HYPOTHESIS TESTING (CHAPTER 8)

GIVEN A RANDOM SAMPLE $X_1, X_2, \ldots X_n$; $X \sim f(x|\theta)$, WE WANT TO TEST IF

$$H_0: \theta \in \Theta_0 \quad \text{OR} \quad H_1: \theta \in \Theta_1$$

WHERE Θ_0 AN Θ_1 ARE SUBSETS OF THE PARAMETER SPACE

EXAMPLE: SUPPOSE THAT THE AVERAGE LIFE OF LIGHT BULBS MADE UNDER A STANDARD MANUFACTURING PROCEDURE IS 1400 HOURS. NOW, SUPPOSE THAT A NEW METHOD OF SEALING IS DESIGNED. AND WE WISH TO DETERMINE IF THIS NEW METHOD INCREASES THE LIFE OF THE BULBS.

IF X_i = LIFETIME OF BULB i, $i=1,2 \ldots n$ AND $X_i \sim \exp(\theta)$

$f(x|\theta) = \frac{1}{\theta} \exp(-x/\theta)$ => WE WISH TO TEST

$$H_0: \theta = 1400 \text{ hrs} \quad \text{OR} \quad H_1: \theta > 1400 \text{ hrs}.$$

IN GENERAL, A HYPOTHESIS IS A STATEMENT ABOUT THE POPULATION PARAMETER θ.

THE TWO HYPOTHESIS IN A HYPOTHESIS TESTING PROBLEM ARE CALLED NULL (H_0) AND ALTERNATIVE (H_1).

WE WILL RESTRICT TO SITUATIONS WITH ONLY TWO HYPOTHESIS.

<u>EXAMPLE</u> IN A SEED CLASIFFICATION PROBLEM, WE WANTED TO FIND IF THE LEVEL OF GERMINATION WAS "POOR", "INTERMEDIATE" TO AND "HIGH"

IF p IS THE PERCENTAGE OF SEEDS THAT GERMINATE ~~OVER~~ ACROSS ALL THE POPULATION.

$$\text{"POOR"} \equiv p \leq 0.4$$
$$\text{"INTERMEDIATE"} \equiv 0.4 \leq p < 0.75$$
$$\text{"HIGH"} \equiv p \geq 0.75$$

OUR VARIABLE OF INTEREST IS:

$$X = \begin{cases} 0 & \text{"NO GERMINATION"} \\ 1 & \text{"GERMINATION"} \end{cases}$$

$X_1, X_2 \ldots X_n$ iid. $X \sim \text{BERNOULLI}(p)$

$H_0: 0 < p < 0.4$; $H_1: 0.4 \leq p \leq 0.75$; $H_2: p \geq 0.75$.

PROBLEM MAY BE SOLVED WITH A LOSS-FUNCTION OPTIMALITY OR A BAYESIAN APPROACH.

BACK TO TWO HYPOTHESIS ONLY.

USUALLY, $\Theta_1 = \Theta_0^c$ AND $\Theta_0 \cap \Theta_1 = \phi$

WE ALSO WILL CLASIFY HYPOTHESIS AS "SIMPLE" OR "COMPOSITE"

SIMPLE HYPOTHESIS: THE DISTRIBUTION OF X IS ~~fully~~ COMPLETELY SPECIFIED. (~~ex~~ LIGHT BULB EX: $\theta = 1400$ IS SIMPLE $\Rightarrow X \sim \text{Exp}(1400)$.

COMPOSITE HYPOTHESIS: DISTRIBUTION OF X NOT COMPLETELY SPECIFIED. ($\text{EX}: \theta > 1400$) SIMPLE VS. ALTERNATIVE.

HYPOTHESIS TESTING PROCEDURE: IS A RULE THAT SPECIFIES FOR WHICH ~~THE~~ SAMPLE VALUES ~~[illegible]~~ H_0 IS CONSIDERED TRUE AND FOR WHICH SAMPLE VALUES, H_1 IS CONSIDERED TRUE.

THE PROCEDURE IS DEFINED BY A SET $\mathscr{C}$ CONTAINED IN $\mathcal{X}$ SUCH THAT:

- IF THE SAMPLE POINT $\underline{X} \in \mathscr{C} \Rightarrow$ WE REJECT H_0.
- IF $\underline{X} \notin \mathscr{C} \Rightarrow$ WE ACCEPT H_0.

EXAMPLE: SUPPOSE THAT $X_1, X_2 \ldots X_n$ FOLLOW A $N(\mu, 1)$ $H_0: \mu = 5$ VS. $H_1: \mu > 5$

WE KNOW $T = \bar{X}$ IS A "GOOD" ESTIMATOR FOR μ.

A POSSIBLE REJECTION REGION IS:

$$C = \{(X_1, X_2, \ldots X_n) : \bar{X} > 5\}.$$ HOW TO FIND C? HOW TO EVALUATE C?

METHODS OF FINDING TESTS: LIKELIHOOD RATIO TEST

RECALL THAT UNDER RANDOM SAMPLING:

$L(\theta|\underline{x}) = f(\underline{x}|\theta) = \prod_{i=1}^{n} f(x_i|\theta)$ FOR $H_0: \theta \in \Theta_0$ VS. $H_1: \theta \in \Theta_0^c$

THE LIKELIHOOD RATIO TEST THE IS BASED ON LIKELIHOOD RATIO STATISTIC:

$$\lambda(\underline{x}) = \sup_{\theta \in \Theta_0} L(\theta|\underline{x}) \Big/ \sup_{\theta \in \Theta} L(\theta|\underline{x})$$

A LIKELIHOOD RATIO TEST IS ANY TEST THAT HAS A REJECTION REGION OF THE FORM $C = \{\underline{x} : \lambda(\underline{x}) \le c\}$ WHERE c IS ANY NUMBER SUCH THAT $0 \le c \le 1$.

NOTICE THAT: $\lambda(\underline{x}) \le 1$ (A RESTRICTED SUPREMUM IS LESS THAN AN UNRESTRICTED SUPREMUM).

INTUITION: IF Θ_0 IS TRUE $\sup_{\theta \in \Theta_0} L(\theta|\underline{x}) \approx \sup_{\theta \in \Theta} L(\theta|\underline{x})$

$\lambda(\underline{x}) \approx 1$. $\Rightarrow$ ACCEPT Θ_0

IF Θ_0 IS NOT TRUE $\sup_{\theta \in \Theta_0} L(\theta|\underline{x})$ SMALL RELATIVE TO $\sup_{\Theta} L(\theta|\underline{x})$ $\Rightarrow \lambda(\underline{x}) \le c$

$\Rightarrow$ REJECT Θ_0

IF $\hat{\theta}_0$ IS THE MLE UNDER Θ_0 AND $\hat{\theta}$ IS THE GENERAL MLE, THEN

$$\lambda(\underline{x}) = L(\hat{\theta}_0|\underline{x}) / L(\hat{\theta}|\underline{x}).$$

EXAMPLE: LET $X_1, X_2, \ldots X_n$ BE A RANDOM SAMPLE FROM THE POISSON DISTRIBUTION:

$$f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0, 1, 2, 3, \ldots$$

SUPPOSE $H_0: \theta = \theta_0$ AND $H_1: \theta \neq \theta_0$ WHERE θ_0 IS SPECIFIED BY THE EXPERIMENTER.

