Spectral Distribution and Density Functions

- We started with the basic model $X_t = Rcos(\omega t) + \epsilon_t$ where ω is the 'dominant' frequency; $f = \omega/2\pi$ is the number of cycles per unit of time and $\lambda = 2\pi/\omega$ is the 'dominant' wavelength or period.
- This model can be generalized to

$$X_t = \sum_{j=1}^{k} R_j \; cos(\omega_j t + \phi_j) + \epsilon_t$$

  which considers the existence of k-relevant frequencies $\omega_1, \omega_2, \ldots, \omega_k$.
- Given the trigonometric identity

$cos(x+y) = cos(x)cos(y) - sin(x)sin(y)$, we have that

$$X_t = \sum_{j=1}^{k} (a_j \ cos(\omega_j t) + b_j \ sin(\omega_j t)) + \epsilon_t$$

with $a_j = R_j cos(\phi_j)$ and $b_j = -R_j sin(\phi_j)$.

- By making $k \rightarrow \infty$, it can be shown that

$$X_t = \int_0^{\pi} cos(\omega t) du(\omega) + \int_0^{\pi} sin(\omega t) dv(\omega)$$

where $u(\omega)$ and $v(\omega)$ are continuous stochastic processes. This is the *spectral* representation of X_t.

- The *Wiener-Khintchine Theorem* says that if $\gamma(k)$ is the autocovariance function of X_t, there must exist a monotonically increasing function $F(\omega)$ such that

$$\gamma(k) = \int_0^{\pi} cos(\omega k) dF(\omega)$$

- The function $F(\omega)$ is the spectral distribution function of the process X_t.
- Notice that for $k = 0$,

$$\gamma(0) = \int_0^{\pi} dF(\omega) = F(\pi) = \sigma_x^2$$

so all other variation in the process is for $0 < \omega < \pi$.
- We can redefine the spectral distribution function as:

$$F^*(\omega) = F(\omega)/\sigma_x^2$$

and so $F^*(\omega)$ is the proportion of variance accounted by ω.

- Also notice that $F^*(0) = 0$, $F^*(\pi) = 1$ and since $F(\omega)$ is monotonically increasing then $F^*(\omega)$ is a cummulative distribution function (CDF).
- The *Spectral Density* function is denoted by $f(\omega)$ and defined as
$$f(\omega) = \frac{dF(\omega)}{d\omega}; \; 0 < \omega < \pi$$
- This function is also known as the *power spectral function* or *spectrum*
- The existence of $f(\omega)$ is under the assumption that the spectral distribution function is differentiable everywhere (except in a set of measure zero).
- This spectral density gives us an alternative

representation for the covariance function

$$\gamma(k) = \int_0^{\pi} cos(\omega k) f(\omega) d\omega$$

This characterization is also known as *Wold's* Theorem.

- If the spectrum has a 'peak' at ω_0, this implies that ω_0 is an important frequency of the process X_t.
- The spectrum or spectral density is a theoretical function of the process X_t . In practice, the spectrum is usually unknown and we use the *periodogram* to estimate it.
- There is an *inverse* relationship between the $f(\omega)$ and $\gamma(k)$,

$$f(\omega) = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \gamma(k) e^{-i\omega k}$$

so the spectrum is the Fourier transformation of the autocovariance function.

- From complex analysis, recall that

$$e^{-i\omega k} = cos(\omega k) - sin(\omega k)i$$

- This implies that

$$f(\omega) = \frac{1}{\pi}\left[\gamma(0) + 2\sum_{k=1}^{\infty}\gamma(k)cos(\omega k)\right]$$

- The *normalized* spectral density $f^*(\omega)$ is defined as:

$$f^*(\omega) = \frac{f(\omega)}{\sigma_x^2} = \frac{dF^*(\omega)}{d\omega}$$

- Then,

$$f^*(\omega) = \frac{1}{\pi}\left[1 + 2\sum_{k=1}^{\infty}\rho(k)cos(\omega k)\right]$$

so the *normalized* spectrum is the Fourier transform of the autocorrelation function (ACF).

