ON SHARP EXTRAPOLATION THEOREMS

by

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M.Sc., Mathematics, Jagiellonian University in Kraków, Poland, 1995

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DISSERTATION

Submitted in Partial Fulfillment of the

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Abstract

Extrapolation is one of the most significant and powerful properties of the weighted theory. It basically states that an estimate on a weighted L^{p_o} space for a single exponent $p_o \geq 1$ and all weights in the Muckenhoupt class A_{p_o} implies a corresponding L^p estimate for all p, $1 < p < \infty$, and all weights in A_p. Sharp Extrapolation Theorems track down the dependence on the A_p characteristic of the weight.

In this dissertation we generalize the Sharp Extrapolation Theorem to the case where the underlying measure is $d\sigma = u_o dx$, and u_o is an A_∞ weight. We also use it to extend Lerner's extrapolation techniques. Such Theorems can then be used to extrapolate some known initial weighted estimates in $L^2(wd\sigma)$. In addition, for some operators this approach allows us to specify the weights $w^{-1} = u_o$ and to use known weighted results in $L^p(wd\sigma)$ to obtain some estimates on the unweighted space.

This work was inspired by the paper [Per1] where the L^2 weighted estimates for

the dyadic square function were considered to obtain the sharp estimates for the so-called Haar Multiplier in L^2.

Contents

Introduction

Extrapolation is one of the most significant and powerful properties of the weighted theory. It basically states that an estimate on a weighted L^{p_o} space for a single exponent p_o and all weights in the Muckenhoupt class A_{p_o} implies a corresponding L^p estimate for all p, $1 < p < \infty$, and all weights in A_p. This is contained in the celebrated theorem due to Rubio de Francia [Ru]. Precise statements of this and other classical theorems can be found in the Preliminaries Chapter 1.

In 1989, Buckley [Buc1] obtained the following *sharp*-estimates for the Hardy-Littlewood Maximal Function, namely for $w \in A_p$

$$\|Mf\|_{L^p(w)} \leq C\, [w]_{A_p}^{\frac{p'}{p}}\, \|f\|_{L^p(w)}.$$

This result is *sharp* in the sense that $[w]^{\frac{p'}{p}}$ cannot be replaced by $\phi([w]_{A_p})$, for any positive non-decreasing function $\phi(t),\ t \geq 1,$ growing slower than $t^{\frac{p'}{p}}$.

In [DrGrPerPet] Buckley's result was used to track down the dependence of the estimates on $[w]_{A_p}$ in the Rubio de Francia Extrapolation Theorem. More precisely, if for a given $1 < r < \infty$ the norm of a sub-linear operator on $L^r(w)$ is bounded by a function of the characteristic constant $[w]_{A_r}$, then for $p > r$ it is bounded on $L^p(v)$ by the same increasing function of the $[v]_{A_p}$, and for $p < r$ it is bounded on $L^p(v)$ by the same increasing function of the $\frac{r}{r'}\frac{p'}{p} = \frac{r-1}{p-1}$ power of the $[v]_{A_p}$. We will refer to this result as the Sharp Extrapolation Theorem.

For some operators the bounds, extrapolated from an initial *sharp*-estimate, are *sharp* but not always. For example, if T is the maximal function and we use Buckley's initial estimates

$$\|Mf\|_{L^r(w)} \leq C\,[w]_{A_r}^{\frac{r'}{r}}\,\|f\|_{L^r(w)},$$

then the sharp extrapolation will give $\|Mf\|_{L^p(w)} \leq C\,[w]_{A_p}^{\max\{\frac{p'}{p},\frac{r'}{r}\}}\,\|f\|_{L^p(w)}$, which is an optimal bound $p < r$. However, for $p > r$, it implies $\|Mf\|_{L^p(w)} \leq C\,[w]_{A_p}^{\frac{r'}{r}}\,\|f\|_{L^p(w)}$, and $\frac{p'}{p} = \frac{1}{p-1} < \frac{1}{r-1} = \frac{r'}{r}$, so sharp extrapolation just preserves the initial estimates and is not optimal for $p > r$ despite the fact that the initial $L^r(w)$ estimate was *sharp*.

If T is any of the Hilbert transform, the Riesz transforms, the Beurling transform, the martingale transform, the dyadic square function or the dyadic paraproduct, then sharp extrapolation guarantees that for any $1 < p < \infty$ there exists a positive constant C_p such that for all weights $w \in A_p$ we have

$$\|T\|_{L^p(w)\to L^p(w)} \leq C_p[w]_{A_p}^{\alpha_p},$$

where $\alpha_p = \max\{1, \frac{p'}{p}\}$. The exponent is sharp for the Hilbert transform, the Riesz transforms, the Beurling and the martingale transforms for all $1 < p < \infty$. For the dyadic square function the exponent is sharp for $1 < p \leq 2$. These estimates are obtained by extrapolating from known linear estimates in $L^2(w)$, namely,

$$\|T\|_{L^2(w)\to L^2(w)} \leq C[w]_{A_2}.$$

These linear estimates were proved using the technique of Bellman functions, [Pet,Witt,HTV,PV], [Pet1], [Wit1], [HTrVo], [PetVo]. My academic sister, Oleksandra Beznosova [Be1], just proved a linear estimate for the dyadic paraproduct in $L^2(w)$, therefore some extrapolated bounds will hold. We do not know yet if any of the extrapolated exponents are sharp for the dyadic paraproduct.

C. Pérez [P2] conjectures a similar result for Calderón-Zygmund operators.

Conjecture 1 (The A_2 conjecture)**.** *Let $1 < p < \infty$ and let T be a Calderón-Zygmund singular integral operator. There is a constant $c = c(n,T)$ such that for any A_p-weight w*

$$\|T\|_{L^p(w)\to L^p(w)} \leq c\, p\, [w]_{A_p}^{\max\{1,\frac{p'}{p}\}}. \tag{0.1}$$

Clearly, as the name suggests, it suffices to prove this conjecture for $p = 2$ and apply the Sharp Extrapolation Theorem to get (0.1). It is not known whether the Bellman function techniques could be also extended to this class of operators.

For $p > 2$ these bounds were improved for S, the square function (both continuous and dyadic), by A. Lerner [Le1] who proved a two weight estimate $\|S\|_{L^2(u)\to L^2(v)}$ and then extrapolated obtaining a better than linear bound

$$\|S\|_{L^p(w)\to L^p(w)} \leq C[w]_{A_p}^{\max\{1,\frac{p}{2}\}\frac{1}{p-1}}.$$

In particular, this holds for the dyadic square function defined as

$$S^d f(x) = \left(\sum_I \frac{|m_{I_l}f - m_{I_r}f|^2}{|I|}\chi_{I(x)}\right)^{\frac{1}{2}},$$

where $m_I f$ is the average of f over the interval $I \in \mathcal{D}$, and $\mathcal{D}$ denotes the dyadic intervals. It is not known yet whether Lerner's estimates are optimal. In fact, Lerner himself conjectures they are not. Finding the optimal power for the square function and $p > 2$ is a very interesting open problem.

In this dissertation we are interested in obtaining Sharp Extrapolation Theorems $d\sigma$ which could be used to extrapolate some known estimates on the weighted space $L^2(wd\sigma)$ to $L^2(wd\sigma)$. We define $d\sigma = u_o dx$, for some $u_o \in A_\infty$, and we consider the class of weights $A_p(d\sigma)$. We also need the Buckley-type *sharp*-estimates for the Maximal Function M_σ, associated with measure σ. First we prove the weighted *weak*-type inequalities and then use interpolation (as in [Buc2]) to obtain the *strong sharp*-estimates. Since this argument is true for any sublinear operator satisfying the same initial *weak*-estimates as the maximal function does, we state the theorem in a more general form, and then give the corresponding result for the maximal operator M_σ.

When trying to implement this outline some technical difficulties were encountered. More precisely, we are seeking *strong*-estimates in $L^p(wd\sigma)$ for the maximal function M_σ, when $w \in A_p(d\sigma)$. It is clear that $w \in A_p(d\sigma)$ implies $w \in A_{p+\varepsilon}(d\sigma)$ for any positive ε, and $[w]_{A_{p+\varepsilon}(d\sigma)} \leq [w]_{A_p(d\sigma)}$. It is a more delicate issue to show that there is an $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}(d\sigma)$. (This is a self-improving integrability condition of weights that we will call the Coifman-Fefferman property). Moreover, we want ε as large as possible with uniform control over $[w]_{A_{p-\varepsilon}(d\sigma)}$ in terms of $[w]_{A_p(d\sigma)}$. Once such an ε has been established, we have *sharp weak*-estimates on $L^{p\pm\varepsilon}(wd\sigma)$ that we can interpolate, keeping track of the constant $[w]_{A_p(d\sigma)}$ to obtain *strong* $L^p(wd\sigma)$ estimates.