FIND A CRITICAL REGION GIVEN BY THE LIKELIHOOD RATIO TEST.

THE LIKELIHOOD FUNCTION IS:

$$L(\theta|\underline{x}) = \prod_{i=1}^{n} \theta^{x_i} e^{-\theta}/x_i! = \theta^{\sum_{i=1}^{n} x_i} e^{-n\theta} / \prod_{i=1}^{n} x_i!$$

$$\sup_{\Theta_0} L(\theta|\underline{x}) = L(\theta_0|\underline{x}) = \theta_0^{\Sigma x_i} e^{-n\theta_0} / \prod x_i!$$

FOR THE MLE : $\log L(\theta|\underline{x}) = \sum_{i=1}^{n} x_i \log\theta - n\theta - \log \prod x_i!$

$$\frac{d \log L(\theta|\underline{x})}{d\theta} = \sum_{i=1}^{n} x_i/\theta - n \stackrel{\text{set}}{=} 0 \Leftrightarrow \hat{\theta} = \bar{x}$$

$$\sup_{\Theta} L(\theta|\underline{x}) = (\bar{x})^{\Sigma x_i} e^{-n\bar{x}} / \prod_{i=1}^{n} x_i!$$

$$\lambda(\underline{x}) = \left(\theta_0^{\sum_{i=1}^{n} x_i} e^{-n\theta_0} / \prod_{i=1}^{n} x_i!\right) \Big/ \left(\bar{x}^{\sum_{i=1}^{n} x_i} e^{-n\bar{x}} / \prod x_i!\right)$$
$$= \left(\frac{\theta_0}{\bar{x}}\right)^{n\bar{x}} e^{-n(\theta_0 - \bar{x})} = e^{-n\theta_0} \left(\frac{\theta_0}{\bar{x}}\right)^{n\bar{x}} e^{n\bar{x}}$$

THE REJECTION REGION IS DEFINED BY:

$$e^{-n\theta_0} \left(\frac{\theta_0}{\bar{x}}\right)^{n\bar{x}} e^{n\bar{x}} < c$$

OBS: THE REJECTION REGION DEPENDS ON $\bar{X}$ A SUFFICIENT STATISTIC FOR θ.

IF $T(\underline{x})$ IS A SUFFICIENT STATISTIC FOR θ BY THE FACTORIZATION THEOREM:

$$f(\underline{x}|\theta) = g(T(\underline{x})|\theta)\, h(\underline{x})$$

$$\lambda(\underline{x}) = \frac{\sup_{\Theta_0} L(\theta|\underline{x})}{\sup_{\Theta} L(\theta|\underline{x})} = \frac{\sup_{\Theta_0} g(T(\underline{x})|\theta)\, h(\underline{x})}{\sup_{\Theta} g(T(\underline{x})|\theta)\, h(\underline{x})} = \frac{h(\underline{x})}{h(\underline{x})} \frac{\sup_{\Theta_0} g(T(\underline{x})|\theta)}{\sup_{\Theta} g(T(\underline{x})|\theta)}$$

THE LIKELIHOOD RATIO TEST TAKES A SIMPLE FORM WHEN $\Theta = \{\theta_0, \theta_1\}$ $\quad H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$

$$\sup_{\Theta_0} L(\theta|\underline{x}) = L(\theta_0|\underline{x}); \quad \sup_{\Theta} L(\theta|\underline{x}) = \begin{cases} L(\theta_0|\underline{x}) & \text{IF } L(\theta_0|\underline{x}) > L(\theta_1|\underline{x}) \\ L(\theta_1|\underline{x}) & \text{IF } L(\theta_0|\underline{x}) \le L(\theta_1|\underline{x}) \end{cases}$$

$$\Rightarrow \lambda(\underline{x}) = \begin{cases} 1 & \text{IF } L(\theta_0|\underline{x}) > L(\theta_1|\underline{x}) \\ \dfrac{L(\theta_0|\underline{x})}{L(\theta_1|\underline{x})} & \text{IF } L(\theta_1|\underline{x}) \le L(\theta_0|\underline{x}) \end{cases}$$

EXAMPLE: SUPPOSE $X_1, X_2, \ldots X_n$ ARE iid OBSERVATIONS WITH $X \sim \text{BERNOULLI}(\theta)$

$$H_0: \theta = 0.5 \quad \text{vs.} \quad H_1: \theta = 0.9$$

$f(x|\theta) = \theta^x (1-\theta)^{1-x}$; $x = 0, 1$; $L(\theta|\underline{x}) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$; $\theta_0 = 0.5$; $\theta_1 = 0.9$.

$\lambda(\underline{x})$ WILL BE DETERMINED BY THE LIKELIHOOD RATIO:

$$\frac{L(\theta = 0.5|\underline{x})}{L(\theta = 0.9|\underline{x})} = \frac{(0.5)^{\sum x_i}(0.5)^{n-\sum x_i}}{(0.9)^{\sum x_i}(0.1)^{n-\sum x_i}} = \left(\frac{0.5}{0.9}\right)^{\sum x_i}\left(\frac{0.5}{0.1}\right)^{n-\sum x_i}$$

$\Rightarrow$ REJECT H_0 WHEN

$$\left(\frac{0.5}{0.9}\right)^{\sum x_i}\left(\frac{0.5}{0.1}\right)^{n-\sum x_i} < c; \quad \text{FOR } 0 < c < 1$$

IF WE RE-EXPRESS THE INEQUALITY, WE GET:

REJECT $H_0 \iff$

$$\sum_{i=1}^{n} x_i > \frac{\log c + n \log(0.1/0.5)}{\log(0.1/0.9)}$$

ANOTHER EXAMPLE OF LRT.

SUPPOSE NOW, WE WANT TO TEST: $X_1, X_2, \ldots X_n$; $X \sim$ Bernoulli(θ)

$$H_0: \theta \leq 0.5 \text{ vs. } H_1: \theta > 0.5$$

$$\sup_{\Theta_0} L(\theta|\underline{x}) = \begin{cases} L(\hat{\theta}|\underline{x}) \ ; & \text{IF } \hat{\theta} < 0.5 \quad \hat{\theta} = \bar{X} \text{ THE MLE} \\ L(\theta = 0.5|\underline{x}), & \text{IF } \hat{\theta} \geq 0.5 \end{cases}$$

$$\sup_{\Theta} L(\theta|\underline{x}) = L(\hat{\theta}|\underline{x}) \Rightarrow$$

$$\lambda(\underline{x}) = \begin{cases} 1 & \text{IF } \bar{X} < 0.5 \quad (\text{DO NO REJECT } H_0) \\ \dfrac{L(\theta = 0.5|\underline{x})}{L(\hat{\theta}|\underline{x})} \ ; & \bar{X} \geq 0.5 \end{cases}$$

$$\frac{L(\theta = 0.5|\underline{x})}{L(\hat{\theta}|\underline{x})} = \frac{(0.5)^{\sum x_i} (0.5)^{n - \sum x_i}}{(\bar{x})^{\sum x_i} (1-\bar{x})^{n - \sum x_i}} = \left(\frac{(1-\bar{x})}{\bar{x}} \right)^{\sum x_i} \left(\frac{0.5}{(1-\bar{x})} \right)^{n}$$

REJECT $H_0 \iff \left(\dfrac{(1-\bar{x})}{\bar{x}} \right)^{\sum x_i} \left(\dfrac{0.5}{(1-\bar{x})} \right)^{n} < C$.