- **Example 1:** White noise process. Suppose that X_t is a purely random process where $E(X_t) = 0$ and $Var(X_t) = \sigma^2$. The autocovariance function is $\gamma(0) = \sigma^2$ and $\gamma(k) = 0; k \neq 0$. Thus, the spectral density function is given by

$$f(\omega) = \sigma^2/\pi$$

- **Example 2:** Consider a first order autoregressive (AR) process

$$X_t = \alpha X_{t-1} + \epsilon_t; \epsilon_t \sim N(0, \sigma^2)$$

The autocovariance function of this process is given by

$$\gamma(k) = \frac{\sigma^2 \alpha^{|k|}}{(1-\alpha^2)} = \sigma_x^2 \alpha^{|k|}; k = 0, \pm 1, \pm 2, \dots$$

Then, the spectral density function is given by

$$f(\omega) = \frac{\sigma^2}{\pi}\left(1 + \sum_{k=1}^{\infty} \alpha^k e^{-ik\omega} + \sum_{k=1}^{\infty} \alpha e^{ik\omega}\right)$$

after some algebra, this gives

$$f(\omega) = \sigma_z^2/[\pi(1 - 2\alpha cos(\omega) + \alpha^2)]$$

- **Example 3:** Define the sequence X_t by

$$X_t = A\ cos(\theta t) + B\ sin(\theta t) + \epsilon_t$$

where ϵ_t is white noise sequence with variance σ^2, A and

B are independent random variables with mean zero and variance τ^2. It can be shown that

$$E(X_t) = 0; Var(X_t) = \tau^2 + \sigma^2$$

Also, for $t \neq s$

$$cov(X_t, X_s) = \tau^2 cos\{\theta(t-s)\}$$

Then X_t is a stationary series with autocovariance

$$\gamma(k) = \begin{cases} \sigma^2 + \tau^2, \ k = 0 \\ \tau^2 cos(k\theta), \ k \neq 0 \end{cases}$$

The spectrum can be evaluated as

$$f(\omega) \quad = \quad \sigma^2 + \tau^2 + 2\tau^2 \sum_{k=1}^{\infty} cos(k\theta) cos(k\omega)$$

$$= \quad \sigma^2 + \tau^2 + 2\tau^2 \sum_{k=1}^{\infty} [cos\{k(\theta + \omega)\} + cos\{k(\theta - \omega)\}]$$

If $\theta = \omega$, then $cos\{k(\theta - \omega)\} = 1$ for all k and the summation is infinite.

This means that the spectrum has a 'spike' at $\omega = \theta$.

The spectrum can only exist if we allow $f(\omega) = \infty$ at isolated values of ω.

Periodogram revisited

- For $0 < \omega < \pi$, the periodogram is defined as

$$
\begin{aligned}
I(\omega) &= \frac{n}{2}(\hat{a}^2 + \hat{b}^2) \\
&= \frac{2}{n}\left[\left(\sum_{t=1}^{n} x_t cos(\omega t)\right)^2 + \left(\sum_{t=1}^{n} x_t sin(\omega t)\right)^2\right]
\end{aligned}
$$

- If $\omega = 2\pi j/n; j < n/2$ is a *Fourier* frequency and since $\sum_t cos(\omega t) = \sum_t sin(\omega t) = 0$ then

$$
I(\omega) = \frac{2}{n}\left[\left(\sum_{t=1}^{n}(x_t - \bar{x}) cos(\omega t)\right)^2 + \left(\sum_{t=1}^{n}(x_t - \bar{x}) sin(\omega t)\right)^2\right]
$$

- Expanding each square term and by the trigonometric identities

$$\left(\frac{n}{2}\right) I(\omega) = \sum_{t=1}^{n}(x_t - \bar{x})^2 + 2\sum_{k=1}^{n-1}\sum_{t=k+1}^{n}(x_t - \bar{x})(x_{t-k} - \bar{x})cos(\omega k)$$

- This gives an alternative expression for the periodogram,

$$I(\omega) = 2\left(g_0 + 2\sum_{k=1}^{n-1} g_k cos(\omega k)\right)$$

- We also have a *normalized periodogram*

$$I^*(\omega) = \frac{I(\omega)}{g_0} = 2\left(1 + 2\sum_{k=1}^{n-1} \rho_k cos(\omega k)\right); \rho_k = g_k/g_0$$

- The last two expressions justify the used of the periodogram as an estimate of the spectral density.
- What is the sampling distribution of $I(\omega)$?