Buckley claims ([Buc1], [Buc2]) that in the case $d\sigma = dx$ one can choose $\varepsilon \sim [w]^{1-p'}$ to get a uniform control $[w]_{A_{p-\varepsilon}} \leq C[w]_{A_p}$. He references the classical [CoFe], however we could not find the justification there. When talking to Carlos Pérez his first reaction was *the proof is not there*([CoFe]); and he sent us a preprint with his own argument which we also extended to the $d\sigma$ case. We had to work on a proof valid also in the general $d\sigma$ case. What seemed to work was an argument using another self-improvement property (proved in [CoFe]) that says that $w \in A_p(d\sigma)$, satisfies the Reverse Hölder condition. More precisely, if $w \in A_p(d\sigma)$, then $w \in RH_{1+\gamma}(d\sigma)$. The game now is to introduce $[w]_{A_p}$ into the picture, and carefully track the dependence of γ on $[w]_{A_p(d\sigma)}$. This knowledge could then be used to track down ε in the Coifman-Fefferman property. However, the proof of the Reverse Hölder Inequality we used, yields the expected order of γ, but it did not seem to guarantee a uniform control over $[w]_{A_{p-\varepsilon}(d\sigma)}$.

Instead we took advantage of the added flexibility of having the $d\sigma$ measure.

There is what we call a tautology $w \in A_p(u_o dx) \iff w^{-1} \in RH_{p'}(wu_o dx)$. We can use now self-improving properties of the $RH_q(d\mu)$ weights. This time $v \in RH_q(d\mu)$ implies immediately $v \in RH_{q-\varepsilon}(d\mu)$ for any $0 < \varepsilon < q-1$. What is now more delicate is to go in the other direction. This is the celebrated Gehring Lemma saying there is an $\varepsilon > 0$ such that $v \in RH_{q+\varepsilon}(d\mu)$. Again, we need ε to be as large as possible with uniform control on $[w]_{RH_{q+\varepsilon}(d\mu)}$ in terms of $[w]_{RH_q(d\mu)}$.

We carefully analyzed Gehring's proof [Ge] in the $d\mu = dx$ case, modulo some refinements made by Iwaniec [Iw], and generalized it to the $d\mu$ case. With the $d\mu$ version of Gehring lemma at hand, we can now obtain *strong*-estimates for M_σ in $L^p(d\sigma)$ and $w \in A_p(d\sigma)$. By the tautology $w^{-1} \in RH_{p'}(d\mu)$, where $d\mu = wd\sigma$, with $v = w^{-1}$, $q = p'$, we get $v \in RH_{p'\pm\varepsilon}(d\mu)$ for an appropriate ε with a uniform control over $[v]_{RH_{p'\pm\varepsilon}(wd\sigma)}$ in terms of $[v]_{RH_{p'}(d\mu)}$. It also means that $w \in A_{(p'\pm\varepsilon)'}(d\sigma)$, and *weak*-estimates hold in $L^{p_o}(wd\sigma)$, $L^{p_1}(wd\sigma)$ for $p_o = (p'+\varepsilon)' < p < p_1 = (p'-\varepsilon)'$. This approach leads to the right power of $[w]_{A_p(d\sigma)}$ (as in [Buc2]) but with an extra constant $D(\mu)$.

We return to the Reverse Hölder Property and use a new proof ([P1]) which allows us to have a uniform control over $[w]_{A_{p-\varepsilon}(d\sigma)}$ in terms of $[w]_{A_p(d\sigma)}$, and relate ε and $[w]_{A_p(d\sigma)}$. This time the constant $D(\sigma)$ appears and we obtain the same power of $[w]_{A_p(d\sigma)}$. This also helps us justify and understand Buckley's classical result for the $d\sigma = dx$ case.

At the end of Chapter 2 we also include the estimates for M_σ inspired by A. Lerner's very recent result (see [Le2]). It allows us to obtain the optimal power of $[w]_{A_p(d\sigma)}$ without using interpolation. Instead it is based on the basic properties of maximal functions and Besicovitch Theorem for the centered maximal function. The doubling constant $D(\sigma)$ also appears but with a different exponent.

Both in the proofs of the *weak*-estimates and the *strong*-ones we encounter different doubling constants, $D(\sigma)$ and $D(\mu)$ respectively. Their role, especially the one of $D(\mu)$, is not fully understood so we decided to keep them both to get some insight. They may be only the artifact of the method used in the proofs. They come directly from the theorem which says that for a doubling measure σ the Maximal Function M_σ is of *weak*-type $(1,1)$ with *weak*-norm at most $D(\sigma)$, or as result of the Calderón-Zygmung decomposition used in the proof. To the best of our knowledge the question whether or not the *weak*-norm can be replaced with a constant independent of the measure is still open (see [StStr], [GrM-S], [GrK]). Even if it has to be there, it is not sure how this may affect the power of the characteristic constant $[w]_{A_p(d\sigma)}$. Moreover, in some of our applications when $d\mu = dx$, $D(\mu)$ is known and even easier to handle than $D(\sigma)$.

In Chapter 3 we use the estimates for M_σ to build the Sharp Extrapolation Theorem $d\sigma$. We follow closely the idea in [DrGrPerPet] tracking down the dependence on $[w]_{A_p(d\sigma)}$. We also state it in an operator-free form (with Tf replaced with an arbitrary L^p integrable function g) which is more convenient in some applications and more general (see p.48 and [CrMPe1]). For example, the theorem in this form can be applied directly to some known inverse estimates for S^d_σ on the weighted $L^2(wd\sigma)$ obtained in [Per4], as it is not important on which side the operator is located.

Moreover, the first lemma used in the proof of Sharp Extrapolation $d\sigma$ will also help us to generalize the above mentioned Lerner's extrapolation technique to the $d\sigma$ case, for any $p_o > 1$ and not just $p_o = 2$, and with different initial two weight estimates.

In [Per1] this extrapolation $d\sigma$ (not *sharp*) was used to obtain L^p estimates for Haar multipliers by comparing to square function S_u. In that case the trick is to think of $d\sigma = udx$, and $d\mu = dx = wd\sigma$ so that $w = u^{-1}$ and $L^p(wd\sigma)$ estimates for S_σ become L^p estimates for S_u. This example was an inspiration to start thinking about this problem.

In the Preliminaries Chapter 1 we list the basic concepts and tools which will be used, some of them already adapted to the $d\sigma$ case. In Chapter 2 we show *weak*-estimates for M_σ, involving both $D(\sigma)$ and $D(\mu)$ constants. We use them to derive *strong sharp*-estimates using interpolation. First we interpolate on $RH_q(d\mu)$ side, and then we obtain similar estimates repeating the procedure on the $A_p(d\sigma)$ side. At the end we show a different way of obtaining such estimates without using interpolation. In Chapter 3 we use the estimates for M_σ to build the Sharp Extrapolation Theorem and then use it to extend Lerner's extrapolation technique to the $d\sigma$ case and for any $p_o > 1$. At the end we show some examples and possible applications of our results.

Chapter 1

Preliminaries

In this chapter we recall the reader basic definitions and well known results. In particular we recall the definitions of weights, doubling measures, distribution functions, Hardy-Littlewood maximal functions. We also state the basic tools such as: interpolation, how to recover L^p-norms from distribution functions, how to construct $A_1(d\sigma)$ weights from locally integrable functions and the maximal function (Coifman-Rochberg $d\sigma$, Lemma 1.4.1), Calderón-Zygmund decomposition and Besicovitch Covering Lemma. We list basic properties of weights, interplay between them and connections with maximal function. We also recall the classical Rubio de Francia Theorem and its *weak*-variant.

All the measures discussed in this dissertation are positive Borel measures. We will simply refer to them as *measures*.

1.1 Weights and doubling measures

A *weight* is a locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere. Given a weight w and a measurable set E, the w-measure of the set E

is denoted by

$$w(E) = \int_E w(x)\,dx.$$

For a measure σ, $\sigma(E) = \int_E d\sigma$. In particular, $|E|$ stands for the Lebesgue measure of E.