HOW TO DETERMINE C? WE'LL SEE THIS IN THE NEXT SECTION

BAYESIAN TESTS.

RECALL THAT IN BAYESIAN, WE HAVE A PRIOR DISTRIBUTION $\pi(\theta)$ WHICH LEADS TO THE POSTERIOR DISTRIBUTION

$$\pi(\theta|\underline{x}).$$

$$H_0: \theta \in \Theta_0 \quad \text{vs.} \quad H_1: \theta \in \Theta_1$$

WE COMPUTE THE POSTERIOR PROBABILITIES

$$P(\theta \in \Theta_0 |\underline{x}) \quad \text{AND} \quad P(\theta \in \Theta_1 |\underline{x})$$

AND COMPARE THEM

IF $P(\theta \in \Theta_0 |\underline{x}) \geq P(\theta \in \Theta_1 |\underline{x}) \Rightarrow$ ACCEPT H_0

IF $P(\theta \in \Theta_1 |\underline{x}) > P(\theta \in \Theta_0 |\underline{x}) \Rightarrow$ REJECT H_0

IN FACT

$$P(\theta \in \Theta_0 | \underline{x}) = \int_{\Theta_0} \pi(\theta | \underline{x}) d\theta$$

$$P(\theta \in \Theta_1 | \underline{x}) = \int_{\Theta_1} \pi(\theta | \underline{x}) d\theta$$

IF $\Theta_1 = \Theta_0^c \Rightarrow P(\theta \in \Theta_1) = 1 - P(\theta \in \Theta_0)$

ACCEPT $H_0 \Leftrightarrow P(\theta \in \Theta_0) > 1/2$

PROBLEM:

IF $\Theta_0 = \{\theta_0\} \rightarrow H_0: \theta = \theta_0$ AND $H_1: \theta \neq \theta_0$.

WITH $\pi(\theta | \underline{x})$ A CONTINUOUS DISTRIBUTION $\Rightarrow$

$P(\theta = \theta_0 | \underline{x}) = 0$ FOR ANY SAMPLE $\underline{x}$

WE WILL ALWAYS REJECT H_0! WITH SIMPLE HYPOTHESIS, THE BAYESIAN APPROACH HAS SOME DIFFICULTIES.

ON THE OTHER HAND, IF WE HAVE MORE THAN 2 HYPOTHESIS, FOR EXAMPLE,

$H_0: \theta \in \Theta_0$; $H_1: \theta \in \Theta_1$; $H_2: \theta \in \Theta_2$

WE COMPUTE

$P(\theta \in \Theta_0 | \underline{x})$; $P(\theta \in \Theta_1 | \underline{x})$; $P(\theta \in \Theta_2 | \underline{x})$

AND SELECT THE HYPOTHESIS WITH THE HIGHEST POSTERIOR PROBABILITY.

EXAMPLE: LET $X_1, X_2, \ldots X_n$ BE iid OBSERVATIONS WITH A BERNOULLI DISTRIBUTION OF PARAMETER θ

$\pi(\theta) \sim U(0,1)$. WE WISH TO TEST:

$$H_0: \theta \leq 0.5 \quad \text{vs.} \quad H_1: \theta > 0.5$$

SINCE A $U(0,1)$ IS A BETA $(1,1)$ DISTRIBUTION THE POSTERIOR DISTRIBUTION IS A BETA(α', β')

WHERE $\alpha' = 1 + \sum_{i=1}^{n} x_i$; $\beta' = 1 + n - \sum_{i=1}^{n} x_i$

$\Rightarrow P(\theta \in \Theta_0 \mid \underline{x}) = P(\theta \leq 0.5 \mid \underline{x}) =$

$$\int_0^{0.5} \frac{\Gamma(n+2)}{\Gamma(1+\sum x_i)\,\Gamma(1+n-\sum x_i)}\, \theta^{\sum x_i + 1 - 1} (1-\theta)^{1+n-\sum x_i - 1}\, d\theta$$

CUMULATIVE BETA DISTRIBUTION.

IF OUR DATA ~~SAYS~~ ESTABLISHES THAT $n = 100$, $\sum_{i=1}^{100} x_i = 40$

THE POSTERIOR IS A BETA (41, 61)

UNDER THIS BETA: $P(\theta \leq 0.5 \mid \underline{x}) = 0.976978$

$\Rightarrow P(\theta > 0.5 \mid x) = 1 - 0.976978 = 0.02302203$

THEN, WE ACCEPT Ho.

IF $\sum_{i=1}^{100} x_i = 70 \Rightarrow$ THE POSTERIOR IS A BETA (71, 31)

AND

$P(\theta \leq 0.5 \mid \underline{x}) = 0.0000276$ REJECT Ho!

ALTERNATIVELY, WE ACCEPT Ho IF

$$1/2 \leq P(\theta \leq 0.5 \mid \underline{x}).$$

$\Leftrightarrow$ $m \leq 0.5$; WHERE m IS THE MEDIAN OF A BETA (α', β').

IN THE FIRST CASE, $m = 0.40131 < 0.5 \Rightarrow$ Reject Ho.

IN THE SECOND CASE, $m = 0.6973644 \geq 0.5 \Rightarrow$ Accept Ho.

METHODS OF EVALUATING TESTS

ERROR PROBABILITIES AND THE POWER FUNCTION

THE FOLLOWING TABLE ILLUSTRATES THE TYPES OF ERRORS IN HYPOTHESIS TESTING.

		DECISION: ACCEPT H_0	REJECT H_0
TRUTH	H_0	CORRECT DECISION	TYPE I ERROR
	H_1	TYPE II ERROR	CORRECT DECISION

TYPE I ERROR $\equiv$ REJECT H_0 WHEN H_0 IS TRUE.

TYPE II ERROR $\equiv$ ACCEPT H_0 WHEN H_0 IS FALSE.

KEY PROBABILITIES:

IF $\mathcal{C}$ (OR R) IS THE REJECTION REGION AND WE WE WISH TO TEST: $H_0: \theta \in \Theta_0$ VS. $H_1: \theta \in \Theta_0^c$

THE PROBABILITY OF A TYPE I ERROR IS:

$$Pr(\underline{x} \in \mathcal{C} \mid \theta) \quad \text{WHERE } \theta \in \Theta_0$$

THE PROBABILITY OF A TYPE II ERROR IS:

$$Pr(\underline{x} \notin \mathcal{C} \mid \theta) \quad \text{WHERE } \theta \in \Theta_1$$

$$= 1 - P(\underline{x} \in \mathcal{C} \mid \theta)$$

BOTH TYPES OF ERRORS DEPEND ON $Pr(\underline{x} \in \mathcal{C} \mid \theta)$

THIS LEADS TO THE FOLLOWING DEFINITION:

<u>DEF</u>: THE POWER FUNCTION OF A HYPOTHESIS TEST WITH REJECTION REGION $\mathcal{C}$ IS A FUNCTION OF θ DEFINED BY

$$\beta(\theta) = P(\underline{x} \in \mathcal{C} \mid \theta)$$

THE IDEAL POWER FUNCTION IS:

$$\beta(\theta) = \begin{cases} 0 & \text{IF } \theta \in \Theta_0 \\ 1 & \text{IF } \theta \in \Theta_0^c \end{cases}$$

USUALLY, THIS IDEAL CANNOT BE ATTAINED.