- By definition, the periodogram satisfies the relation:

$$nI(\omega) = A(\omega)^2 + B(\omega)^2$$

where

$$A(\omega) = \sum_{i=1}^{n} x_t cos(\omega t); B(\omega) = \sum_{i=1}^{n} x_t sin(\omega t)$$

- To understand the sampling distribution of the periodogram, lets suppose x_t is a realization of a white noise process (i.i.d. $X_t \sim N(0, \sigma^2)$).
- What is the distribution of $A(\omega)$ and $B(\omega)$?
- Linear combinations of normal variables are normal.
- In fact, $E(A(\omega)) = E(B(\omega)) = 0$

- Additionally, by the trigonometric identities,

$$\begin{aligned} Var(A(\omega)) &= \sigma^2 \sum_{t=1}^{n} cos^2(\omega t) = (n\sigma^2)/2 \\ Var(B(\omega)) &= \sigma^2 \sum_{t=1}^{n} sin^2(\omega t) = (n\sigma^2)/2 \end{aligned}$$

- Also,

$$\begin{aligned} Cov(A(\omega), B(\omega)) &= E\left[\sum_{t=1}^{n}\sum_{s=1}^{n} X_t X_s cos(\omega t) sin(\omega s)\right] \\ &= \sigma^2 \sum_{t=1}^{n} cos(\omega t) sin(\omega t) = 0 \end{aligned}$$

- It follows that $A(\omega)\sqrt{2/n\sigma^2}$ and $A(\omega)\sqrt{2/n\sigma^2}$ are independent Normal random variables.

- Therefore,

$$2[\{A(\omega)\}^2 + \{B(\omega)\}^2]/(n\sigma^2) \ \sim \ \chi_2^2$$

- so $2I(\omega)/\sigma^2$ is a chi-square distribution with 2 degrees of freedom or

$$I(\omega) \sim \sigma^2 \chi_2^2/2$$

- In particular,

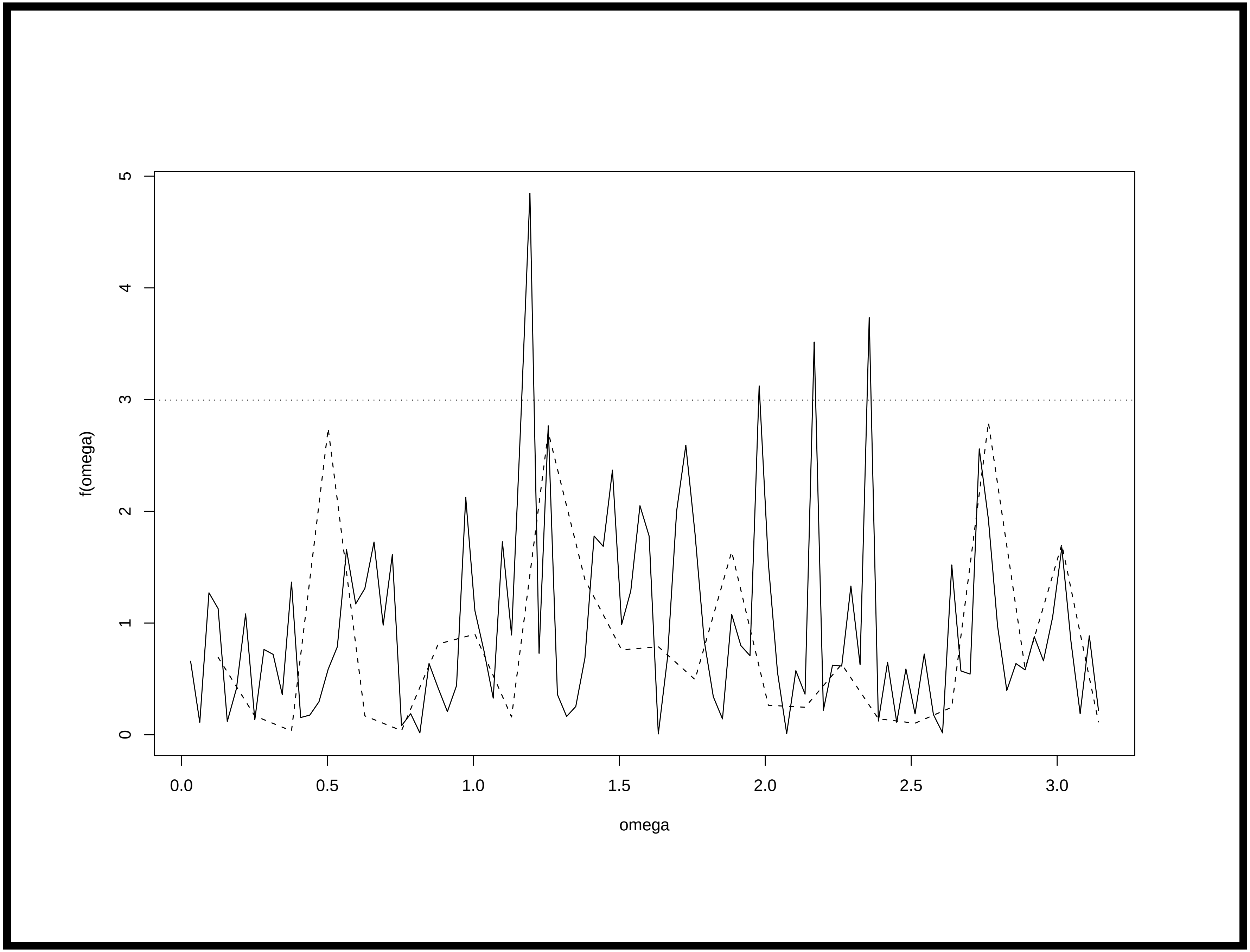
$$\begin{aligned} E(I(\omega)) &= \sigma^2 \\ V(I(\omega)) &= \sigma^4 \end{aligned}$$

