) = \prod_{i=1}^{n}(1 - R_i B)$ so the $R_i' s$ are the reciprocal roots of the equation $C(B) = 0$, we have that the solution is $Z_t = \sum_{i=1}^{n} A_i R_i^t$ (without proof).

- For the PACF we can apply Cramer's rule for $k = 1, \ldots, p$ which can gives us an expression for P_{kk}.
- If $k > p$, then $P_{kk} = 0$ so the PACF of an AR(p) must cut down to zero after lag $k = p$, where p is the order of the AR model.

ACF and PACF for Moving Average models

- Lets start with the MA(1) given the equation

$$X_t = \omega_t + \theta\omega_{t-1}$$

  with θ the model parameter and $\omega_t \sim N(0, \sigma^2)$

- Lets find an expression for the ACF, ρ_k
- For lag $k = 0$

$$\gamma_0 = Var(X_t) = Var(\omega_t) + \theta^2 Var(\omega_{t-1}) = \sigma^2(1 + \theta^2)$$

- For lag 1, consider the product $X_t X_{t-1}$. Using the MA(1) equation, we obtain

$$\begin{aligned} X_t X_{t-1} &= (\omega_t + \theta\omega_{t-1})(\omega_{t-1} + \theta\omega_{t-2}) \\ &= \omega_t\omega_{t-1} + \theta{\omega_{t-1}}^2 + \theta\omega_{t-2}\omega_t + \theta^2\omega_{t-1}\omega_{t-2} \end{aligned}$$

- If we take expected value at both sides of the equation,

$$E(X_t X_{t-1}) = \gamma_1 = \theta\sigma^2$$

- Additionally for any value of k

$$X_t X_{t-k} = \omega_t\omega_{t-k} + \theta\omega_t\omega_{t-k-1} + \theta\omega_{t-1}\omega_{t-k} + \theta^2\omega_{t-1}\omega_{t-k-1}$$

- Then, if $k > 1$, $E(X_t X_{t-k}) = \gamma_k = 0$

- The ACF of an MA(1) is given by

$$\rho_k = \begin{cases} \theta/(1+\theta^2); & k = 1 \\ 0 & k > 1 \end{cases}$$

- Using $\rho_k = 0$ for $k > 1$,we can show that the PACF

$$P_{kk} = \phi_{kk} = \frac{\theta^k(1-\theta^2)}{1-\theta^{2(k+1)}} \quad k \geq 1$$

- Contrary to its ACF, which cuts off after lag 1, the PACF of an MA(1) model decays exponentially.

- For a general MA(q) process, the ACF "cuts down" to zero after lag q and the PACF will have exponential behavior depending on the characteristic roots of $\Theta(B) = (1 + \theta_1 B + \theta_2 B^2 \ldots + \theta_q B^q) = 0$.

- Instead of trying to find equations in the general case, we will look at somes examples using simulation.
- Firstly, we consider an MA(1) process with parameters $\theta = 0.9$ and $\theta = -0.9$.
- Then, we consider an MA(2) process with parameters $(\theta_1 = 0.85, \theta_2 = 0.5)$ and with parameters $(\theta_1 = -0.85, \theta_2 = -0.5)$
- Finally, we consider an MA(4) process with parameters $(\theta_1 = 0.9, \theta_2 = -0.8, \theta_3 = 0.75, \theta_4 = -0.4)$
- Again, we are using this function *arima.sim*