Definition 1.1.1. *We say a positive Borel measure σ is a doubling measure, $\sigma \in \mathbf{D}$, if there exists a constant C, independent of Q, such that $\sigma(3Q) \leq C\sigma(Q)$, where $3Q$ denotes the cube concentric to Q with side length three times as long. The smallest such constant is called the doubling constant of σ and denoted by $D(\sigma)$. We say w is a doubling weight if wdx is a doubling measure. Its doubling constant will be denoted by $D(w)$.*

For any positive quantities X, Y "$X \sim Y$" will mean $\frac{1}{C} \leq \frac{X}{Y} \leq C$, for some positive constant C depending only on n and p. For any $p > 0$ and $p \neq 1$, p' denotes the dual exponent i.e. $p' = \frac{p}{p-1}$.

Remark 1.1.1. *Sometimes the doubling condition is defined as above but with $2Q$ instead of $3Q$. However, $3Q$ has the advantage that if we consider the dyadic parent of Q, say $\widetilde{Q}$, so that Q is one of the 2^n children of $\widetilde{Q}$ in $\mathbb{R}^n$ obtained by subdividing each side of $\widetilde{Q}$ into 2^n equal pieces, then $\widetilde{Q} \subset 3Q$, but never in $2Q$.*

1.2 Distribution Function and Real Interpolation

The distribution function $d_f(\lambda)$ provides information about the size of f but not about the behavior of f near any given point.

Definition 1.2.1. *Let (X, σ) be a measurable space and let $f : X \to \mathbb{C}$ be a measurable function. We call the function $d_f : (0, \infty) \to [0, \infty)$, given by*

$$d_f(\lambda) = \sigma\left(\{x \in X : |f(x)| > \lambda\}\right),$$

the distribution function of f (associated with σ).

Proposition 1.2.1. *Let $\phi : [0,\infty) \to [0,\infty)$ be differentiable, increasing and such that $\phi(0) = 0$. Then*

$$\int_X \phi\left(|f(x)|\right) d\sigma = \int_0^\infty \phi'(\lambda)\, d_f(\lambda)\, d\lambda \tag{1.1}$$

In particular, if $\phi(t) = t^p$ and $f \in L^p(d\sigma)$, then will get

$$\|f\|^p_{L^p(d\sigma)} = p \int_0^\infty \lambda^{p-1} d_f(\lambda)\, d\lambda. \tag{1.2}$$

If f is a finite function and $d_f(\lambda) < \infty$ for all $\lambda > 0$, then

$$\int_X |f(x)|^p \, d\sigma = -\int_0^\infty \lambda^p \, \mathsf{d} d_f(\lambda). \tag{1.3}$$

Proofs can be found in [Sa] p.163. Below we prove a variation of (1.3) which will be used in Section 2.1. More precisely,

$$\int_X g^r \, d\sigma = -\int_0^\infty s^{r-1} dh(s), \tag{1.4}$$

where

$$h(s) = \int_{E(s)} g\, d\sigma \quad \text{and} \quad E(s) = \{x \in X :\ g(x) > s\},$$

and the function g is positive and locally integrable, as in Lemma 2.2.2.

Proof. If $h(t)$ is finite for all positive t and $g(t)$ also finite, then the integral at the right hand side of (1.4) is well defined. The integral at the left can be expressed by the Lebesgue sums as follows. Taking an ε-subdivision, $0 < \varepsilon < 2\varepsilon ... < m\varepsilon < ...$ and letting $X_j = \{x \in X :\ (j-1)\varepsilon \le g(x) < j\varepsilon\}$, with the measure $d\mu_o := g d\sigma$, we have $\mu_o(X_j) = h((j-1)\varepsilon) - h(j\varepsilon)$ and

$$\begin{aligned} \int_X g^r \, d\sigma &= \int_X g^{r-1}\, g d\sigma = \int_X g^{r-1}\, d\mu_o = \lim_{\varepsilon \to 0} \sum_{j=1}^\infty (j\varepsilon)^{r-1} \mu_o(X_j) \\ &= -\lim_{\varepsilon \to 0} \sum_{j=1}^\infty (j\varepsilon)^{r-1} [h(j\varepsilon) - h((j-1)\varepsilon)] = -\int_0^\infty s^{r-1} dh(s), \end{aligned}$$

where the last integral is a Riemann-Stieltjes Integral (see [R]). In particular, if $X = E(t)$, then $h(s) = h(t)$ for all $0 < s \leq t$, which implies that $dh(s) = 0$ for all $s \leq t$. Thus,

$$\int_{E(t)} g^r \, d\sigma = -\int_t^\infty s^{r-1} dh(s). \tag{1.5}$$

We will use this equation in the proof of Lemma 2.2.2. □

Definition 1.2.2. *We say an operator T is of weak-type (p,p), with respect to the measure σ, if*

$$d_{Tf}(\lambda) = \sigma\left(\{x \in X : \ |Tf| > \lambda\}\right) \leq \left(\frac{C\|f\|_{L^p(d\sigma)}}{\lambda}\right)^p \tag{1.6}$$

The smallest such C is called *weak*-type $L^p(d\sigma)$ norm of T.

The following theorem is a useful tool which allows to deduce L^p boundedness from *weak* inequalities.

Theorem 1.2.1 (Marcinkiewicz interpolation theorem with respect to *sigma*)**.** *Suppose $1 \leq p_0 < p_1 < \infty$ and that T is a sublinear operator of weak-type (p_0, p_0) and (p_1, p_1), with respect to the measure σ, with norms M_0 and M_1, then T is actually of strong type (p,p) for all $p_0 < p < p_1$ and for any $0 < t < 1$,*

$$\|Tf\|_{L^{p_t}(d\sigma)} \leq K_t M_0^{1-t} M_1^t \|f\|_{L^{p_t}(d\sigma)}, \tag{1.7}$$

where $\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}$ *and* $K_t^{p_t} = \frac{2^{p_t}}{p_t}\left(\frac{p_1}{p_1 - p_t} + \frac{p_0}{p_t - p_0}\right)$.

The proof can be found for example in [Sa] or [Gr].

We will need this theorem only in two very particular cases. Firstly, when $p_0 := p - \varepsilon$ and $p_1 := p + \varepsilon$, for some $\varepsilon > 0$ such that $\max\{M_{p-\varepsilon}, M_{p+\varepsilon}\} \leq c\, M_p$, for some absolute constant $c \geq 1$ and $t = \frac{\varepsilon}{2p} + \frac{1}{2}$. Secondly, when $p_0 := (p' + \varepsilon)'$ and $p_1 := (p' - \varepsilon)'$, in which case $t = \frac{1}{2} - \frac{\varepsilon}{2}\frac{(p-1)}{p}$. In both cases $p_t = p$.

Then, the inequality (1.7) becomes

$$\|Tf\|_{L^p(d\sigma)} \leq \frac{K_p\, M_p}{(\varepsilon)^{\frac{1}{p}}} \, \|f\|_{L^p(d\sigma)}, \tag{1.8}$$

where $K_p = 2^{\frac{p+1}{p}}\, c$ in the first case, and in the second case $K_{p'} = \frac{2^{\frac{p+1}{p}}}{(p-1)^{\frac{2}{p}}} c$ (and blows up as p approaches 1).

1.3 Hardy-Littlewood Maximal Function

The study of averages of functions is better understood and simplified by the introduction of the maximal function.

Definition 1.3.1. *Given a locally measurable function f, and a measure $\sigma \in D$, the (uncentered) Hardy-Littlewood maximal function (associated with σ), $M_\sigma f$, is defined by*

$$M_\sigma f(x) = \sup_{Q \ni x} \frac{1}{\sigma(Q)} \int_Q |f(y)|\, d\sigma,$$

for all cubes Q in $\mathbb{R}^n$ with sides parallel to the axes.

The function

$$M_\sigma^c f(x) = \sup_{Q_x} \frac{1}{\sigma(Q_x)} \int_{Q_x} |f(y)|\, d\sigma,$$

where Q_x denotes the cubes centered at x, is called the centered Hardy-Littlewood maximal function.

Remark 1.3.1. *Let us observe that*

$$M_\sigma^c f(x) \leq M_\sigma f(x) \leq D(\sigma)\, M_\sigma^c f(x). \tag{1.9}$$

The first inequality is obvious. If we take $x \in Q$, and Q_x denotes a cube congruent to Q and centered at x, then $Q \subset 3\,Q_x$ and

$$\frac{1}{\sigma(Q)} \int_Q |f|\, d\sigma \leq \frac{1}{\sigma(Q)} \int_{3Q_x} |f|\, d\sigma \;\leq \frac{\sigma(3Q_x)}{\sigma(Q)} \frac{1}{\sigma(3Q_x)} \int_{3Q_x} |f|\, d\sigma \leq D(\sigma) M_\sigma^c f(x),$$

from which the second inequality follows. In particular,

$$\|M_\sigma f\|_{L^p(d\mu)} \leq D(\sigma) \|M_\sigma^c f\|_{L^p(d\mu)}. \tag{1.10}$$

Theorem 1.3.1. *The maximal operator M_σ is bounded on $L^p(d\sigma)$, $1 < p \leq \infty$, and is weak $(1,1)$, with respect to σ, and a weak-norm at most $D(\sigma)$.*

The proof can be found in [Gr] or [Du].