- Recall that if X_t is white noise, the spectrum $f(\omega) = \sigma^2$ so in this case $I(\omega)$ is unbiased but an inconsistent estimator of $f(\omega)$

- In fact there is a theorem presented in Diggle's book (page 97) that generalizes these results to the case of a Gaussian and stationary process.

- Let X_t be a stationary and Gaussian process with spectrum $f(\omega)$. Let $x_t; t = 1 \ldots, n$ be a partial realization of this process and $I(\omega)$ the periodogram of x_t.
- Let $\omega_j = 2\pi j/n$ for $j < n/2$, then as $n \to \infty$
  1. $I(\omega_j) \sim f(\omega_j)\chi_2^2/2$
  2. $I(\omega_j)$ independent of $I(\omega_k)$ for all $j \neq k$
- As an example, consider $n = 200$ observations of a white noise process N(0,1) and its corresponding periodogram $I(\omega)$ for $n = 50$, $n = 200$.
- The obtain the periodogram, I used the following commands:

```
x <- rnorm(200)
per <- spec.pgram(x,plot=FALSE)
plot(2*pi*per$freq,per$spec,type='l',
xlab="omega",ylab="f(omega)")
per <- spec.pgram(x[1:50],plot=FALSE)
lines(2*pi*per$freq,per$spec,lty=2)
abline(h=qchisq(0.95,2)/2,lty=3)
```


f(omega)
omega
0
1
2
3
4
5
0.0
0.5
1.0
1.5
2.0
2.5
3.0

- The solid line is the periodogram for all 200 observations and the dashed line is the periodogram only for the first 50 observations.
- The horizontal line is the .95% quantile of $\chi^2_2/2$ random variable.
- Notice that variability for the periodogram based on 50 obsevations is similar to the periodogram obtained with all 200 observations.
- Only a few values of $I(\omega)$ are greater than the 0.95 quantile. These values are scattered through the frequency range.
- The quantile value gives a valid test of significance of the $\chi^2_2/2$ distribution for a prespecified value of ω.

A Test for White Noise

- To test for white noise, the proposed test statistic is to use the *maximum periodogram ordinate*

$$T = max\{I_1, I_2, \dots I_m\}$$

where $I_j = I(2\pi j/n); j < n/2$ and m is the largest integer less than $n/2$.

- We known that under the null hypothesis (i.e. X_t white noise) the periodogram ordinates I_j are a random sample with a scaled χ^2_2 distribution.

- The distribution of I_j is

$$G(u) = Pr[I_j \leq u] = 1 - exp(-u/\sigma^2)$$

- Given the mutual independence of I_j, under the white noise hypothesis, the distribution function for T is:

$$H(t) = G(t)^m = (1 - exp(-u/\sigma^2))^m$$

- In practice, usually σ^2 is unknown. We can substitute and estimaor of the variance in $H(t)$, $s^2 = \sum(x_i - \overline{x})^2/(n-1)$, to obtain an approximate test.
- Fisher(1929) deduced the **exact** distribution for $T_0 = T/\{\sum_{i=1}^m I_j/m\}$ under a white noise process:

$$Pr[T_0 > mx] = \sum_{k=1}^{r} [m!/k!(m-r)!](-1)^{k-1}(1-kx)^{m-1}$$

where r is the largest integer less than x^{-1}.
- **Example** For $n = 200$ observations following a $N(0,1)$ distribution, I obtained a value of $t = 5.867294$ and

$s^2 = 1.037762$

1. For the approximate test, the p-value is 0.296048.
2. For Fisher's test, $t_0 = 5.619912$ and the p-value is 0.2907733

```
x <- rnorm(200)
I <- spec.pgram(x,plot=F)$spec
t <- max(I)
t0 <- max(I)/mean(I)
s2 <- var(x)
m <- length(I)
1-(1-exp(-t/s2))^m
# k!=gamma(k+1)
```

Tapering

- This is an option that is available within this function spec.pgram.