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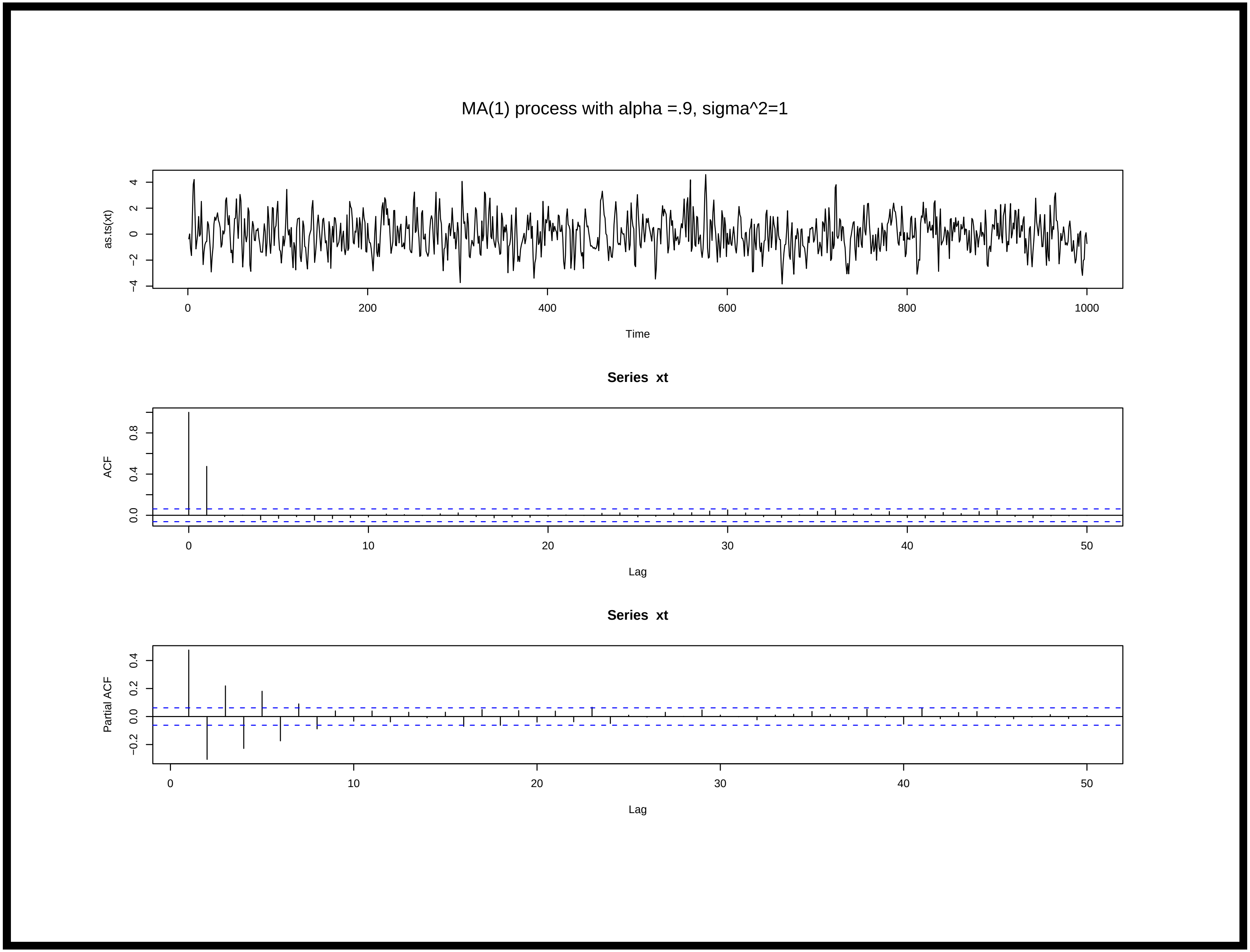
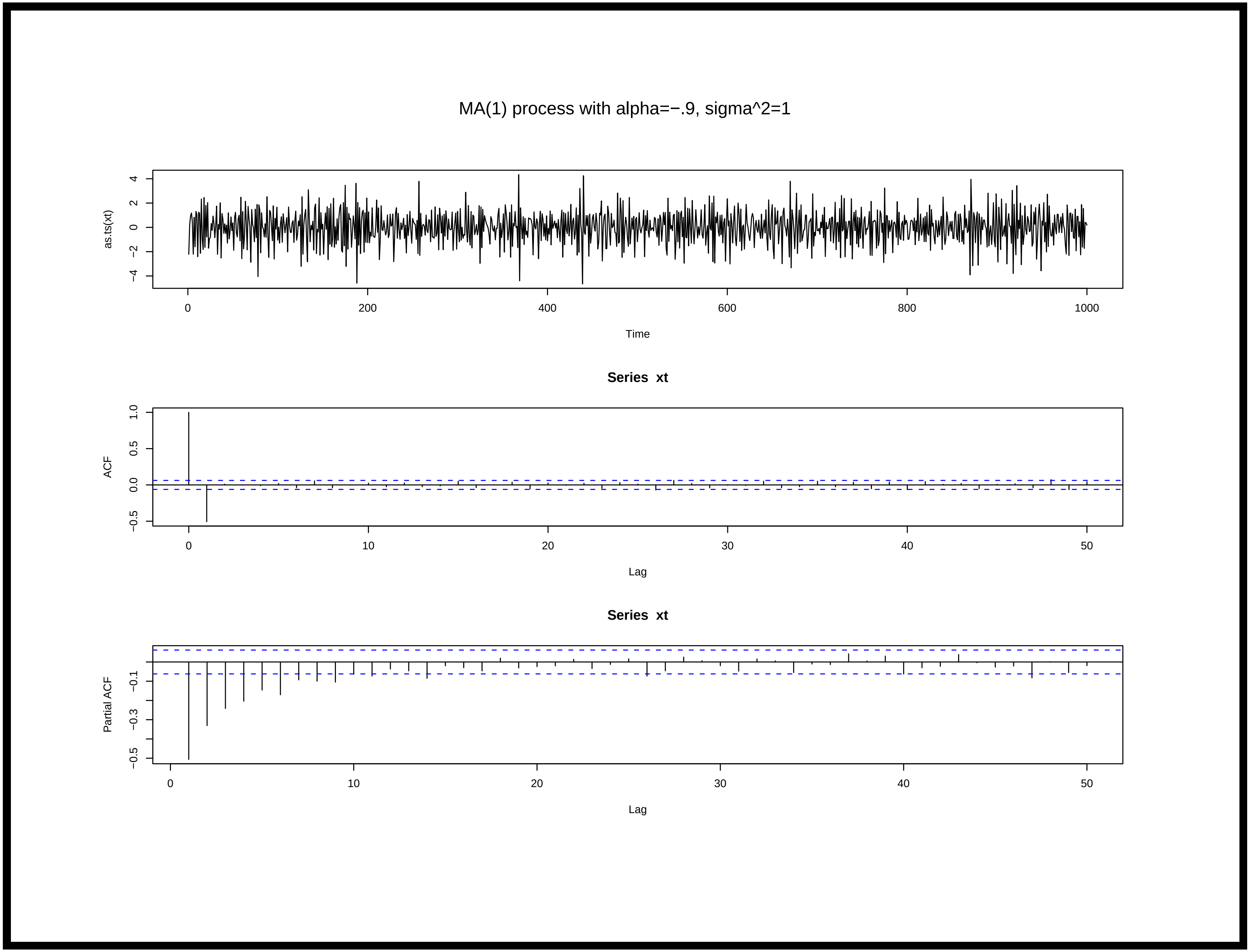
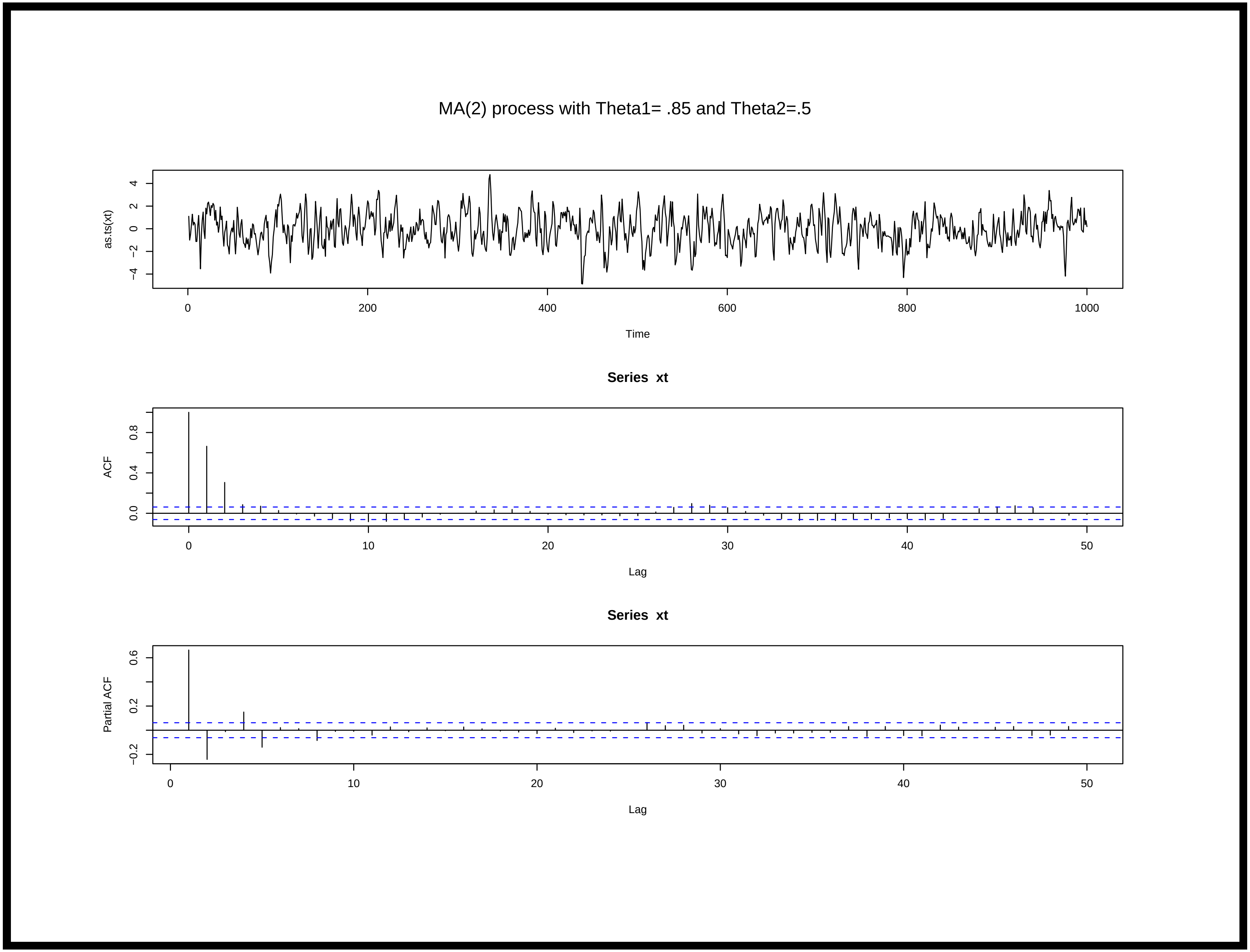
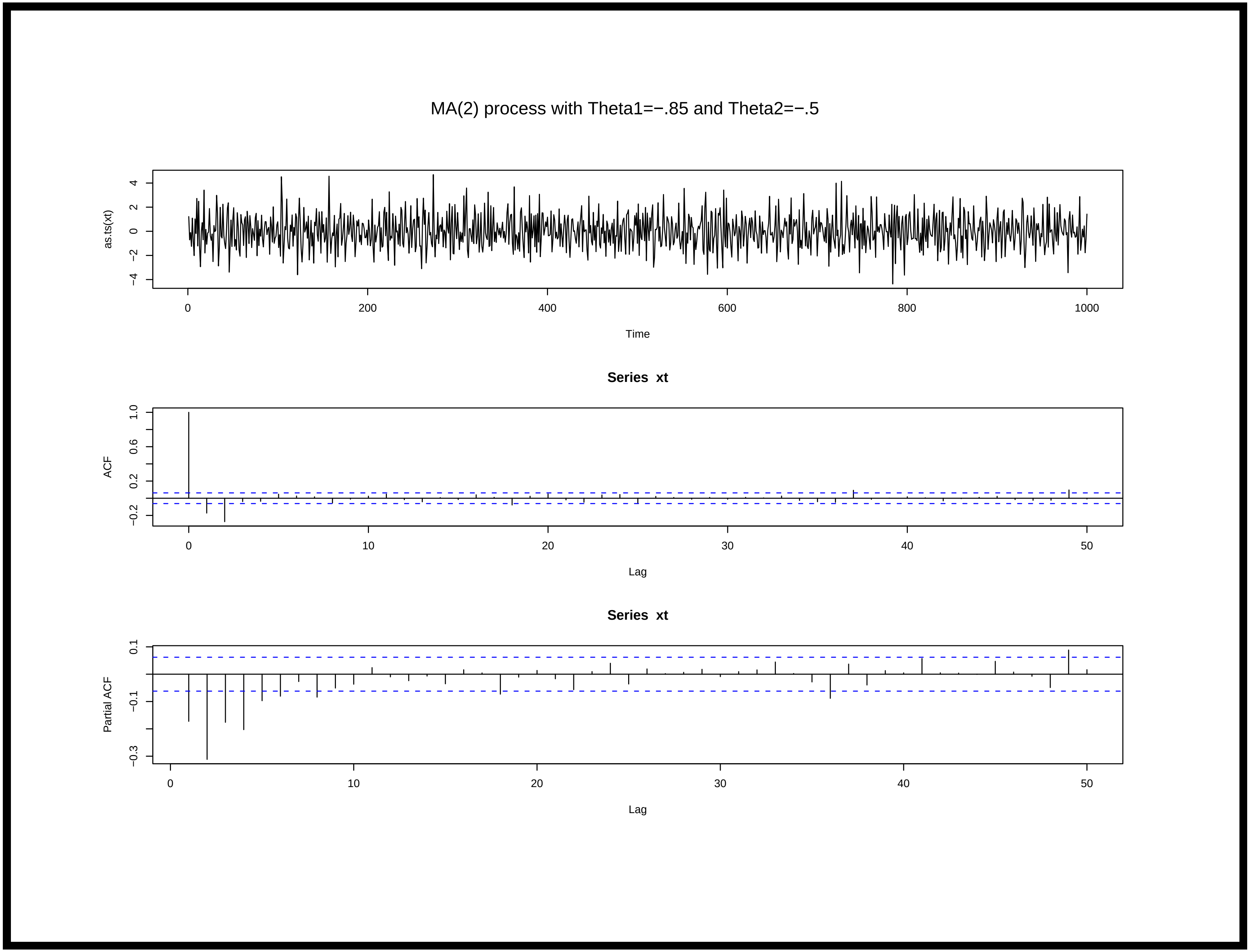
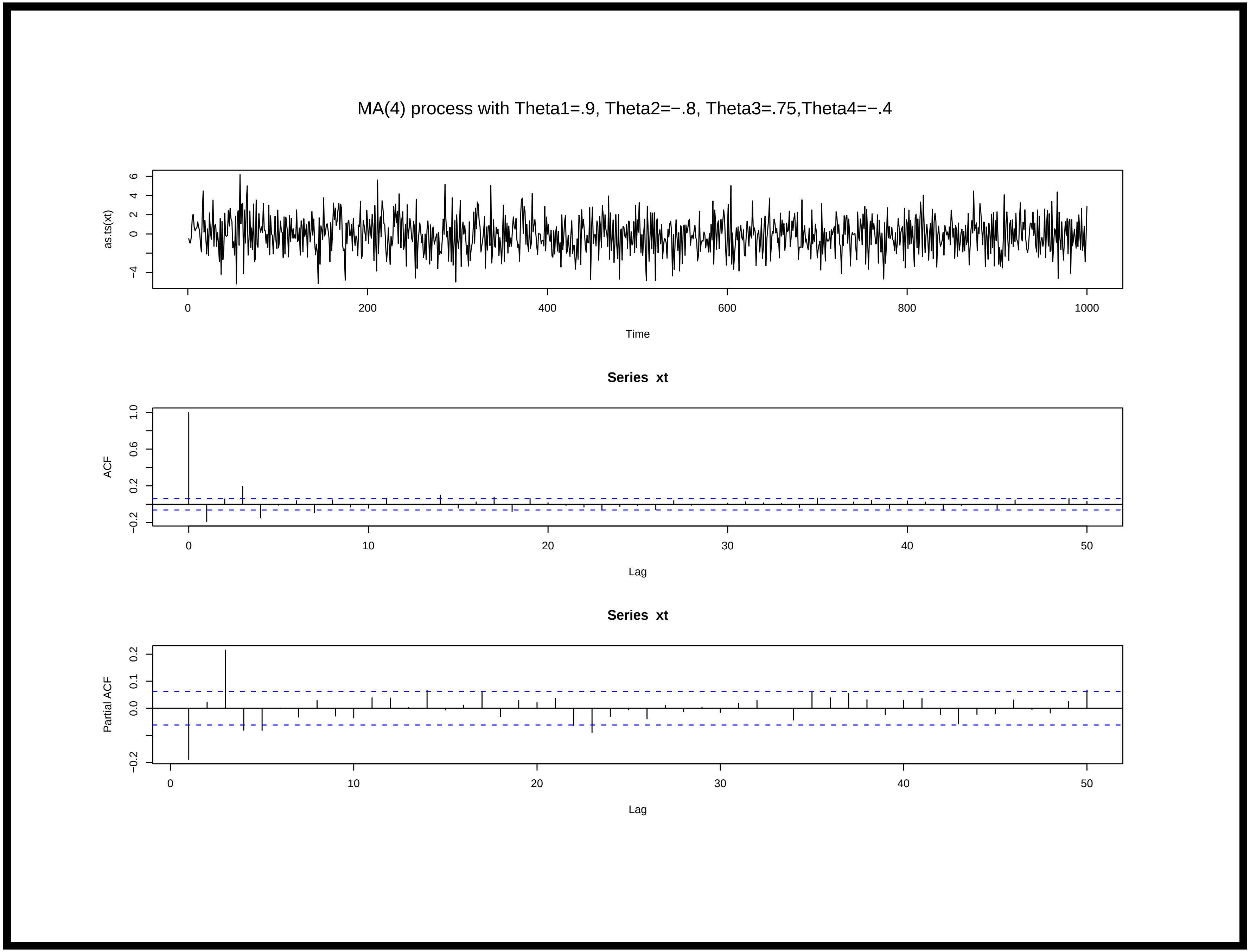
xt=arima.sim(1000,model=list(ma=0.9))
```


MA(1) process with alpha =.9, sigma^2=1
as.ts(xt)
Time
Series xt
ACF
Lag
Series xt
Partial ACF
Lag


MA(1) process with alpha=−.9, sigma^2=1
as.ts(xt)
Time
Series xt
ACF