Definition 1.3.2. *A dyadic interval in $\mathbb{R}$ is an interval of the form*

$$[m2^{-k},\, (m+1)2^{-k}),$$

where m, k are integers. A dyadic cube in $\mathbb{R}^n$ is a product of dyadic intervals of the same length.

Any two dyadic cubes are either disjoint, or one contains the other.

Theorem 1.3.2. *The dyadic maximal operator, with respect to Borel measure σ, and defined as*

$$M^d_\sigma f(x) = \sup_{\substack{Q \ni x \\ Q \text{ dyadic cube}}} \frac{1}{\sigma(Q)} \int_Q |f(y)|\, d\sigma,$$

is weak $(1,1)$, with respect to σ, and a weak-norm at most 1. (See [Gr], p. 555).

1.4 $A_1(d\sigma)$ weights

In this section we introduce the class $A_1(d\sigma)$ and prove a very useful result which allows to construct such weights from $L^1_{loc}(d\sigma)$ functions.

Definition 1.4.1. *A weight w is said to be an $A_1(d\sigma)$ weight, if*

$$M_\sigma(w)(x) \leq C\, w(x), \text{for } \sigma\text{-a.e. } x \in \mathbb{R}^n \text{ and some constant } C. \tag{1.11}$$

The infimum of all such C's is called $[w]_{A_1(d\sigma)}$.

The following characterization first appeared in [CoRo] and below we follow the idea from [GaRu] and adapt it to the $d\sigma$ case.

Lemma 1.4.1 (Coifman-Rochberg for $d\sigma$). *Let $f \in L^1_{loc}(d\sigma)$ be such that $M_\sigma f < \infty$ σ-a.e. If $0 \le \gamma < 1$, then $(M_\sigma f)^\gamma$ is an $A_1(d\sigma)$ whose $A_1(d\sigma)$ constant depends only on γ and $D(\sigma)$.*

Proof. Let Q be a fixed cube and $\widetilde{Q} := 3Q$. We write $f = f_1 + f_2$, where $f_1 = f\,\chi_{\widetilde{Q}}$ is the restriction of f to $\widetilde{Q}$. Since M_σ is sublinear and $0 \le \gamma < 1$, we have

$$(M_\sigma f)^\gamma \le (M_\sigma f_1)^\gamma + (M_\sigma f_2)^\gamma .$$

It suffices to show that for all x, and $Q \ni x$, there exists $C > 0$ such that

$$\frac{1}{\sigma(Q)} \int_Q (M_\sigma f_i)^\gamma \; d\sigma \le C\,(M_\sigma f)^\gamma\,(x),$$

for $i = 1, 2$.

We carry out the estimates for $M_\sigma f_1$ and $M_\sigma f_2$ separately.
We use Proposition 1.2.1, then split the integral at $R = \frac{\|f_1\|_{L^1(d\sigma)}}{\sigma(Q)}$ to get

$$\begin{aligned}\frac{1}{\sigma(Q)} \int_Q (M_\sigma f_1)^\gamma \; d\sigma &= \frac{1}{\sigma(Q)} \int_0^\infty \gamma\, t^{\gamma-1} \sigma(\{x \in Q : M_\sigma f_1 > t\})\, d\sigma \\ &= \frac{1}{\sigma(Q)} \int_0^R \gamma\, t^{\gamma-1} \sigma(\{x \in Q : M_\sigma f_1 > t\})\, d\sigma \\ &\quad + \frac{1}{\sigma(Q)} \int_R^\infty \gamma\, t^{\gamma-1} \sigma(\{x \in Q : M_\sigma f_1 > t\})\, d\sigma\end{aligned}$$

In the first integral, the distribution function is clearly less than $\sigma(Q)$.
In the second, using the *weak*-type estimate stated in Theorem 1.3.1, the distribution function is less than $D(\sigma)\, t^{-1} \|f_1\|_{L^1(d\sigma)}$.

Thus

$$\begin{aligned}\frac{1}{\sigma(Q)}\int_Q (M_\sigma f_1)^\gamma\, d\sigma &\le \frac{1}{\sigma(Q)}\left(\sigma(Q)\, R^\gamma + D(\sigma)\,\|f_1\|_{L^1(d\sigma)}\int_R^\infty \gamma\, t^{\gamma-2}\, dt\right)\\ &= R^\gamma\left(1+\frac{D(\sigma)\gamma\,\|f_1\|_{L^1(d\sigma)}}{(1-\gamma)\,R\,\sigma(Q)}\right) = R^\gamma\left(1+\frac{D(\sigma)\,\gamma}{(1-\gamma)}\right)\\ &= \left(1+\frac{D(\sigma)\,\gamma}{(1-\gamma)}\right)\left(\frac{\int_{3Q}|f|\,d\sigma}{\sigma(Q)}\right)^\gamma\\ &\le \left(1+\frac{D(\sigma)\,\gamma}{(1-\gamma)}\right)\left(D(\sigma)\frac{\int_{3Q}|f|\,d\sigma}{\sigma(3Q)}\right)^\gamma = C_{\sigma,\gamma}\,(M_\sigma f(x))^\gamma.\end{aligned}$$

where $C_{\sigma,\gamma} = \left(1+\frac{D(\sigma)\,\gamma}{(1-\gamma)}\right) D(\sigma)^\gamma$, and for any $x \in Q$.
Now we can estimate the second term. By construction of f_2, for any $x, y \in Q$ we have

$$M_\sigma f_2(y) \le D(\sigma) M_\sigma f_2(x). \tag{1.12}$$

In fact, if Q' is a cube containing y and intersecting the complement of $\widetilde{Q}$, then $x \in Q \subset 3\,Q'$, and

$$\frac{1}{\sigma(Q')}\int_{Q'} f_2\, d\sigma \le \frac{D(\sigma)}{\sigma(3Q')}\int_{3Q'} f_2\, d\sigma \le D(\sigma)\, M_\sigma f_2(x). \tag{1.13}$$

Now, take the supremum over all cubes Q' containing y, on the left hand side, and we get (1.12). (Note that if $y \in Q' \subset \widetilde{Q}$, then $\frac{1}{\sigma(Q')}\int_{Q'} f_2\, d\sigma{=}0$). Thus, raising the inequality (1.12) to the power γ, and taking σ-average over Q with respect to y, for every $x \in Q$ we have

$$\frac{1}{\sigma(Q)}\int_Q (M_\sigma f_2(y))^\gamma\, d\sigma \le D(\sigma)^\gamma\,(M_\sigma f_2(x))^\gamma \le D(\sigma)^\gamma\,(M_\sigma f(x))^\gamma.$$

Putting together both estimates we conclude that for all $x \in Q$

$$\frac{1}{\sigma(Q)}\int_Q (M_\sigma f)^\gamma\, d\sigma \le \left(2+\frac{D(\sigma)\,\gamma}{(1-\gamma)}\right) D(\sigma)^\gamma\,(M_\sigma f(x))^\gamma.$$

□

This proves that $(M_\sigma f)^\gamma \in A_1(d\sigma)$.

Remark 1.4.1. *The converse statement is also true, see for example [Du] or [Gr] p. 690.*

1.5 $A_p(d\sigma)$ weights

In this section we define the class of $A_p(d\sigma)$ weights and state their main properties.

Definition 1.5.1. *If σ is a positive measure on $\mathbb{R}^n$, we say w is an $A_p(d\sigma)$ weight, $1 < p < \infty$, if*

$$\sup_{Q\in\mathbb{R}^n}\left(\frac{1}{\sigma(Q)}\int_Q w(x)\,d\sigma\right)\left(\frac{1}{\sigma(Q)}\int_Q w(x)^{-\frac{1}{p-1}}\,d\sigma\right)^{p-1} < \infty. \tag{1.14}$$

The expression in (1.14) is called the $A_p(d\sigma)$ characteristic constant of w and is denoted by $[w]_{A_p(d\sigma)}$.

The main properties of these classes of weights include the following:

(a) The classes $A_p(d\sigma)$ are increasing as p increases; precisely, for $1 < p < q < \infty$ we have $A_p(d\sigma) \subset A_q(d\sigma)$ and $[w]_{A_q(d\sigma)} \le [w]_{A_p(d\sigma)}$, (This is just a simple consequence of Hölder's inequality.)