A "GOOD" POWER FUNCTION IS SUCH THAT:

$\beta(\theta) \approx 1$ IF $\theta \in \Theta_0^c$ AND $\beta(\theta) \approx 0$ IF $\theta \in \Theta_0$

EXAMPLE: LET $X_1, X_2, \ldots X_n$ BE A RANDOM SAMPLE FROM A NORMAL $(\theta, 25)$. CONSIDER

$H_0: \theta \leq 17$ VS. $H_1: \theta > 17$.

REJECT IF AND ONLY IF $\bar{X} > 17 + 5/\sqrt{n}$ (THIS A LRT SEE EXERCISE 8.37)

FIND THE POWER FUNCTION $\beta(\theta)$

$\beta(\theta) = Pr\left[\bar{X} > 17 + 5/\sqrt{n} \mid \theta\right]$. NOTICE THAT

GIVEN θ, $\bar{x} \sim N(\theta, 25/n) \Rightarrow Z = \dfrac{\bar{X} - \theta}{5/\sqrt{n}} \sim N(0,1)$

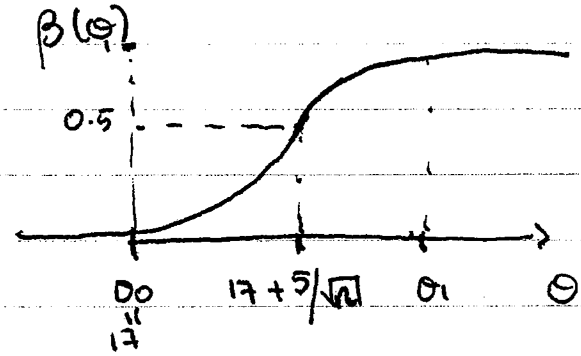
THEN

$$\beta(\theta) = Pr\left(Z > \frac{17 + 5/\sqrt{n} - \theta}{5/\sqrt{n}}\right) = 1 - \Phi\left(\frac{17 + 5/\sqrt{n} - \theta}{5/\sqrt{n}}\right)$$

WHERE Φ IS THE CDF OF A $N(0,1)$.

AS θ INCREASES FROM $-\infty$ TO ∞ $\Phi\left(\frac{17+5/\sqrt{n}-\theta}{5/\sqrt{n}}\right)$ DECREASES ~~INCREASES~~ FROM 1 TO 0, THEN $\beta(\theta)$ INCREASE FROM ZERO TO ONE.

SKETCH OF $\beta(\theta)$



$\theta < 17$ POWER ≈ 0 ✓ THIS IS WHAT WE WANT

~~POWER~~ $\theta > \theta_1$ POWER ≈ 1 ✓

FOR $17 < \theta < 5/\sqrt{n} + 17$ POWER < 0.5 LESS THAN 1/2 OF REJECTING H_0.

NOTICE THAT THE MAXIMUM TYPE I ERROR PROBABILITY IS REACHED AT $\theta = \theta_0 = 17$

WHAT IS THIS MAXIMUM TYPE I ERROR PROBABILITY?

$$\beta(\theta=17) = 1 - \Phi(1) = 0.1586$$

WHAT IS THE MAXIMUM TYPE II ERROR PROB. IF $\theta \geq 17+5$?

RECALL THAT IF $\theta \in \Theta_1$,

$$\Pr[\text{Type II ERROR}] = 1-\beta(\theta)$$

THE MAXIMUM TYPE II ERROR PROB. FOR $\theta \geq 17+5$ IS

$$1-\beta(\theta=22) = \Phi\left(\frac{5/\sqrt{n} - 5}{5/\sqrt{n}}\right) \leftarrow \text{DEPENDS ON } n!$$

IF n FIXED (SAY $n=25$) $\Rightarrow 1-\beta(\theta=22) = \Phi(-4) \approx 0$

IF WE FIX THIS ERROR (SAY AT 0.01) WE CAN FIND n THAT YIELDS THIS DESIRED PROBABILITY.

$$\Rightarrow \quad \Phi\left(\frac{5/\sqrt{n}-5}{5/\sqrt{n}}\right) = 0.01$$

$$\Rightarrow \quad \frac{5/\sqrt{n}-5}{5/\sqrt{n}} = -2.33 \qquad \text{SOLVE FOR } n$$

$n \approx 11.0889 \Rightarrow n=12$ GIVES THE DESIRED PROBABILITY.

GENERALLY ~~THE POWER FUNCTION~~ THE ERRORS WILL DEPEND STRONGLY ON n.

~~THE~~ ~~α~~ ~~LEVEL~~ SIZE OF A TEST IS $\alpha = \sup_{\theta \in \Theta_0} \beta(\theta)$

~~THE LEVEL OF THE TEST IS~~

* A TEST WITH ~~WHOSE~~ POWER $\beta(\theta)$ IS OF LEVEL α IF

$$\sup_{\Theta_0} \beta(\theta) \le \alpha$$

IN COMPLICATED TESTING SITUATIONS, IT IS OFTEN IMPOSSIBLE TO BUILD A SIZE α TEST.

A TEST WITH POWER $B(\theta)$ IS OF SIZE α IF

$$\sup_{\Theta_0} \beta(\theta) = \alpha \quad *$$

IN OUR EXAMPLE, $\alpha = 0.1586$ IS THE SIZE OF TEST

BACK TO LRT,

$$\text{Reject } H_0 \iff \lambda(\underline{x}) \le c$$

TO DETERMINE C, FIRST WE SPECIFY A LEVEL α (SAY. 0.01, 0.05)

THE POWER FUNCTION IS: $\beta(\theta) = \Pr[\lambda(\underline{x}) \le c \mid \theta]$

C IS SUCH THAT:

$$\sup_{\theta \in \Theta_0} P[\lambda(\underline{x}) \le c \mid \theta] \le \alpha$$

EXAMPLE: LET $X_1, X_2 \ldots X_n$ BE iid BERNOULLI(θ)

$$H_0: \theta = 0.5 \quad \text{vs.} \quad H_1: \theta = 0.9$$

THE LRT ESTABLISHES THAT WE REJECT Ho IF AND ~~ONLY~~ ONLY IF:

$$\sum_{i=1}^{n} x_i > c$$

$\alpha = 0.05$

THE NULL HYPOTHESIS IS FORMED OF ONE POINT $\theta_0 = 0.5$

$$\sup_{\Theta_0} P\left(\sum_{i=1}^{n} x_i > c \mid \theta\right) = P\left(\sum_{i=1}^{n} x_i > c \mid \theta = 0.5\right).$$

IF n IS LARGE ENOUGH. BY CLT $\sum_{i=1}^{n} x_i \sim N(n(0.5), n(0.5)^2)$

$\Rightarrow$

$$P\left(\sum_{i=1}^{n} x_i > c \mid \theta = 0.5\right) = \Pr\left(z > \frac{c - n(0.5)}{\sqrt{n}\, 0.5}\right) = .05$$