  `spec.pgram(x,taper=0.2)`

- A *data taper* is a transformation of x_t into a new series by multiplying it by constants and to reduce the effect of extreme observations,

$$y_t = c_t x_t; \quad t = 1, 2, \ldots n$$

- The sequence c_t is chosen to be close to zero at the end sections of the series, but close to one towards the central part. $(0 < c_t \leq 1)$.
- If p is the proportion of observations to be tapered, n is

total number of observations and $m = np$, the *split cosine bell* taper is defined as :

$$c_t = \begin{cases} .5(1 - cos(\pi(t - .5)/m)) & t = 1, \ldots, m \\ 1 & t = m+1, \ldots, n-m \\ .5(1 - cos(\pi(n - t - .5)/m)) & t = n-m+1, \ldots, n \end{cases}$$

Smoothing the Periodogram

- If we have the spectrum $f(\omega)$ is a smooth function of ω, another periodogram based estimator of $f(\omega)$ is:

$$\hat{f}(\omega_j) = (2p+1)^{-1} \sum_{l=-p}^{p} I(\omega_{j+l})$$

- $\hat{f}(\omega)$ is a simple moving average of $I(\omega)$

- If X_t is a stationary random process with spectrum $f(\omega)$ for any Fourier frequency ω_j as $n \to \infty$
  - $\hat{f}(\omega_j) \sim f(\omega_j)\chi^2_{2(2p+1)}/(2(2p+1))$
  - $\hat{f}(\omega_j)$ is independent of $\hat{f}(\omega_k)$ whenever $j - k \geq 2p + 1$
- A general version of this estimator is defined as

$$\hat{f}(\omega_j) = \sum_{l=-p}^{p} w_l I(\omega_{j+l})$$

with $\sum_{l=-p}^{p} w_l = 1$.

- The asymptotic distribution of $\hat{f}(\omega)$ is given by

$$\hat{f}(\omega) \sim f(\omega)\chi^2_{\nu}/\nu$$

but now the degrees of freedom are defined as

$\nu = 2/\sum_{i=-p}^{p} w_l^2$

- Now recall that the periodogram can be expressed as

$$I(\omega) = g_0 + 2\sum_{k=1}^{n-1} g_k cos(k\omega)$$

- A possible explanation of why $I(\omega)$ is not such a great estimator of the spectrum is because g_k can be large when $r_k \equiv 0$, particularly for high values of k.

- As we showed before, the variability of $I(\omega)$ is not a function of the number of data points.

- Alternatively, we could use a *truncated Periodogram* defined as

$$I_K(\omega) = g_0 + 2\sum_{k=1}^{K} g_k cos(k\omega)$$

for a value of K that is less than $n-1$.

- We also have a *Lag window* Periodogram,

$$\hat{f}_\lambda(\omega) = g_0 + 2\sum_{k=1}^{n-1} \lambda_k g_k cos(k\omega)$$

where λ_k is a sequence of constants that needs to be specified by the user.

- Bartlett (1950) proposed that

$$\lambda_k = \begin{cases} 1 - k/S & k \leq S \\ 0 & k > S \end{cases}$$

- Daniell (1946) proposed a sequence which corresponds to the "spans" option of spec.pgram in R/Splus.

$$\lambda_k = sin(\pi k/S)/(\pi k/S)$$

- Parzen (1961) proposed that

$$\lambda_k = \begin{cases} 1 - 6(k/S)^2 + 6(k/S)^3 & k \leq S/2 \\ 2(1 - k/S)^3 & S/2 < k \leq S \\ 0 & k > S \end{cases}$$

where large values of S correspond to less smoothing.