(b) $w \in A_p(d\sigma)$ if and only if $w^{1-p'} \in A_{p'}(d\sigma)$ and

$$[w^{1-p'}]_{A_{p'}(d\sigma)} = [w]^{\frac{1}{p-1}}_{A_p(d\sigma)}, \tag{1.15}$$

This type of statement is what we will call a tautology, and it follows directly from the definitions.

(c) If $w \in A_p(d\sigma)$, then $w \in A_{p-\varepsilon}(d\sigma)$ for some $\varepsilon > 0$. (This is a deeper result due to Coifman-Fefferman which can be found in [CoFe] when $d\sigma = dx$, and in the general case in [Buc2], [OrPe]). We will prove and analyze closely its $d\sigma$ version, in particular the relation between ε, $[w]_{A_p(d\sigma)}$ and $D(\sigma)$ in Subsection 2.4.2.

Remark 1.5.1. *The function $w(x) = |x|^\alpha$ belongs to A_p, $p > 1$ if and only if $-n < \alpha < n(p-1)$; w is an A_1 weight if and only if $-n < \alpha \le 0$. However, w is doubling in the larger range $-n < \alpha < \infty$.*

1.6 A_p weights and classical results

In the case $d\sigma = dx$, the reference to the measure is usually suppressed, and A_p, M will be used. Below we list the classical results about the A_p weights.

The most important theorem in the theory of the A_p weights, due to Muckenhoupt, states that the A_p class fully characterizes the boundedness of M on $L^p(w)$. The A_p class can be also used to characterize many other classical operators. Furthermore, it is of interest to determine how the operator norms are bounded in terms of the $[w]_{A_p}$ constant.

Theorem 1.6.1 (Muckenhoupt)**.** *Let $1 \le p < \infty$. The inequality*

$$w\left(x : Mf(x) > \lambda\right) \le \frac{C}{\lambda^p} \int_Q |f(x)|^p w(x)\, dx$$

holds if and only if $w \in A_p$.

Furthermore,

Theorem 1.6.2. *If $1 < p < \infty$, then M is bounded on $L^p(w)$ if and only if $w \in A_p$.*

S. Buckley [Buc1] obtained the following sharp result for M in terms of $[w]_{A_p}$.

Theorem 1.6.3. *If $1 < p < \infty$, $w \in A_p$, then*

$$\|Mf\|_{L^p(w)} \leq C[w]_{A_p}^{\frac{p'}{p}} \|f\|_{L^p(w)}. \tag{1.16}$$

The power $[w]_{A_p}^{\frac{p'}{p}}$ is best possible.

We will revisit these Theorems and their proofs in Chapter 2 when trying to obtain the $d\sigma$ version. The following factorization theorem provides an interesting representation of the A_p weights.

Theorem 1.6.4 (Jones). *Suppose w is an A_p weight, for some $1 < p < \infty$. Then there are A_1 weights w_1, w_2 such that*

$$w = w_1 \, w_2^{1-p}$$

The proof can be found in [Jo], and most textbooks [Du], [Gr], [GaRu]. The A_p condition can be also formulated for pairs of weights as follows.

Definition 1.6.1. *A pair of weights (u, v) is said to belong to the class A_p, $1 < p < \infty$, if*

$$\sup_{Q \in \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q u(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q v(x)^{-\frac{1}{p-1}}\, dx \right)^{p-1} < \infty. \tag{1.17}$$

The expression in (1.17) is called the A_p characteristic constant of (u, v) and is denoted by $[u, v]_{A_p}$. We will consider pairs of the A_p weights when proving Lerner's $d\sigma$ Extrapolation Theorem in Section 3.3.

Definition 1.6.2. *A pair of weights (u, v) is said to be an A_1 weight, if*

$$M(u)(x) \leq Cv(x) \;\; \text{a.e. } x \in \mathbb{R}^n, \; \text{for some constant } C. \tag{1.18}$$

When $u = v$ these are exactly the A_1 and A_p conditions defined in (1.3) and (1.4).

Necessary and sufficient conditions on u and v, such that the *weak-type* inequality (1.19) holds, are similar to the one-weighted case, presented in Theorem 1.6.1.

Theorem 1.6.5 (Muckenhoupt)**.** *Given p, $1 \leq p < \infty$, the weak-type inequality*

$$u(\{x : Mf(x) > \lambda\}) \leq \frac{C}{\lambda^p} \int_Q |f(x)|^p v(x)\, dx \tag{1.19}$$

holds if and only if $(u, v) \in A_p$.

This was proved in [Mu]. However, for *the strong* (p, p) boundedness the A_p condition is necessary but not sufficient. The characterization of *strong* inequalities with two weights is due to E. Sawyer and is called S_p condition. The pair $(u, v) \in S_p$ if and only (1.20) holds.

Theorem 1.6.6 (Sawyer)**.** *M is bounded from $L^p(v)$ to $L^p(u)$, $1 < p < \infty$, if and only if*

$$\int_Q \Big(M(\chi_Q v^{1-p'})(x)\Big)^p u(x)\, dx \leq C \int_Q v(x)^{1-p'}\, dx. \tag{1.20}$$

for all cube Q, and C independent of Q, χ_Q the characteristic function of the cube Q.

For the proof we refer the reader to [Saw1]. For equal weights, A_p and S_p conditions are equivalent [HuKN].

Remark 1.6.1. *A few years before the A_p theory, in 1960, H. Helson and G. Szegö (see [HeSz]) gave necessary and sufficient condition for the boundedness of the Hilbert transform in $L^2(\mu)$.*

Theorem 1.6.7. *The Hilbert Transform H is bounded in $L^2(\mu)$ if and only if μ is absolutely continuous with respect to Lebesgue measure, $d\mu = w(x)dx$, and w is of the form*

$$\log w = u + H\,v, \text{ with } u, v \in L^\infty \text{ and } \|v\|_\infty < \frac{\pi}{2}.$$

This condition works only for L^2 and must be equivalent to A_2 but there is not direct proof of this equivalence (see [Du2] or [Sa]). The L^p versions and other important generalizations were proved by Cotlar and Sadosky in a sequence of papers (see [Sa]).

1.7 $A_\infty(d\sigma)$ weights

Definition 1.7.1. *We say w is an $A_\infty(d\sigma)$ weight if, for all cubes Q, and all $E \subset Q$, we have*

$$\frac{\mu(E)}{\mu(Q)} \le C \left(\frac{\sigma(E)}{\sigma(Q)}\right)^{\varepsilon} \tag{1.21}$$

for some C, $\varepsilon > 0$, where $d\mu = w\,d\sigma$.

Lemma 1.7.1. *The following is equivalent to $w \in A_\infty(d\sigma)$*
For all cubes Q, there exists a constant C independent of Q such that,

$$\frac{1}{\sigma(Q)} \int_Q w\,d\sigma \le C \exp\left(\frac{1}{\sigma(Q)} \int_Q \log w\,d\sigma\right).$$

The proof can be found in [Buc2]. Thus, the $A_\infty(d\sigma)$ characteristic constant can be defined as follows.

$$[w]_{A_\infty(d\sigma)} := \sup_{Q\in\mathbb{R}^n} \left\{\left(\frac{1}{\sigma(Q)} \int_Q w\,d\sigma\right) \exp\left(\frac{1}{\sigma(Q)} \int_Q \log w^{-1}\,d\sigma\right)\right\} < \infty. \tag{1.22}$$

Theorem 1.7.1. *(Jensen's Inequality)*
Let σ be a positive measure on a sigma-algebra $\mathcal{M}$ in a set Ω, so that $\sigma(\Omega) = 1$. If f is a real function in $L^1(\sigma)$, if $a < f(x) < b$ for all $x \in \Omega$, and if ϕ is convex on (a, b), then

$$\phi\left(\int_\Omega f\,d\sigma\right) \le \int_\Omega (\phi \circ f)\,d\sigma. \tag{1.23}$$

The cases $a = -\infty$ and $b = \infty$ are not excluded.

The proof can be found for example in [R].

Remark 1.7.1. *If we take $\phi(x) = e^x$, and $f = \log g$, then (1.23) becomes*

$$\exp\left\{\int_\Omega \log g\, d\sigma\right\} \leq \int_\Omega g\, d\sigma \tag{1.24}$$

Corollary 1.7.1. *For all $1 \leq p < \infty$ we have*

$$[w]_{A_\infty(d\sigma)} \leq [w]_{A_p(d\sigma)}. \tag{1.25}$$

This follows from the definitions and Jensen's inequality (1.24), applied to $g = w^{\frac{-1}{p-1}}$.