Lag
Series xt
Partial ACF
Lag


MA(2) process with Theta1= .85 and Theta2=.5
as.ts(xt)
Time
Series xt
ACF
Lag
Series xt
Partial ACF
Lag


MA(2) process with Theta1=−.85 and Theta2=−.5
as.ts(xt)
Time
Series xt
ACF
Lag
Series xt
Partial ACF
Lag


MA(4) process with Theta1=.9, Theta2=−.8, Theta3=.75,Theta4=−.4
as.ts(xt)
Time
Series xt
ACF
Lag
Series xt
Partial ACF
Lag

- To obtain the ACF and PACF for an ARMA(p,q) process, we need to follow the same strategy used to obtain the ACFs and PACFs for AR and MA models.

- Recall that an ARMA(p,q) process is defined by the equation,

$$X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + \omega_t + \theta_1 \omega_{t-1} + \ldots + \theta_q \omega_{t-q}$$

- As before, if we multiply by X_{t-k} both sides of the equation and take "expected value", we obtain

$$\begin{aligned} \gamma_k &= \phi_1 \gamma_{k-1} + \ldots + \phi_p \gamma_{k-p} + E(X_{t-k}\omega_t) - \theta_1 E(X_{t-k}\omega_{t-1}) - \\ & \ldots - \theta_q E(X_{t-k}\omega_{t-q}) \end{aligned}$$

- Since

$$E(X_{t-k}\omega_{t-i}) = 0; \quad k > i$$

we obtain that

$$\gamma_k = \phi_1 \gamma_{k-1} + \ldots + \phi_p \gamma_{k-p}; \quad k \ \geq \ q+1$$

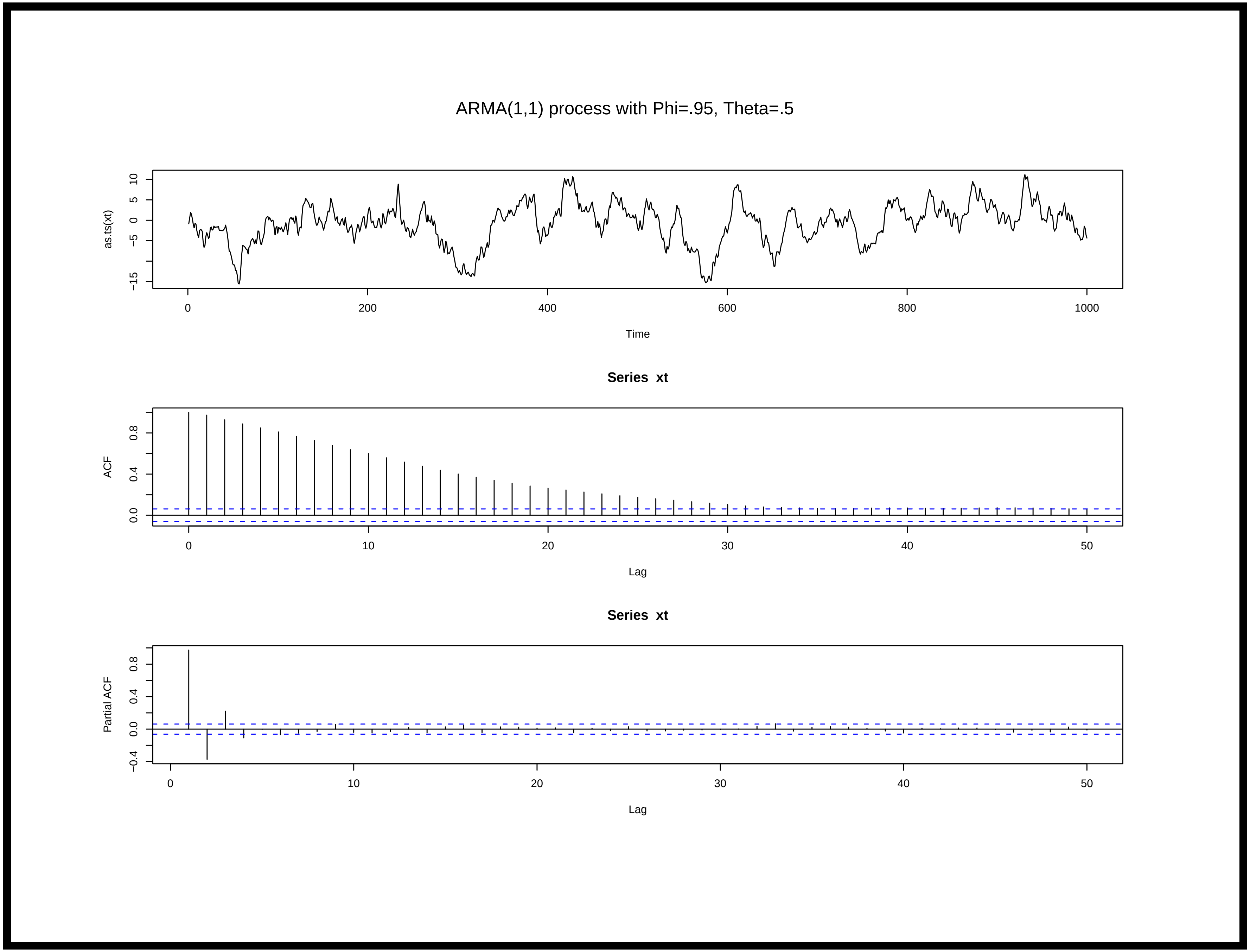
- We know (divide by γ_0) that this equivalent to

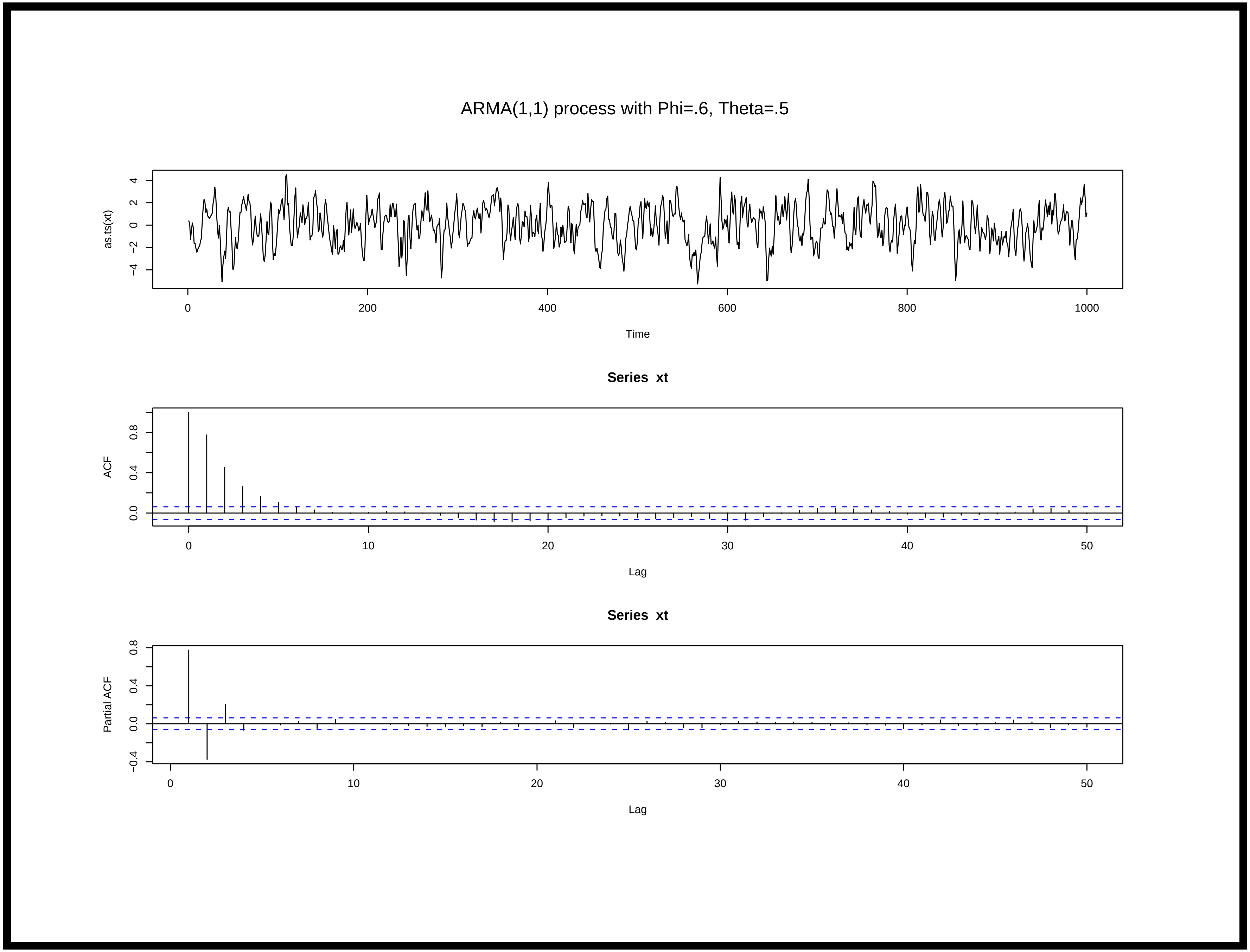
$$\rho_k = \phi_1 \rho_{k-1} + \ldots + \phi_p \rho_{k-p}; \ k \ \geq \ q+1$$