$\Rightarrow C = -1.64\sqrt{n(0.9)(.1)} + n(.0.9)$

THEN SOLVING FOR n AND C

$$n = 10.75\ ;\ C = 8.0688.$$

(I'M ASSUMING THAT THE CLT IS VALID HERE, OTHERWISE WE NEED TO USE THE CDF OF A BINOMIAL)

MOST POWERFUL TEST:

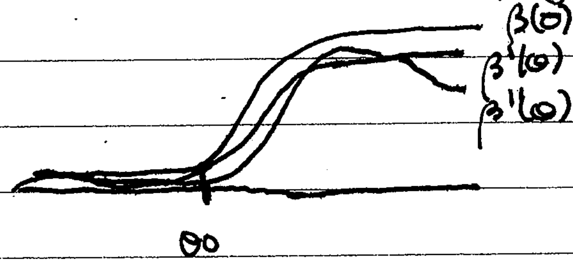
SUPPOSE WE WISH TO TEST

$$H_0: \theta \in \Theta_0 \quad \text{vs.} \quad H_1: \theta \in \Theta_0^c$$

A LEVEL α TEST IS THE UNIFORMLY MOST ~~POWERFUL~~ POWERFUL IF: ITS POWER FUNCTION $\beta(\theta)$ SATISFIES:

$$\beta(\theta) \geq \beta'(\theta) \quad \text{FOR ALL } \theta \in \Theta_0^c$$

WHERE $\beta'(\theta)$ IS THE POWER FUNCTION OF ANY OTHER α-LEVEL TEST.



TO EASIEST SITUATION TO FIND THE MOST POWERFUL TEST IS WHEN $H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$ $\theta_0 \neq \theta_1$ (BOTH HYPOTHESIS ARE SIMPLE). THE POWER FUNCTION IS GIVEN BY 2 POINTS

NEYMANN-PEARSON LEMMA. CONSIDER TESTING $H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$ A TEST WITH REJECTION REGION R (OR $\mathcal{C}$) SUCH THAT

- $\underline{x} \in R \Leftrightarrow f(\underline{x} | \theta_1) > K f(\underline{x} | \theta_0)$
- $\underline{x} \in R^c \Leftrightarrow f(\underline{x} | \theta_1) < K f(\underline{x} | \theta_0)$; $0 < K$

AND $\alpha = P(\underline{X} \in R | \theta_0)$ IS A UNIFORMLY MOST POWERFUL TEST. (UMP). OF LEVEL THAT α.

<u>PROOF</u> ONLY FOR THE CASE WHERE $f(\underline{x}|\theta)$ IS A pdf OF A CONTINUOUS RANDOM VARIABLE.

DEFINE $\phi(\underline{x})$ BE THE TEST FUNCTION DEFINED BY R

$$\Rightarrow \quad \phi(\underline{x}) = \begin{cases} 1 & \text{IF } \underline{x} \in R \\ 0 & \text{IF } \underline{x} \notin R. \end{cases}$$

LET $\phi'(\underline{x})$ BE THE TEST FUNCTION OF ANY OTHER α LEVEL TEST $\quad (\phi'(\underline{x}) = 1;\ \underline{x} \in R'\ ;\ \phi'(\underline{x}) = 0.$ IF $x \notin R')$.

LET $\beta(\theta)$ BE THE POWER FUNCTION OF $\phi(\underline{x})$ AND $\beta'(\theta)$ BE THE POWER FUNCTION OF $\phi'(\underline{x})$.

SINCE $0 \le \phi'(\underline{x}) \le 1$. ~~AND~~ $\phi(\underline{x}) = 1$ IF $f(\underline{x}|\theta_1) > k f(\underline{x}|\theta_0)$ AND $\phi = 0$ IF $f(\underline{x}|\theta_1) < k f(\underline{x}|\theta)$

$$\Rightarrow$$

$$[\phi(\underline{x}) - \phi'(\underline{x})]\,[f(\underline{x}|\theta_1) - k f(\underline{x}|\theta_0)] \geq 0$$

$\Rightarrow$

$$0 \le \int_{\mathcal{X}} [\phi(\underline{x}) - \phi'(\underline{x})]\,[f(\underline{x}|\theta_1) - k f(\underline{x}|\theta_0)]\,d\underline{x}$$

$$= \int_{\mathcal{X}} \phi(\underline{x}) f(\underline{x}|\theta_1)\,d\underline{x} - \int_{\mathcal{X}} \phi'(\underline{x}) f(\underline{x}|\theta_1)\,d\underline{x} - k \int_{\mathcal{X}} \phi(\underline{x}) f(\underline{x}|\theta_0)\,d\underline{x}$$

$$+ \int_{\mathcal{X}} k\, \phi'(\underline{x}) f(\underline{x}|\theta_0)\,d\underline{x} = \beta(\theta_1) - \beta'(\theta_1) - k\beta(\theta_0) - k\beta'(\theta_0)$$

WE KNOW THAT $\beta(\theta_0) = \alpha$ BECAUSE $\alpha = P(\underline{x} \in R | \theta = \theta_0)$

AND $\beta'(\theta_0) \le \alpha \quad \Rightarrow k(\beta(\theta_0) - \beta'(\theta_0)) \geq 0$

$\Rightarrow -k(\beta(\theta_0) - \beta'(\theta_0)) \le 0$

$$0 \le \beta(\theta_1) - \beta'(\theta_1) - k(\beta(\theta_0) - \beta'(\theta_0)) \le \beta(\theta_1) - \beta'(\theta_1)$$

$\Rightarrow \quad \beta^*(\theta_1) \geq \beta'(\theta_1) \quad \Rightarrow$ R DEFINES THE UMP.

$\Rightarrow$ BY THE NORMAL CDF $\dfrac{C - n(0.5)}{\sqrt{n}(0.5)} = 1.64$

$\Rightarrow \quad C = 1.64\sqrt{n}(0.5) + n(0.5)$

IF CLT DOES NOT APPLY, RECALL THAT $\sum_{i=1}^{n} X_i \sim BIN(n, \theta)$ UNDER $\theta_0 \Rightarrow BIN(n, 0.5)$, C HAS TO BE OBTAINED FROM THE BINOMIAL CDF. ($n = 10$, $C = 7$, $\alpha = .05468$
$C = 8$, $\alpha = 0.1074$)

OBSERVATION: IF

$$H_0: \theta = \theta_0 \quad vs. \quad H_1: \theta = \theta_1$$

AND $\mathcal{C}$ IS A REJECTION REGION FOR TEST.