1.8 $RH_p(d\sigma)$ weights

The Reverse Hölder property is a fundamental feature of the $A_p(d\sigma)$ weights. It is of independent interest and plays an important role in the theory weights.

Definition 1.8.1. *If $\sigma \in \mathbf{D}$ and $1 < p < \infty$, we say w is a $RH_p(d\sigma)$ weight or $w \in RH_p(d\sigma)$ if*

$$\left(\frac{1}{\sigma(Q)}\int_Q w^p\, d\sigma\right)^{\frac{1}{p}} \leq C\frac{1}{\sigma(Q)}\int_Q w\, d\sigma \tag{1.26}$$

for all cubes Q. The smallest such C is referred to as $RH_p(d\sigma)$-characteristic of w and is denoted by $[w]_{RH_p(d\sigma)}$.

Some properties of these classes include the following:

(a) The classes $RH_p(d\sigma)$ are decreasing as p increases; precisely, for $1 < p < q < \infty$ we have $RH_p(d\sigma) \supset RH_q(d\sigma)$.
(This is just a consequence of Hölder inequality.)

(b) If $w \in RH_p(d\sigma)$, then $w \in RH_{p+\varepsilon}(d\sigma)$ for some $\varepsilon > 0$.
This is a much deeper result and it is usually referred to as the Gehring Lemma.

It was first proved in [Ge], and for general measures in [St], [Buc2], [Mil] or [Iw]. We will revisit the proof of Gehrings Lemma $d\sigma$ in Section 2.2, tracking very carefully the relation between ε, $[w]_{RH_p(d\sigma)}$ and $D(\sigma)$.

Remark 1.8.1. *In the study of partial differential equations or quasi-conformal mappings the inequality (1.26) is usually replaced with a weaker and a more natural condition (sometimes called the weak-$RH_p(d\sigma)$, see [UrNeu])*

$$\left(\frac{1}{\sigma(Q)}\int_Q w^p\, d\sigma\right)^{\frac{1}{p}} \leq \frac{C}{\sigma(2Q)}\int_{2Q} w\, d\sigma. \tag{1.27}$$

More details can be found in [UrNeu], [Mil] or [Iw].

Remark 1.8.2. *The function $w(x) = |x|^\alpha$ belongs to RH_p if and only if $\alpha > -\frac{n}{p}$.*

Definition 1.8.2. *If σ_1 and σ_2 are positive doubling measures, we say σ_1 is comparable to σ_2 if there exist α, $\beta \in (0,1)$ such that $\frac{\sigma_1(E)}{\sigma_1(Q)} < \beta$ whenever $\frac{\sigma_2(E)}{\sigma_2(Q)} < \alpha$ for every $E \subset Q$, and every Q.*

Proposition 1.8.1. *The following conditions are equivalent.*

a) There exist $\delta > 0$, $C > 0$ such that for every measurable set A contained in a cube Q

$$\frac{\sigma_2(A)}{\sigma_2(Q)} \leq C\left(\frac{\sigma_1(A)}{\sigma_1(Q)}\right)^\delta,$$

b) σ_2 is comparable to σ_1,

c) σ_1 is comparable to σ_2,

d) $d\sigma_2 = w\, d\sigma_1$ with $w \in RH_p(d\sigma_1)$ for some $p > 1$.

This was proved in [CoFe].

Corollary 1.8.1. *The comparability of measures is an equivalence relation.*

Corollary 1.8.2. *If σ_1 and σ_2 are comparable, then $\sigma_1(A) = 0 \iff \sigma_2(A) = 0$.*

Corollary 1.8.3. *If $u_o \in A_\infty$ and $d\sigma := u_o dx$, and if $d\mu := w d\sigma$, where $w \in A_p(d\sigma)$, then the measures: $d\mu$, $d\sigma$ and dx are comparable.*

In particular, when talking about a.e. statements, this will allow us to omit the underlying measure: σ, μ.

Lemma 1.8.1. *Let σ be an arbitrary but fixed doubling measure on $\mathbb{R}^n$.*

If $w \in A_\infty(d\sigma)$, then $w \in \mathbf{D}$. Furthermore,

$$A_\infty(d\sigma) = \bigcup_{1 \le p < \infty} A_p(d\sigma) = \bigcup_{1 \le q < \infty} RH_q(d\sigma)$$

When $d\sigma = dx$ this a result of Coifman and Fefferman [CoFe]. The general case can be found in [Buc2]. Thus, $w \in A_p(d\sigma) \iff w \in RH_q(d\sigma)$ for some p and q. There is no possible relationship between p and q as the example of a power function $w(x) = |x|^\alpha$ demonstrates. In Section 2.3 we will prove and analyze closely the $d\sigma$ version of what we call Reverse Hölder Inequality (or the $RH_{1+\gamma}(d\sigma)$ property) for the $A_p(d\sigma)$ weights. In particular, the relation between γ, $[w]_{A_p(d\sigma)}$ and $D(\sigma)$.

However, for positive functions u, v on $\mathbb{R}^n$ and $1 < p < \infty$ the following tautology holds (see [Gr]), where $\frac{1}{p} + \frac{1}{p'} = 1$,

$$[uv^{-1}]_{RH_{p'}(vdx)} = [vu^{-1}]^{\frac{1}{p}}_{A_p(udx)} \tag{1.28}$$

that is uv^{-1} satisfies a reverse Hölder condition of order p' with respect to vdx if and only if vu^{-1} is in $A_p(udx)$.

Proof.

$$
\begin{aligned}
[uv^{-1}]_{RH_{p'}(vdx)} &= \sup_{Q\ni x} \frac{\left(\frac{1}{v(Q)}\int_Q (uv^{-1})^{p'}\, vdx\right)^{\frac{1}{p'}}}{\frac{1}{v(Q)}\int_Q (uv^{-1})vdx} = \sup_{Q\ni x} \frac{\left(\frac{1}{v(Q)}\int_Q u^{p'}v^{1-p'}dx\right)^{\frac{1}{p'}}}{\frac{u(Q)}{v(Q)}} \\
&= \sup_{Q\ni x} \frac{v(Q)}{u(Q)} \left(\frac{1}{v(Q)}\int_Q u^{p'}v^{1-p'}dx\right)^{\frac{1}{p'}} \\
&= \sup_{Q\ni x} \frac{[v(Q)]^{\frac{1}{p}}}{u(Q)} \left(\int_Q (vu^{-1})^{1-p'}\, udx\right)^{\frac{p-1}{p}} \\
&= \sup_{Q\ni x} \left[\frac{v(Q)}{u(Q)} \left(\frac{1}{u(Q)}\int_Q (vu^{-1})^{1-p'}\, udx\right)^{p-1}\right]^{\frac{1}{p}} \\
&= \left[\sup_{Q\ni x} \left(\frac{1}{u(Q)}\int_Q (vu^{-1})\, udx\right)\left(\frac{1}{u(Q)}\int_Q (vu^{-1})^{1-p'}\, udx\right)^{p-1}\right]^{\frac{1}{p}} \\
&= [vu^{-1}]^{\frac{1}{p}}_{A_p(udx)}
\end{aligned}
$$

□

In particular,

$$w \in RH_{p'}(dx) \iff w^{-1} \in A_p(wdx), \tag{1.29}$$

$$w \in A_p(dx) \iff w^{-1} \in RH_{p'}(wdx). \tag{1.30}$$

1.9 Rubio de Francia Extrapolation Theorem

In this section we present the classical Rubio de Francia and its *weak*-version.

Theorem 1.9.1 (Rubio de Francia extrapolation)**.** *Given an operator T, suppose that for some p_o, $1 \le p_o < \infty$, and every $w \in A_{p_o}$, there exists a constant C depending on $[w]_{A_{p_0}}$ such that*

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_o} w(x)\, dx \le C \int_{\mathbb{R}^n} |f(x)|^{p_o} w(x)\, dx. \tag{1.31}$$

Then for every p, $1 < p < \infty$*, and every* $w \in A_p$ *there exists a constant depending on* $[w]_{A_p}$ *such that*

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx. \tag{1.32}$$

The original proof was rather complex and nonconstructive. García-Cuerva gave a more direct proof of this result, involving only weighted norm inequalities (see [GaRu]). For $p_o > 1$ a simpler proof is due to Duoandikoetxea (see [Du]). In a very recent monograph on extrapolation (see [CrMPe1]) the authors presented the history of this theorem and provided a new and more direct proof. In addition, they included a number of their results on the generalizations of this theorem in various directions.

