Stat. Inference 553 Name: ____________________

Spring 2003

Solutions TEST No. 2

1. Let X be a single observation from the Bernoulli density $f(x|\theta) = \theta^x(1-\theta)^{1-x}$ where $0 < \theta < 1$. Let $T_1 = X$ and $T_2 = 1/2$.

   (a) Are both T_1 and T_2 unbiased estimators for θ? Is either? Justify your answer.
   Answer: $E(T_1) = E(X) = \theta$ so T_1 is unbiased. $E(T_2) = E(1/2) = 1/2$ so in general, T_2 is not unbiased except when $\theta = 1/2$.

   (b) Compare the mean square error of T_1 with that of T_2.
   Answer Since T_1 is unbiased, $MSE(T_1) = Var(T_1) = \theta(1-\theta)$. Since T_2 is constant then, $Var(T_2) = 0$ and $MSE(T_2) = (\theta - 1/2)^2$. Notice that both MSEs are quadratic functions on $(0,1)$. $MSE(T_1) < MSE(T_2)$ for θ near 0 or 1. $MSE(T_2) < MSE(T_1)$ when θ is near $1/2$.

2. Let X be a single observation with pdf $f(x|\theta) = (\theta/2)^{|x|}(1-\theta)^{1-|x|}$ where $x = -1, 0, 1$, $0 \le \theta \le 1$.

   (a) What is a maximum likelihood estimator for θ? Justify your answer.
   Answer: We have two cases. If $|x| = 0$ then $f(x|\theta) = (1-\theta)$ which is a decreasing function on θ. The maximum is reached at $\hat{\theta} = 0$. If $|x| = 1$ then $f(x|\theta) = (\theta/2)$ which is an increasing function on θ. The maximum is reached at $\hat{\theta} = 1$. Either way, the MLE is $\hat{\theta} = |X|$.
   Notice that taking the *log* of $f(x|\theta)$ and then its derivative with respect to θ, is not correct because *log* $f(x|\theta)$ is not defined for $\theta = 0$ or $\theta = 1$. Still, if you follow this approach, you may obtain that the MLE is $\hat{\theta} = |X|$. Is one of this situations where you get the right answer with the wrong procedure.!?

   (b) Is the estimator in part (a) a uniformly minimum variance unbiased estimator (UMVUE)? Justify your answer.
   Answer We know that in this problem, $T = |X|$ is a sufficient and complete statistic for θ (exponential family of distribution result). Also,

$$E(|X|) = 2(\theta/2) + 0(1-\theta) = \theta$$

   so $|X|$ is an unbiased estimator of θ. By the Lehmann-Scheffé theorem, $|X|$ is an UMVUE.
   Notice that we may use the Crámer-Rao lower bound (CRLB) approach if we impose the additional condition that $0 < \theta < 1$. Then,

$$\frac{d^2 log\ f(x|\theta)}{d\theta^2} = -\frac{|x|}{\theta^2} - \frac{(1-|x|)}{(1-\theta)^2}$$

and

$$E\left(-\frac{d^2 log\ f(x|\theta)}{d\theta^2}\right) = \frac{1}{\theta(1-\theta)}.$$

The CRLB for the class of unbiased estimators of θ is $\theta(1-\theta)$.
Finally, notice that $E(|X|^2) = \theta$ and $Var(|X|) = \theta(1-\theta)$, so $T = |X|$ is unbiased and reaches the lower bound.

(c) What is the form of the likelihood ratio test statistic for $H_0 : \theta = 0.5$ versus $H_1 : \theta \neq 0.5$.
Answer: Under H_0, the supremum of $f(x|\theta)$ is $(1/4)^{|x|}(1/2)^{1-|x|}$. Without any restrictions the supremum is $(|x|/2)^{|x|}(1-|x|)^{1-|x|}$. Then, the likelihood ratio statistic is:

$$\lambda(x) = \left(\frac{1}{2|x|}\right)^{|x|}\left(\frac{1}{2(1-|x|)}\right)^{1-|x|}$$

We reject the null hypothesis if $\lambda(x) < c$.

3. An experimenter observes X, a random sample of size one, where X follows a $N(\theta, 1)$. She wishes to test: $H_0 : \theta = 0$ versus $H_1 : \theta = 1$.

(a) Find the uniformly most powerful α level test.
Answer: This is a case of simple null hypothesis versus a simple alternative hypothesis. The UMP will be given by the Neyman-Pearson lemma. Here, $f(x|\theta) = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}(x-\theta)^2\right)$. Then

$$\frac{f(x|\theta=0)}{f(x|\theta=1)} = exp\left(-\frac{1}{2}[x^2-(x-1)^2]\right) = exp\left(-\frac{1}{2}(2x+1)\right).$$

We reject the null hypothesis if and only if $exp(-0.5(2X+1)) < c$, if and only if $X > k$.
Under the null hypothesis X is a $N(0,1)$, then k must satisfy that:

$$Pr[X > k|\theta = 0] = \alpha$$

$k = Z_{1-\alpha}$, the quantile $(1-\alpha)$ under the normal CDF. (You may also write this quantile as Z_α.)

(b) For the test of part (a), what is the probability of the type II error? You may express this probability as a function of a quantile of the N(0,1) distribution.

Answer: If β is the probability of type II error:

$$\beta = Pr[X < Z_{1-\alpha}|\theta = 1] = Pr[X - 1 < Z_{1-\alpha}] = Pr[Z < Z_{1-\alpha}] = \Phi(Z_{1-\alpha} - 1)$$

where $\Phi(\cdot)$ is the CDF of the $N(0,1)$.

(c) Now suppose that she wishes to test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. Also, suppose that H_0 is rejected if and only if $X > 1.64$. Make a sketch of the power function. Justify your answer.

Answer The power function is:

$$\beta(\theta) = Pr[X > 1.64|\theta] = Pr[Z > 1.64 - \theta] = 1 - \Phi(1.64 - \theta)$$

The sketch must show that $\beta(\cdot)$ is an strictly increasing function of θ for values of θ greater (it is a function of $\Phi(1.64 - \theta)$). Also $\beta(\theta = 0) = .05$ so the probability of the type I error is 0.05.