Statistical Inference 553. Spring 2003

Homework 6 sols. and Class Exercises 7-8

May, 2003

1. **Exercise 9.10:**

   (a) By the form (9.2.11), $f(t|\theta) = g(Q(t;\theta))\left|\frac{dQ(t;\theta)}{dt}\right|$. Since $Q(t;\theta)$ is a montone function of t, if we make $Y = Q(T;\theta)$, then $T = Q^{-1}(Y;\theta)$. It all reduces to show that the pdf of Y does not depend on θ. By applying Theorem 2.1.5 and using the form 9.2.11

   $$f_Y(y) = g(y)\left|\frac{dQ^{-1}(y;\theta)}{dt}\right|^{-1}\left|\frac{dQ^{-1}(y;\theta)}{dt}\right| = g(y)$$

   Then $Q(T;\theta)$ is a pivotal quantity.

   (b) If $g = 1$ and $Q(t;\theta) = F_\theta(t)$, then

   $$f(t|\theta) = \left|\frac{dF_\theta(t)}{dt}\right|$$

   which is a widely known relation between a pdf and a cdf.

2. **Exercise 9.13**

   (a) We need to evaluate $Pr[Y/2 < \theta < Y]$. This is equal to $Pr[-log(X)^{-1} < \theta < -log(X)^{-1}] = Pr[1/2 < -\theta log(X) < 1] = Pr[1/2 < Z < 1]$ where $Z = -\theta log(X)$. By the change of variable theorem 2.1.5, Z follows an exponential (1) distribution. Then,

   $$Pr[1/2 < Z < 1] = e^{-1/2} - e^{-1} = 0.2386$$

   This is the desired confidence coefficient.

   (b) If $Z = -\theta log(X)$, $X = exp(-Z/\theta)$. Then

   $$f_Z(z) = \theta exp(-Z/\theta)^{\theta-1}(1/\theta)exp(-Z/\theta) = exp(-Z)$$

   Then Z is a pivotal quantity and it can lead to the same confidence interval $(y/2, y)$.

   (c) The intervals are the same. Although, since pivotal quantities are not unique, if you used another pivotal, you might obtained a different answer. In fact, with the same pivotal Z if we make, $q = 0.2726$, $Pr[Z < q] = 0.2386$. This leads to a "one sided" confidence interval $(0, qY)$ with the same confidence level as in (a).

3. **Exercise 9.17**

(a) Consider $T = X_{(n)} = max\{X_1, X_2, \ldots, X_n\}$, in this problem, the pdf of T is:

$$f_T(t|\theta) = n(t - \theta + 1/2)^{n-1}; \theta - 1/2 < t < \theta + 1/2$$

If we make $Y = T - \theta$, then

$$f_Y(y) = n(y + 1/2)^{n-1}; -1/2 < y < 1/2$$

T is a pivotal quantity. Hence, by finding a and b such that

$$\int_a^b n(y + 1/2)^{n-1} dy = 1 - \alpha,$$

The confidence interval for θ is $(T - b, T - a)$.

(b) As in part (a), consider $T = X_{(n)}$, the pdf of T is:

$$f_T(t|\theta) = 2n\frac{t^{2n-1}}{\theta^{2n}}; 0 < t < \theta$$

If we make $Y = T/\theta$, then

$$f_Y(y) = 2ny^{2n-1}; 0 < y < 1$$

T is a pivotal quantity. Hence, by finding a and b such that

$$\int_a^b 2ny^{2n-1} dy = 1 - \alpha$$

The confidence interval for θ is $(T/b, T/a)$.

4. **Exercise 9.23**

(a) The interval in Example 9.2.15 is:

$$\left(\frac{1}{2n}\chi^2_{2Y,1-\alpha/2}, \frac{1}{2n}\chi^2_{2Y,\alpha/2}\right)$$

which for $Y = 558, n = 9$, data on part (b), it results in $(57.744, 66.379)$.
Inverting the LRT gives confidence intervals of the form:

$$\{\lambda : Y(\lambda n/Y) + Y - n\lambda \geq k\} = [a_1 Y/n, a_2 Y/n]$$

where a_1 and a_2 satisfy the equation $log(a_i) - a_i = (k/Y - 1)$. Using the asymtotic result that $-2\ log(\lambda(x)) \sim \chi^2_{(1)}$, then $k = -\chi^2_{.01}/2 = -1.353$. Using a k value in this range gives a LRT inverted interval of $(57.7, 66.4)$.