Both García-Cuerva (see [Cu]) and Rubio de Francia (see [Ru] or [CrMPe1]) were able to adapt their proofs for the *strong*-type inequalities to get the *weak*-type versions.

Theorem 1.9.2. *Given an operator* T*, suppose that for some* p_o*,* $1 \leq p_o < \infty$*, and all* $w \in A_{p_o}$*,* T *is of* $L^{p_o}(w)$ *weak-type* (p_o, p_o)*, then* T *is of* $L^p(w)$ *weak-type* (p, p) *for all* $1 < p < \infty$*, and all* $w \in A_p$*.*

Proof. If $E_\lambda := \{x \in \mathbb{R}^n : \ |Tf(x)| > \lambda\}$, then we can write the *weak*-type (p_o, p_o) inequality as

$$\|\lambda \chi_{E_\lambda}\|_{L^{p_o}(w)} \leq C \|f\|_{L^{p_o}(w)}.$$

Now, we can apply the Extrapolation Theorem to get

$$\|\lambda \chi_{E_\lambda}\|_{L^p(w)} \leq C \|f\|_{L^p(w)},$$

for all p and $w \in A_p$, which is equivalent to T being of *weak*-type (p, p), for any $1 < p < \infty$. $\square$

As J. Duoandikoetxea points out (see [Du2]) it is possible to start with *weak* $(1,1)$ inequalities with respect to all A_1 weights to deduce *strong* $L^p(w)$ inequalities with $w \in A_p$. However, it is not possible to obtain *weak* $(1,1)$ inequalities in Theorem 1.9.1. There exist operators bounded on $L^p(w)$ for all $w \in A_p$, and $1 < p < \infty$, which are not *weak* $(1,1)$ (even unweighted). In [CrMPe] Rubio de Francia A_p Theorem was generalized to A_∞ weights in the context of Muckenhoupt bases, which allows to prove *weak*-endpoint inequalities starting from *strong*-type inequalities.

1.10 Calderón-Zygmund Decomposition and Besicovitch Covering Lemma

Calderón-Zygmund Decomposition was invented by Calderón and Zygmund and proved to be extremely useful in real variable analysis of singular integrals.

Theorem 1.10.1. *Given $\sigma \in \mathbf{D}$, and a function f which is σ-integrable and non-negative, and given a positive number λ, there exists a sequence $\{Q_j\}$ of disjoint dyadic cubes such that*

(i) $f(x) \leq \lambda$ *for* σ*-a.e.* $x \notin \bigcup_j Q_j$;

(ii) $\sigma\left(\bigcup_j Q_j\right) \leq \frac{D(\sigma)\,\|f\|_{L^1(d\sigma)}}{\lambda}$;

(iii) $\lambda < \frac{1}{\sigma(Q_j)} \int_{Q_j} f\, d\sigma \leq D(\sigma)\lambda$.

The proof can be found in [Cu] or [Gr].

The Besicovitch covering lemma has the advantage of being applicable even if the underlying measure is non-doubling.

Lemma 1.10.1 (Besicovitch covering lemma)**.** *Suppose $A \subset \mathbb{R}^n$ is bounded and that for each $x \in A$, Q_x is a cube centered at x. Then we can choose, from among*

$\{Q_x : x \in A\}$, a possibly finite sequence $\{Q_i\}$ and an associated sequence of integers $\{m_i\}$ such that

(a) $A \subset \bigcup_i Q_i$;

(b) $1 \leq m_i \leq N_n$, *where* N_n *depends only on* n;

(c) Q_i *and* Q_j *are disjoint if* $m_i = m_j$.

The proof can be found in [Gu]. The sub-points (b) and (c) say that the sequence of cubes can be distributed into a finite number of disjoint families. More precisely, given j,

$1 \leq j \leq N_n$ we can define $F_j := \{k \in \mathbb{N} : \ m_k = j\}$, and then write

$$\bigcup_i Q_i = \bigcup_{j=1}^{N_n} \bigcup_{k \in F_j} Q_k,$$

where the family $\{Q_k\}_{k \in F_j}$ consists of pairwise disjoint cubes.

As a consequence of Besicovitch lemma we get a very useful theorem.

Theorem 1.10.2. *The centered Hardy-Littlewood maximal function M_σ^c, defined in Section 1.3, is of weak type $(1,1)$ with respect to σ, with the weak norm at most $C(n)$, where the constant $C(n)$ depends only n.*

The proof can be found for example in [Gu] p. 37.

Remark 1.10.1. *The theory of A_p weights has an analogue when the underlying measure σ is nondoubling but satisfies $\sigma(\delta Q) = 0$ for all cubes Q in $\mathbb{R}^n$ with sides parallel to the axes, where δQ is the boundary of Q, see [OrPe].*

Chapter 2

Buckley type estimates for Maximal Function

In this chapter we obtain *sharp*-estimates for the uncentered Maximal Function associated with measure σ. We are interested in the Buckley-type estimates, where the dependence on the $A_p(d\sigma)$ (or $RH_q(d\mu)$, where $d\mu = wd\sigma$) characteristic constant is established, with possibly the best power. As in the classical case (see [Buc2]), we start first with the *weak*-type estimates and then use the self-improvement properties of the $A_p(d\sigma)$ (or $RH_q(d\mu)$) weights and the Interpolation Theorem 1.2.1 to get the *strong*-type estimates. The Interpolation Theorem, as clearly seen in (1.8), requires a careful study of ε as a function of $A_p(d\sigma)$ (or $RH_q(d\mu)$) characteristic constant. Since the Buckley's Theorem, especially the fact used in his theorem, stated without a proof, that the ε appearing in the Coifman-Fefferman-Buckley Theorem should be of order $[w]_{A_p}^{1-p'}$ was not that clear to us, we first took advantage of the tautology (1.28), and obtained the Gehring type estimates. Then, we used them to interpolate on the $RH_q(d\mu)$ side instead. We obtained the same power as in Theorem 1.6.3 but the constant $D(\mu)$ appeared as a result of the Calderón-Zygmung decomposition used in the proof.

In the second part of this chapter, as we try to avoid the constant $D(\mu)$, we go back to Theorem 2.3.1, to obtain the Coiffman-Fefferman-Buckley Theorem $d\sigma$. We arrive at the same conclusions but with a different doubling constant i.e. $D(\sigma)$ this time. Finally, we present yet another proof, based on A. Lerner's very recent result (see [Le1]). It allows us to obtain the *strong*-estimates without using interpolation. Instead, it relies only on the properties of the Maximal Function and the Theorem 1.10.2. It also yields a doubling constant $D(\sigma)$ but with a different exponent.

These estimates will be later used in Chapter 3 to obtain Sharp Extrapolation Theorems.

2.1 Weak Estimates

In this section we generalize Muckenhoupt-Buckley type *weak*-estimates to the case $d\sigma$, where $d\sigma = u_o dx$ for some $u_o \in A_\infty$.

Lemma 2.1.1. *If $f \in L^p(d\mu)$, where $d\mu = w d\sigma$ and $w \in A_p(d\sigma)$, and if*

$\frac{1}{\sigma(Q_k)} \int_{Q_k} f\, d\sigma \geq \lambda > 0$ for each of the disjoint cubes Q_k, then

$$\sum_k \mu(Q_k) \leq C_n [w]_{A_p(d\sigma)} \left(\frac{\|f\|_{L^p(d\mu)}}{\lambda} \right)^p \tag{2.1}$$

Proof. Without loss of generality, we can assume $f \geq 0$, and normalized in $L^p(d\mu)$, i.e.

$\|f\|_{L^p(d\mu)} = \left(\int f^p w\, d\sigma\right)^{\frac{1}{p}} = 1.$

Since $w \in A_p(d\sigma) \iff w^{-\frac{1}{p-1}} \in A_{p'}(d\sigma)$, and

$$\frac{\upsilon(Q_k)\mu(Q_k)^{p'-1}}{\sigma(Q_k)^{p'}} \leq [w]_{A_p(d\sigma)}^{p'-1} \tag{2.2}$$

for any such Q_k, where $\upsilon(Q_k) := \int_{Q_k} w^{-\frac{p'}{p}}\, d\sigma$. (Note that $\frac{p'}{p} = \frac{1}{p-1}$).