which gives the Yule-Walker equations but with the restriction $k \geq q + 1$.

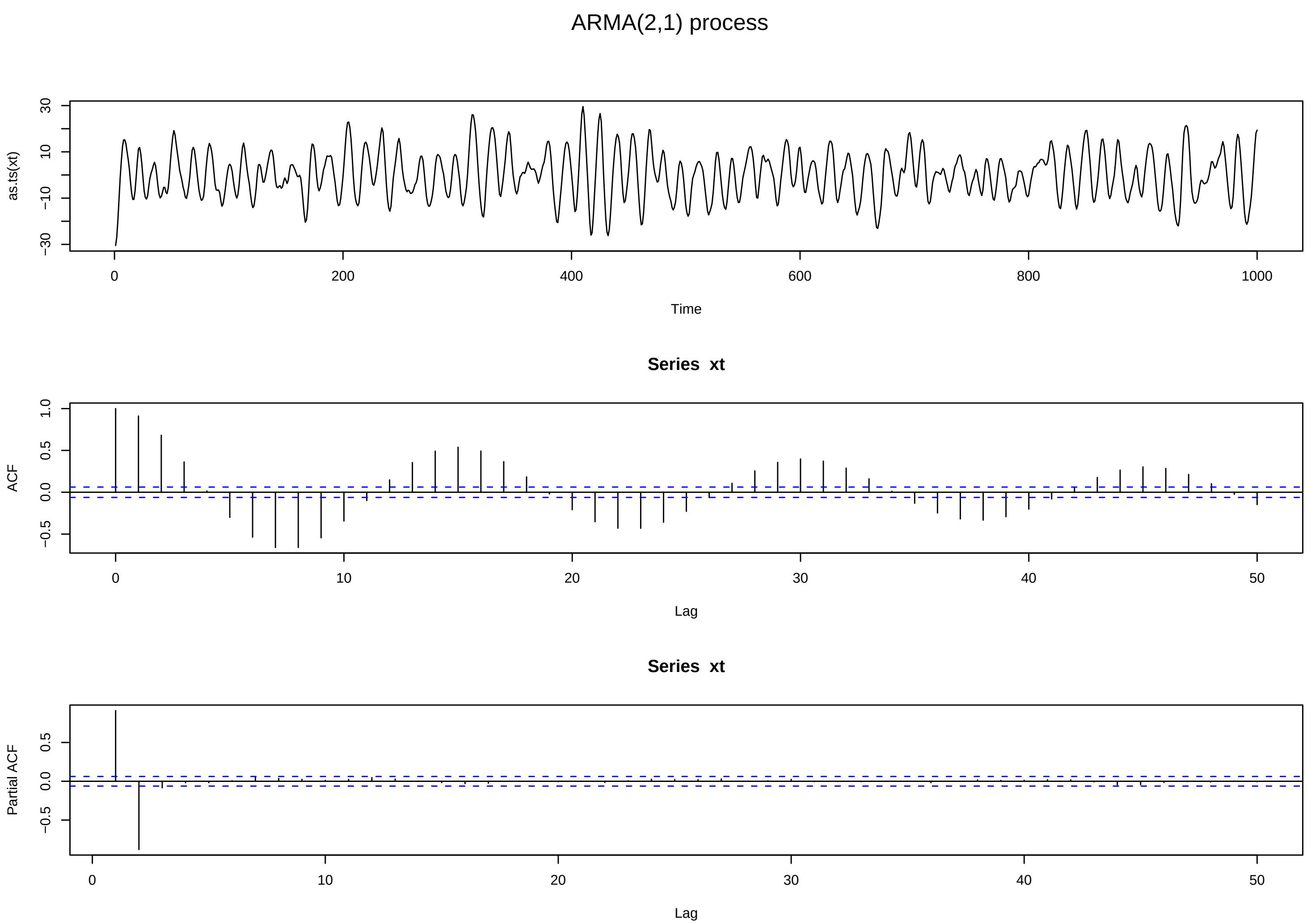
- For a lag $k \geq q + 1$, the autocorrelation function of an ARMA(p,q) process has a similar behavior to the ACF of a pure AR(p) process.
- However, the first q autocorrelations ρ_1,ρ_2,...ρ_q depend on both autoregressive and moving average parameters.
- The PACF for an ARMA (p,q) is complicated and usually not needed.

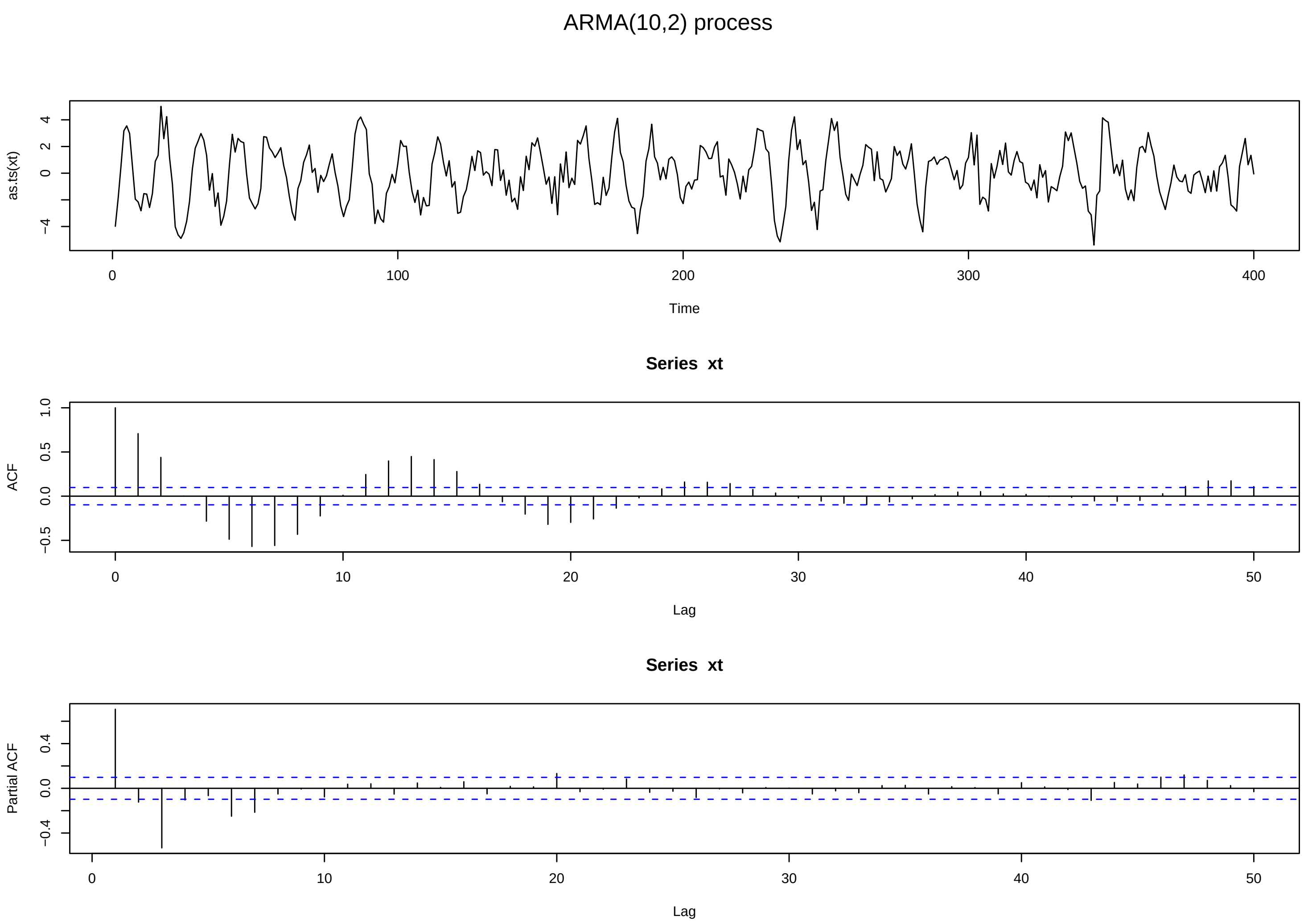