Error probs: $\alpha = P(\underline{x} \in \mathcal{C} \mid \theta = \theta_0)$

$\beta = P(\underline{x} \notin \mathcal{C} \mid \theta = \theta_1) = 1 - P(\underline{x} \in \mathcal{C} \mid \theta = \theta_1)$

IN THE PREVIOUS EX, SAY $n = 50$, $\alpha = 0.05$

$\Rightarrow C = 1.64 \cdot \sqrt{50} \cdot (0.5) + 50(0.5)$

$= 30.80$

WHAT IS β?

$$\beta = 1 - Pr\left(\sum_{i=1}^{n} X_i > 30.80 \mid \theta = 0.9\right)$$

$$= 1 - P\left(Z > \frac{30.8 - 50(.9)}{\sqrt{50(.9)(.1)}}\right) = 1 - P(Z > -6.7) \simeq 0$$

IF α, β ARE DETERMINED, WE CAN FIND n THAT LEADS TO THESE ERRORS. RECALL BLLI EX. $H_0: \theta = 0.5$ vs. $H_1: \theta = 0.9$

SUPPOSE WE WANT THAT $\alpha = \beta = 0.05$ FOR THE BLLI EX.

Reject H_0 IF $\sum X_i > C$

$P(\sum X_i > C \mid \theta = 0.5) = .05 \Rightarrow C = 1.64\sqrt{n}(0.5) + n(0.5)$

$0.05 = 1 - P(\sum X_i > C \mid \theta = 0.9) \Rightarrow \dfrac{C - n(0.9)}{\sqrt{n(0.9)(0.1)}} = -1.64$

NOTES:

- REVERSE OF NEYMAN-PEARSON LEMMA IS TRUE.
IF ϕ' IS A MOST POWERFUL TEST OF SIZE α, NECESSARILY $\phi' = \phi$; ϕ TEST FUNCTION FOR NEYMAN-PEARSON.
(SEE END OF THEOREM 8.3.12).
- THE N-P LEMMA SAYS: REJECT Ho $\Leftrightarrow$ $\underline{k f(\underline{x}|\theta_0)} < f(\underline{x}|\theta_1)$
WHICH IS THE SAME TO

$$\text{Reject } H_0 \Leftrightarrow \frac{f(\underline{x}|\theta_0)}{f(\underline{x}|\theta_1)} < \frac{1}{k} = c \; (\text{L.R.T.!})$$

EXAMPLE:

LET $X_1, X_2, \ldots X_n$ BE A RANDOM SAMPLE FROM $f(x;\theta) = \theta e^{-\theta x}$ WHERE $\theta = \theta_0$ OR $\theta = \theta_1$; θ_0 AND θ_1 ARE KNOWN FIXED NUMBERS. ASSUME THAT $\theta_1 > \theta_0$.
NOW,

$$f(\underline{x}|\theta_0) = \theta_0^n \exp\left(-\theta_0 \sum_{i=1}^n x_i\right); \; f(\underline{x}|\theta_1) = \theta_1^n \exp\left(-\theta_1 \sum_{i=1}^n x_i\right)$$

ACCORDING TO THE NEYMANN-PEARSON LEMMA, THE MOST POWERFUL TEST HAS THE FORM:

$$\text{Reject } H_0 \Leftrightarrow K < \left(\frac{\theta_1}{\theta_0}\right)^n \exp\left(-(\theta_1 - \theta_0)\sum_{i=1}^n x_i\right)$$

WHICH IS EQUIVALENT TO

$$\sum_{i=1}^n x_i < \left(\frac{1}{\theta_1 - \theta_0}\right) \log\left(\left(\frac{\theta_1}{\theta_0}\right)^n \frac{1}{K}\right) = K'$$

THE INEQUALITY OF THE N-P LEMMA HAS BEEN SIMPLIFIED TO:

$$\sum_{i=1}^n x_i < K'$$

ALSO, $\alpha = P_r[\text{reject } H_0 \mid \theta=\theta_0] = P_r\left[\sum_{i=1}^{n} x_i < k' \mid \theta=\theta_0\right]$

WE KNOW THAT $\sum_{i=1} x_i \sim \text{GAMMA}(n, \theta_0^{-1})$

HENCE,

$$P_r\left[\sum x_i < k' \mid \theta=\theta_0\right] = \int_{-\infty}^{k'} \frac{\theta_0^n}{\Gamma(n)} x^{n-1} e^{-x\theta_0}\, dx = \alpha$$

k' IS THE α-QUANTILE OF A GAMMA (n, θ_0^{-1}).

HOW TO FIND A UMP IF AT LEAST THE ALTERNATIVE IS COMPOSITE.

EXAMPLE LET. $X_1, X_2, \ldots X_n$ BE iid SUCH THAT $X_i \sim N(\mu, \sigma^2)$ WITH σ^2 KNOWN. THE HYPOTHESIS TO TEST ARE

$$H_0: \mu = \mu_0 \quad \text{vs.} \quad H_1: \mu > \mu_0$$

IDEA: FIX A VALUE AND PERFORM A SIMPLE VS. SIMPLE TEST

TAKE AN ARBITRARY $\mu_1 > \mu_0$ IN THE ALTERNATIVE.

FIND N-P REJECTION REGION.

$$\text{Reject } H_0 \iff \frac{f(\underline{x} \mid \mu_0)}{f(\underline{x} \mid \mu_1)} < \frac{1}{k}$$

$$f(\underline{x} \mid \mu_0) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_0)^2\right)$$

$$f(\underline{x} \mid \mu_1) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_1)^2\right)$$

RECALL THAT $\sum_{i=1}^{n} (x_i - \mu)^2 = n(\bar{x} - \mu)^2 + \sum_{i=1}^{n} (x_i - \bar{x})^2$

$$\Rightarrow \frac{f(\underline{x} \mid \mu_0)}{f(\underline{x} \mid \mu_1)} = \exp\left(-\frac{n}{2\sigma^2}\left[(\bar{x} - \mu_0)^2 - (\bar{x} - \mu_1)^2\right]\right) < \frac{1}{k}$$

AFTER SOME ALGEBRA...

THE INEQUALITY IS EQUIVALENT TO

$$\frac{n}{\sigma^2}(\mu_0-\mu_1)\bar{x} < \log\left(\frac{1}{k}\right) + \frac{n}{2\sigma^2}(\mu_0^2-\mu_1^2)$$

SINCE $(\mu_0-\mu_1) < 0$ THEN OUR REGION HAS THE FORM

$$\bar{X} > k'$$

THIS FORM DOES NOT DEPEND ON μ_1 AND k' IS FIXED WITH $\alpha = Pr[\bar{X} > k' | \mu = \mu_0]$ $\left(k' = \mu_0 + z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right)\right)$

HENCE, THE REGION $\bar{X} > k'$ DEFINES THE MOST POW. TEST!

WHAT HAPPENS WHEN WE CONSIDER THE TEST:

$$H_0: \mu \le \mu_0 \quad vs. \quad H_1: \mu > \mu_0.$$

AND LETS SAY WE USE THE SAME REJECTION REGION.

$$R = \{\underline{x};\ \bar{x} > k'\} \qquad \alpha = P_0[\bar{X} > k' | \mu = \mu_0]$$

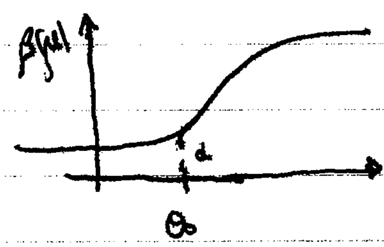
IS THIS A α-LEVEL UMP?