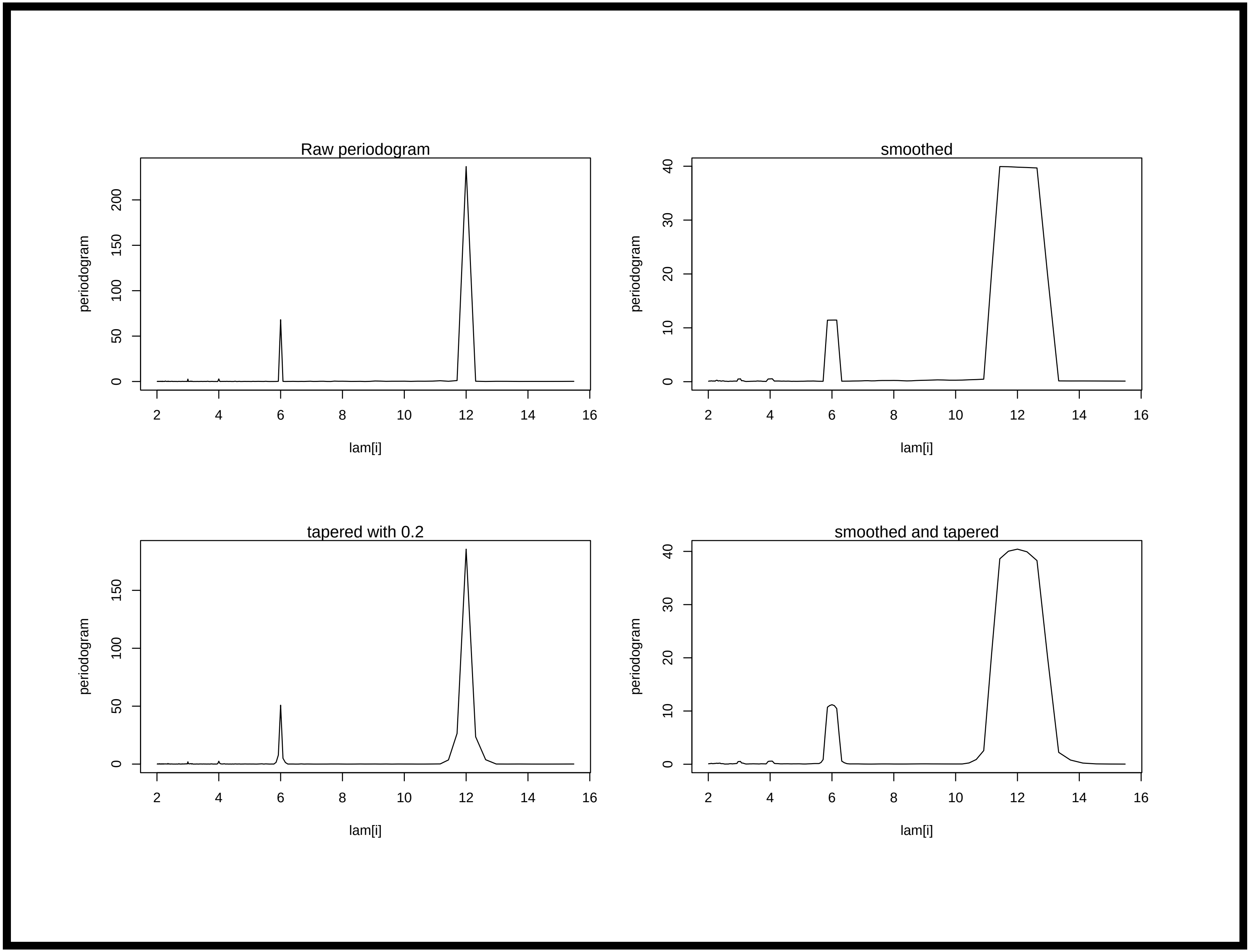
- We consider again the CO2 data and we will look into different versions of the periodogram. Here is the R code.

```
data(co2)
co2diff <- as.vector(diff(co2))
par(mfrow=c(2,2))
per<-spec.pgram(co2diff,taper=0,pad=0,detrend=F,
demean=F,plot=F)
lam<-1/per$freq
plam<-per$spec
i<-2<lam & lam<16
plot(lam[i],plam[i],type='l',ylab='periodogram')
mtext("Raw periodogram")
```

```
per<-spec.pgram(co2diff,spans=c(6),taper=0,pad=0,
detrend=F,demean=F,plot=F)

per<-spec.pgram(co2diff,taper=0.3,pad=0,detrend=F,
demean=F,plot=F)

per<-spec.pgram(co2diff,spans=c(6),taper=0.2,pad=0,
detrend=F,demean=F,plot=F)
```


Raw periodogram
smoothed
tapered with 0.2
smoothed and tapered
periodogram
lam[i]