- This PACF will have a similar behavior as the PACF of a MA(q) process.
- Lets look at some examples for simulated data of an ARMA(1,1) processes.
- The examples consider 1000 simulations. The AR coefficient is 0.95 (0.6) and MA coefficient is 0.5.
- We will also consider an ARMA(2,1) process where the AR part is built with $r = 0.95$ $\omega = 0.42$ and the MA parameter is $\theta = 0.7$.
- Finally, we will show an ARMA(10,2) process where AR part is defiened with 10 complex reciprocal roots.


ARMA(1,1) process with Phi=.95, Theta=.5
as.ts(xt)
Time
Series xt
ACF
Lag
Series xt
Partial ACF
Lag



ARMA(1,1) process with Phi=.6, Theta=.5
as.ts(xt)
Time
Series xt
ACF
Lag
Series xt
Partial ACF
Lag


ARMA(2,1) process
as.ts(xt)
Time
Series xt
ACF
Lag
Series xt
Partial ACF
Lag



ARMA(10,2) process
as.ts(xt)
−4
0
2
4
0
100
200
300
400
Time
Series xt
ACF
−0.5
0.0
0.5
1.0
0
10
20
30
40
50
Lag
Series xt
Partial ACF
−0.4
0.0
0.4
0
10
20
30
40
50
Lag