ALL WE NEED TO CHECK IS THAT:

$$\alpha = \sup_{\Theta_0} \beta(\mu)$$

WHERE

$$\beta(\mu) = Pr[\bar{X} > k' | \mu] = Pr\left[Z > \frac{k'-\mu}{\sigma/\sqrt{n}} \Big| \mu\right]$$



$$= 1 - \Phi\left(\frac{k'-\mu}{\sigma/\sqrt{n}}\right)$$

WHAT IF WE NOW WANT TO TEST

$$H_0: \mu = \mu_0 \quad vs. \quad H_1: \mu < \mu_0 \qquad k' = \mu_0 - z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right)$$

THE REJECTION REGION IS NOW OF THE FORM.

$$\bar{X} < k'$$

THIS IS ALSO A UMP.

$$\alpha = Pr[\bar{X} < k' | \mu = \mu_0]$$

WHAT ABOUT $H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$?

FOR $\mu_1 < \mu_0$ THE MOST POWERFUL TEST IS GIVEN BY R_1:

~~[illegible]~~ $\bar{X} < K_1$ ($\approx \mu_0 - z_\alpha \sigma/\sqrt{n}$) WITH A POWER FUNCTION $\beta_1(\mu) = P_r[\bar{X} < K_1 \mid \mu]$ (t_1)

FOR $\mu_2 > \mu_0$ THE MOST POWERFUL TEST IS GIVEN BY $\bar{X} > K_2$.

WITH A POWER FUNCTION $\beta_2(\mu) = P[\bar{X} > \underbrace{K_2}_{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}} \mid \mu]$ t_2.

AT μ_1

$$\beta_1(\mu_1) \geq \beta_2(\mu_1)$$

AT μ_2 $\quad \beta_2(\mu_2) \geq \beta_1(\mu_1)$

THERE IS NOT A UMP TEST FOR THIS PROBLEM.

TRADEOFF:

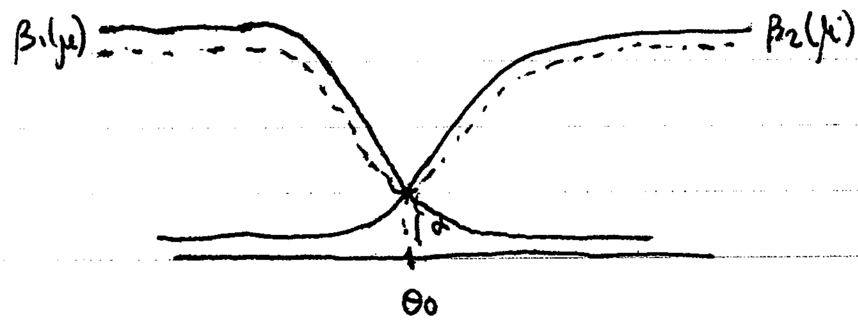
Reject H_0 IF $\bar{X} < K''$ OR $\bar{X} > K'$

(TWO TAILED TEST)

$$\alpha = P_r\left\{ \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} < -z_{\alpha/2} \text{ or } \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > z_{\alpha/2} \;\middle|\; \mu = \mu_0 \right\}$$

$$\Rightarrow K'' = \mu_0 - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) \qquad K' = \mu_0 + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right).$$

THIS IS NOT A UMP TEST α-LEVEL



LAST TIME: $X_1, X_2, \ldots X_n$ iid OBS. $X \sim N(\mu, \sigma^2)$ σ^2 KNOWN.

$$H_0: \mu = \mu_0 \quad \text{vs.} \quad H_1: \mu \neq \mu_0$$

IF WE USE AS OUR REJECTION REGION $\varphi = \{ \underline{x} : \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > z_{\alpha/2}$ OR $\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} < -z_{\alpha/2} \}$ THIS LEADS TO A SIZE α TEST BUT NOT A UMP TEST.

ANOTHER ANGLE, φ IS A L.R.T.!

$$L(\mu; \underline{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right)$$

$$\sup_{H_0} L(\mu; \underline{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_0)^2\right)$$

$$\sup_{\mu} L(\mu; \underline{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2\right)$$

THE RATIO IS:

$$\lambda(\underline{x}) = \frac{\exp\left(-\frac{1}{2\sigma^2}\left(n(\mu_0 - \bar{x})^2 + \sum_{i=1}^{n}(x_i - \bar{x})^2\right)\right)}{\exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)\right)} = \exp\left(\frac{-n}{2\sigma^2}(\bar{x} - \mu_0)^2\right)$$

REJECT $H_0 \iff \lambda(\underline{x}) \leq c \iff \exp\left(\frac{-n}{2\sigma^2}(\bar{x} - \mu_0)^2\right) \leq c$

$$\iff \frac{(\bar{x} - \mu_0)^2}{\sigma^2/n} \geq -2\log(c) \iff \frac{|\bar{x} - \mu_0|}{\sigma/\sqrt{n}} \geq (-2\log(c))^{1/2}$$

$(-2\log(c))^{1/2}$ FIXED TO $z_{\alpha/2}$

WITH σ^2 UNKNOWN.

UNDER H_0:

THE MLE FOR σ^2 IS $\hat{\sigma}_0^2 = \sum_{i=1}^{n} (x_i - \mu_0)^2 / n$ $\quad \mu = \mu_0$

RECALL THAT THE UNRESTRICTED MLE ARE:

$$\hat{\mu} = \bar{x} \quad \text{AND} \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$\sup_{\Theta_0} L(\mu,\sigma^2 \mid \underline{x}) = \left(\frac{1}{2\pi\hat{\sigma}_0^2}\right)^{n/2} \exp\left(-\frac{1}{2\hat{\sigma}_0^2}\sum_{i=1}^{n}(x_i-\mu_0)^2\right)$$

$$= \left(\frac{1}{2\pi\hat{\sigma}_0^2}\right)^{n/2} \exp\left(-\frac{n}{2}\right)$$

$$\sup_{\Theta} L(\mu,\sigma^2 \mid \underline{x}) = \left(\frac{1}{2\pi\hat{\sigma}^2}\right)^{n/2} \exp\left(-\frac{1}{2\hat{\sigma}^2}\sum(x_i-\bar{x})^2\right)$$

$$= \left(\frac{1}{2\pi\hat{\sigma}^2}\right)^{n/2} \exp\left(-\frac{n}{2}\right)$$

$\lambda(\underline{x}) = \dfrac{\sup_{H_0} L(\mu,\sigma^2 \mid \underline{x})}{\sup L(\mu,\sigma^2 \mid \underline{x})} = \left(\dfrac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{n/2}$; REJECT $H_0 \iff \left(\dfrac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{n/2} \leq c$

RECALL THAT: $\sum_{i=1}^{n}(x_i-\mu_0)^2 = n(\bar{x}-\mu_0)^2 + \sum_{i=1}^{n}(x_i-\bar{x})^2$

REJECT $H_0 \iff$

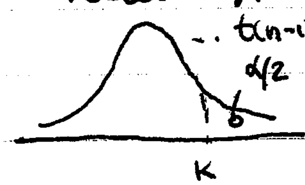
$$\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n(\bar{x}-\mu_0)^2+\sum_{i=1}^{n}(x_i-\bar{x})^2} \leq c^{2/n}$$

DIVIDE BY $\sum_{i=1}^{n}(x_i-\bar{x})^2$ NUM. AND DEN. OF $\lambda(\underline{x})$

REJECT $H_0 \iff c^{n/2}-1 \leq \dfrac{(\bar{x}-\mu_0)^2}{\sum(x_i-\mu_0)^2/n}$ IF WE MULTIPLY BY $(n-1)$

REJECT $H_0 \iff \dfrac{|\bar{x}-\mu_0|}{s/\sqrt{n}} > k$ WHERE $s^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2/(n-1)$

GIVEN H_0 $\dfrac{\bar{x}-\mu_0}{s/\sqrt{n}}$ FOLLOWS A $t(n-1) \Rightarrow k = t_{(n-1)}^{\alpha/2}$



NOTE: BY A SIMILAR ARGUMENT TO σ^2 KNOWN CASE $\Rightarrow$ THIS IS NOT A UMP.

