Stat 461/561, Math 441. Probability.
Fall 2010

Name: ____________________

Take Home Final Exam. Due on Tuesday December 14 at noon HUM 443

Please read the exam questions carefully. Be clear, concise and complete to answer. The exam has 6 problems and each problem is worth equally. Students enrolled at the 400 level only need to answer Problems 1-5. Please write your answers in the space provided between questions. If you need extra space, you may write on the back or attach more pages, but please make sure you number those pages accordingly to the corresponding exam question. Justify your answers completely to receive full credit. The exam is individual so you are not allowed to discuss the exam questions with other classmates or anyone else, but you are allowed to use the textbook, web, class notes or other related materials. Please STAPLE the exam and write your name on the first page before you turn it in.

1. Show that if $X_1, X_2, \ldots, X_n$ are independent random variables and if $Y_1 = g_1(X_1)$, $Y_2 = g_2(X_2), \ldots, Y_n = g_n(X_n)$, then $Y_1, Y_2, \ldots, Y_n$ are also independent random variables. Assume that each of the functions $g_i(X_i); i = 1, 2, \ldots, n$ has an inverse function. If needed, first consider particular values $n = 2$ or $n = 3$ and then generalize the result for an arbitrary value of n.

2. Suppose that the random variable Y has a binomial distribution with n trials and success probability X, where n is a given constant and X is a Uniform(0,1) random variable.

   (a) Find $E(Y)$ and $Var(Y)$.

   (b) Find the joint distribution of X and Y.

   (c) Find the marginal distribution of Y.

3. Let $X \sim N(0, 1)$ and $Y \sim N(0, 1)$. Suppose X and Y are independent and define $U = X + Y$ and $V = X - Y$.

   (a) Find the mean and variance of U and V respectively without deriving the joint or marginal distributions of these variables.

   (b) Find the joint distribution of U and V.

   (c) Find the marginal distribution of U and V respectively.

4. A court is investigating the possible ocurrance of an unlikely event T. The reliability of two independent witnesses called Alf and Bob is known to the court: Alf tells the truth with probability α and Bob tells the truth with probability β, and there is no collusion (agreement) between the two of them. Let A and B be the events that Alf and Bob assert (repectively) that T occurred, and let $\tau = P(T)$.

   (a) What is the probability that T occurred given that both Alf and Bob declared that T occurred?

   (b) Find the probability of a) when $\alpha = \beta = 9/10$ and $\tau = 10^{-3}$ What do you think about this value as basis for a judicial conclusion.

5. Let the joint density for the random variables X and Y be defined by

$$f_{X,Y}(x, y) = 6\,(1 - y), \quad 0 < x < y < 1$$

and zero otherwise.

(a) Evaluate $E(X)$ and $E(X^2)$ and hence find an expression for $Var(X)$, where the expectations are taken with respect to $f_X(x)$.

(b) Derive the conditional density $f_{Y|X}(y|x)$ and the conditional expectation

$$E_{Y|X}[(1 - Y)|X = x]$$

for general x, $0 < x < 1$.

(c) Evaluate $P(Y < 2X)$.

6. Let the joint distribution of (X, Y) be a Bivariate Normal Distribution defined by:

$$\begin{aligned} f(x,y) &= (2\pi\sigma_x\sigma_y\sqrt{1-\rho^2})^{-1} \times \\ & exp\left[\frac{-1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right)\right] \end{aligned}$$

where $-\infty < \mu_x < \infty$; $-\infty < \mu_y < \infty$; $\sigma_x > 0$, $\sigma_y > 0$; $-1 < \rho < 1$. Lets define

$$W = \left(\frac{X-\mu_x}{\sigma_x}\right); Z = \left(\frac{Y-\mu_y}{\sigma_y}\right)$$

(a) Find the joint distribution of (W, Z), $f(w, z)$.

(b) Find the marginal distribution of W and from this derive the marginal distribution of X.

(c) Find the conditional distribution of Y given X.