5. **Exercise 9.26** The likelihood function is:

$$f(x|\theta) = \theta^n (\prod_{i=1}^n x_i)^{\theta-1} = \theta^n exp\left((\theta - 1)\sum_{i=1}^n log(x_i)\right)$$

The prior distribution is:

$$\pi(\theta|r,\lambda) = \frac{1}{\Gamma(r)\lambda^r}\theta^{r-1}exp(-\theta/\lambda)$$

Then, the posterior distribution is:

$$\pi(\theta|x) \propto f(x|\theta)\pi(\theta|r,\lambda) \propto \theta^{n+r-1}exp(-\theta(1/\lambda + \sum_{i=1}^{n} log(x_i)))$$

which corresponds to a Gamma with parameters:

$$r^* = n + r; \lambda^* = (1/\lambda + \sum_{i=1}^{n} log(x_i))^{-1}$$

The $(1-\alpha)$ credible set for θ is given by two numbers a_1 and a_2, such that:

$$Pr[a_1 < \theta < a_2|x] = \int_{a_1}^{a_2} Gamma(\theta|r^*,\lambda^*)d\theta = 1-\alpha$$

.

6. **Exercise 9.39** By Theorem 9.3.2, we know that $f(a) = f(b)$ and $a \le x^* \le b$ where x^* is the unique mode of $f(x)$. Without loss of generality assume that $x^* = 0$, since $f(a) = f(b)$, then $a = -\delta$ and $b = \delta$ with a certain $\delta > 0$. Also, the symmetry of $f(x)$ implies

$$\int_{-\infty}^{-\delta} f(x)dx = \int_{\delta}^{\infty} f(x)dx$$

Given that

$$\int_{-\delta}^{\delta} f(x)dx = 1-\alpha$$

, then

$$\int_{-\infty}^{-\delta} f(x)dx = \int_{\delta}^{\infty} f(x)dx = \alpha/2$$

7. **Exercise 7, class list**. Let $Y = \sum_{i=1}^{n} X_i$, then θY follows a $Gamma(n,1)$ so it is a pivotal quantity. Then, we can find q_1 and q_2 such that

$$Pr[q_1 < \theta Y < q_2] = 1-\alpha$$

(a) A confidence interval for $1/\theta$ is $(Y/q_2, Y/q_1)$.

(b) A confidence interval for $1/\theta^2$ is $((Y/q_2)^2, (Y/q_1)^2)$.

(c) A confidence interval for $e^{-\theta}$ is $(e^{-q_2/Y}, e^{-q_2/Y})$.

(d) The pdf of Y_1 is

$$f_{Y_1}(y) = (n\theta)e^{-n\theta y}$$

If we make $Z = \theta Y_1$ then $f_Z(z) = ne^{-nz}$, then Z is a pivotal quantity. If we select q_1 and q_2 such that $Pr[q_1 < \theta Y_1 < q_2] = 1-\alpha$, a confidence interval for θ is $(q_1/Y_2, q_2/Y_1)$.

8. **Exercise 8, class list**

(a)

$$Pr(X < 1/\theta < 2X) = Pr(1/2\theta < X < 1/\theta) = \int_{1/2\theta}^{1/\theta} \theta e^{-\theta x} dx = 0.2386$$

The length of the interval is X so the its expected length is $E(X) = 1/\theta$.

(b) Consider an interval of the type $(0, aX)$ where $0 < a < 1$. The expected length of this interval is $a(1/\theta)$ which is smaller than the expected length of the interval in a). Also,

$$Pr(0 < 1/\theta < aX) = Pr(1/a\theta < X < \infty) = e^{-1/a}$$

If we make this last probability equal to 0.2386, $a = 0.6978$. The confidence interval is $(0, 0.6978X)$.