IN GENERAL FOR $H_0: \theta = \theta_0$ (OR $\theta \leq \theta_0$) VS $H_1: \theta > \theta_0 \Rightarrow$ A UMP WILL EXIST

FOR $H_0: \theta = \theta_0$ VS. $H_1: \theta \neq \theta_0$ USUALLY A UMP WILL NOT EXIST.

A LARGE CLASS OF PROBLEMS THAT HAVE AN UMP α-LEVEL TEST FOR ONE SIDED HYPOTHESES INVOLVE THE MONOTONE LIKELIHOOD RATIO PROPERTY.

WHAT IS A MONOTONE LIKELIHOOD RATIO?

SUPPOSE WE HAVE A STATISTIC T WITH A FAMILY OF pdfs OR pmfs $\{ g(t|\theta); \theta \in \Theta \}$ θ A REAL PARAMETER

T HAS A MLR IF FOR EVERY $\theta_1 > \theta_2$ THE RATIO $g(t|\theta_1)/g(t|\theta_2)$ IS A MONOTONE (INC. OR DEC.) FUNCTION OF t.

EXAMPLE: IF $X_1, X_2 \ldots X_n$ ARE iid POISSON(λ) RVs. WE KNOW $T = \sum_{i=1}^{n} x_i \sim$ POISSON $(n\lambda)$.

$g(t|\lambda) = \frac{e^{-n\lambda}(n\lambda)^t}{t!}$, $t = 0, 1, 2, \ldots$ CONSIDER $\lambda_1 > \lambda_2$

$g(t|\lambda_1)/g(t|\lambda_2) = e^{-n(\lambda_1 - \lambda_2)} \left(\frac{\lambda_1}{\lambda_2} \right)^t$ INCREASING IN T.

- IF $X_1, X_2, X_3 \ldots X_n$ ARE iid BERNOULLI(θ), WE HAVE $T = \sum_{i=1}^{n} x_i \sim$ BIN(n, θ), $g(t|\theta) = \binom{n}{t} \theta^t (1-\theta)^{n-t}$

IF $\theta_1 > \theta_2$. $\Rightarrow g(t|\theta_1)/g(t|\theta_2) = \left(\frac{\theta_1}{\theta_2}\right)^t \left(\frac{(1-\theta_1)}{(1-\theta_2)}\right)^{n-t}$

$= \left(\frac{\theta_1}{\theta_2} \frac{(1-\theta_2)}{(1-\theta_1)} \right)^t \left(\frac{(1-\theta_1)}{(1-\theta_2)} \right)^n$ INCREASING IN t.

IN FACT, IF $g(t|\theta) = h(t) c(\theta) e^{w(\theta) t}$ (EXP. FAMILY) HAS A MLR IF $w(\theta)$ IS A NON-DECREASING FUNCTION

IF $\theta_1 > \theta_2 \Rightarrow g(t|\theta_1)/g(t|\theta_2) = \left(\frac{c(\theta_1)}{c(\theta_2)} \right) e^{(w(\theta_1) - w(\theta_2)) t}$

SINCE $w(\cdot)$ NON DECREASING $\Rightarrow$ RATIO NON DECREASING IN t.

THE MLR GOES BEYOND THE EXPONENTIAL FAMILY.

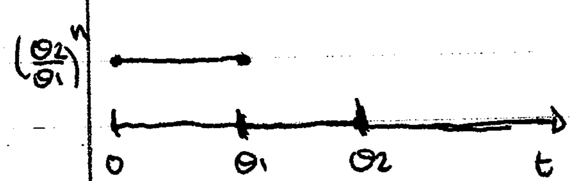
EXAMPLE: LET $X_1, X_2, \ldots X_n$, iid OBS. $X \sim U(0,\theta)$

$T = X_{(n)} = \text{MAX}\{X_1, X_2, \ldots X_n\}$ $\quad g(t|\theta) = \dfrac{n\, t^{n-1}}{\theta^n}, \; 0<t<\theta$

IF $\theta_1 > \theta_2 \Rightarrow g(t|\theta_1)/g(t|\theta_2) = \left(\dfrac{\theta_2}{\theta_1}\right)^n I_{(0,\theta_1)}(t) / I_{(0,\theta_2)}(t)$

NON DECREASING FUNCTION OF t.

(RATIO ONLY DEFINED FOR $g(t|\theta_1) > 0$ OR $g(t|\theta_2)$).



MAIN RESULT FOR M.L.R. (KARLIN – RUBIN THEOREM 8.3.17)

$$H_0: \theta \le \theta_0 \quad \text{-VS.} \quad H_1: \theta > \theta_0$$

T IS A SUFFICIENT STATISTIC FOR θ WITH FAMILY $\{g(t|\theta); \theta \in \Theta\}$ WITH A MLR. THEN THE RULE

REJECT $H_0 \iff T > t_0$; WHERE $\alpha = P[T > t_0 | \theta = \theta_0]$

DEFINES A UMP OF SIZE α.

($H_0: \theta \ge \theta_0$ VS. $H_1: \theta < \theta_0 \iff T < t_0$ DEFINES THE REJECTION REGION).

i) $H_0: \lambda \le \lambda_0$ VS. $H_1: \lambda > \lambda_0$ REJECT $H_0 \iff \sum_{i=1}^n x_i > t_0$

$\alpha = P[\sum_{i=1}^n x_i > t_0 \mid \sum_{i=1}^n x_i \sim \text{POISSON}(n\lambda_0)]$

ii) $H_0: \theta \le \theta_0$ VS. $H_1: \theta > \theta_0$ REJECT $H_0 \iff \sum_{i=1}^n x_i > t_0$

$\alpha = P[\sum_{i=1}^n x_i > t_0 \mid \sum_{i=1}^n x_i \sim \text{BIN}(n, \theta_0)]$.

iii) $H_0: \theta \le \theta_0$ VS. $H_1: \theta > \theta_0$ REJECT $H_0 \iff X_{(n)} > t_0$

$$\alpha = P[X_{(n)} > t_0 \mid \theta = \theta_0] = \int_{t_0}^{\theta_0} \frac{n t^{n-1}}{\theta_0^n} dt = \frac{1}{\theta_0^n} t^n \Big|_{t_0}^{\theta_0}$$

$$\alpha = \frac{(\theta_0^n - t_0^n)}{\theta_0^n} \Rightarrow \quad t_0 = (\theta_0^n - \alpha\theta_0^n)^{1/n}$$