Now, using first the hypothesis, then Hölder inequality plus the fact that $\|f\|_{L^p(d\mu)} = 1$, and finally using (2.2) and the fact that Q_k's are disjoint we get :

$$\begin{aligned}
\sum_k \mu(Q_k) &\le \sum_k \frac{\mu(Q_k)}{\lambda\, \sigma(Q_k)} \int_{Q_k} f\, d\sigma \\
&= \int \sum_k \frac{\mu(Q_k)}{\lambda\, \sigma(Q_k)} \chi_{Q_k} (f w^{\frac{1}{p}}) w^{-\frac{1}{p}}\, d\sigma \\
&\le \left(\int f^p w\, d\sigma\right)^{\frac{1}{p}} \left(\int \left|\sum_k \frac{\mu(Q_k)}{\lambda\, \sigma(Q_k)} \chi_{Q_k}\right|^{p'} w^{-\frac{p'}{p}}\, d\sigma\right)^{\frac{1}{p'}} \\
&\le \left(\sum_k \frac{\upsilon(Q_k)\mu(Q_k)^{p'}}{\lambda^{p'}\, \sigma(Q_k)^{p'}}\right)^{\frac{1}{p'}} \le \frac{[w]^{\frac{1}{p}}_{A_p(d\sigma)}}{\lambda} \left(\sum_k \mu(Q_k)\right)^{\frac{1}{p'}}
\end{aligned}$$

Thus,

$$\sum_k \mu(Q_k) \le \frac{[w]_{A_p(d\sigma)}}{\lambda^p}$$

□

Now we can use Lemma 2.1.1 to prove the following lemma.

Lemma 2.1.2. *The centered maximal function M^c_σ is weak (p,p), with respect to measure μ, where $d\mu = w d\sigma$, and $w \in A_p(d\sigma)$,*

$$\mu(\{x \in \mathbb{R}^n :\ M^c_\sigma f > \lambda\}) \le C_n [w]_{A_p(d\sigma)} \left(\frac{\|f\|_{L^p(d\mu)}}{\lambda}\right)^p.$$

Proof. Without loss of generality, we can assume $f \ge 0$ and $\|f\|_{L^p(d\mu)}$=1. Suppose now $M^c_\sigma f(x) > \lambda$, then there must exist some maximal Q_x centered at x such that $\frac{1}{\sigma(Q_x)} \int_{Q_x} f\, d\sigma > \lambda$. (By letting $\widetilde{Q}$ be a large enough dilation of a cube Q, we can make $\mu(\widetilde{Q})$ very large because $d\mu$ is *comparable* to dx). By the Besicovitch covering

Lemma 1.10.1

$$A_r = \{x : |x| < r,\ M^c_\sigma f > \lambda\} \subset \bigcup_{j=1}^{N_n} \bigcup_{k\in F_j} Q_k$$

where, each family $\{Q_k\}_{k\in F_j}$ is formed by pairwise disjoint cubes. Thus,

$$\mu(A_r) \leq \mu(\bigcup_{j=1}^{N_n} \bigcup_{k\in F_j} Q_k) \leq \sum_{j=1}^{N_n} \mu(\bigcup_{k\in F_j} Q_i) \leq N_n \max_{1\leq j\leq N_n} \mu(\bigcup_{k\in F_j} Q_k) = N_n \sum_{k\in F_0} \mu(Q_k),$$

where F_0 is a collection whose union has maximal μ-measure. By letting $r \to \infty$ and using the Lemma 2.1.1 we obtain the desired inequality:

$$\mu(\{x \in \mathbb{R}^n :\ M^c_\sigma f > \lambda\} \leq N_n\, \mu\left(\bigcup_k Q_k\right) \leq \frac{C_n\,[w]_{A_p(d\sigma)}}{\lambda^p} \|f\|_{L^p(d\mu)}. \tag{2.3}$$

□

By (1.9) we get a similar result for the uncentered maximal function.

Corollary 2.1.1.

$$\mu(\{x \in \mathbb{R}^n :\ M_\sigma f > \lambda\} \leq \frac{C_n\, D(\sigma)^p\,[w]_{A_p(d\sigma)}}{\lambda^p} \|f\|_{L^p(d\mu)}. \tag{2.4}$$

2.1.1 Strong Doubling

In this section we present a lemma comparing the measures μ and σ, which will be used a few times in the next sections.

Lemma 2.1.3 (Strong Doubling)**.** *Let $w \in A_p(d\sigma)$ for some $1 \leq p < \infty$ and let $0 < \alpha < 1$. Then there exists $0 < \beta < 1$ such that whenever S is a measurable subset of a cube Q that satisfies $\sigma(S) \leq \alpha\sigma(Q)$, we have $\mu(S) \leq \beta\mu(Q)$.*

Proof. Applying Hölder's inequality with exponents p and p' we obtain

$$\left(\frac{1}{\sigma(Q)}\int_Q f(x)\,d\sigma\right)^p = \left(\frac{1}{\sigma(Q)}\int_Q f(x)w(x)^{\frac{1}{p}}w(x)^{-\frac{1}{p}}\,d\sigma\right)^p$$

$$\begin{aligned}
&\le \left(\frac{1}{\sigma(Q)}\int_Q f(x)^p w(x)\,d\sigma\right)\left(\frac{1}{\sigma(Q)}\int_Q w(x)^{-\frac{p'}{p}}\,d\sigma\right)^{\frac{p}{p'}}\\
&= \left(\frac{1}{\mu(Q)}\int_Q f(x)^p d\mu\right)\left(\frac{1}{\sigma(Q)}\int_Q w(x)\,d\sigma\right)\left(\frac{1}{\sigma(Q)}\int_Q w(x)^{-\frac{1}{p-1}}\,d\sigma\right)^{p-1}\\
&\le [w]_{A_p(d\sigma)}\left(\frac{1}{\mu(Q)}\int_Q f(x)^p d\mu\right)
\end{aligned}$$

Thus,

$$\left(\frac{1}{\sigma(Q)}\int_Q f(x)\,d\sigma\right)^p \le [w]_{A_p(d\sigma)}\left(\frac{1}{\mu(Q)}\int_Q f(x)^p d\mu\right) \tag{2.5}$$

If we set $f = \chi_A$ in (2.5), we obtain

$$\left(\frac{\sigma(A)}{\sigma(Q)}\right)^p \le [w]_{A_p(d\sigma)}\frac{\mu(A)}{\mu(Q)} \tag{2.6}$$

If we write $S = Q\backslash A$ we get

$$\left(1-\frac{\sigma(S)}{\sigma(Q)}\right)^p \le [w]_{A_p(d\sigma)}\left(1-\frac{\mu(S)}{\mu(Q)}\right). \tag{2.7}$$

Given $0 < \alpha < 1$, set

$$\beta = 1-\frac{(1-\alpha)^p}{[w]_{A_p(d\sigma)}} \tag{2.8}$$

and use (2.7) to obtain the required conclusion. More precisely, if $\frac{\sigma(S)}{\sigma(Q)} \le \alpha$, then

$$(1-\alpha)^p \le [w]_{A_p(d\sigma)}\left(1-\frac{\mu(S)}{\mu(Q)}\right).$$

From here we deduce

$$\frac{\mu(S)}{\mu(Q)} \le 1-\frac{(1-\alpha)^p}{[w]_{A_p(d\sigma)}} = \beta.$$

□

Corollary 2.1.2. *Using (2.5) with the function* $f = \chi_Q$ *and averaging over* $3Q$ *in the place of* Q *we get*

$$\frac{\mu(3Q)}{\mu(Q)} \leq [w]_{A_p(d\sigma)} \left(\frac{\sigma(3Q)}{\sigma(Q)}\right)^p, \tag{2.9}$$

which implies $\mu \in \mathbf{D}$ *and* $D(\mu) \leq [w]_{A_p(d\sigma)} [D(\sigma)]^p$.

As a consequence of (2.5) we also obtain an alternative and a much simpler proof of the *weak*-estimates for M_σ but with $D(\mu)$ constant.

Corollary 2.1.3. *If* $d\mu = w d\sigma$, $w \in A_p(d\sigma)$, *then*

$$\mu(\{x \in \mathbb{R}^n :\ M_\sigma f > \lambda\}) \leq \frac{D(\mu)\,[w]_{A_p(d\sigma)}}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p \, d\mu. \tag{2.10}$$

Proof. By (2.5) we get

$$M_\sigma f(x) \leq [w]^{\frac{1}{p}}_{A_p(d\sigma)} \, [M_\mu(f^p)(x)]^{\frac{1}{p}} \quad \text{for a.e. } x. \tag{2.11}$$

By Theorem 1.3.1 M_μ is of *weak*-type $(1,1)$, with respect to measure μ, thus we obtain

$$\begin{aligned} \mu(\{x \in \mathbb{R}^n :\ M_\sigma f(x) > \lambda\}) &\leq \mu\left(\{x \in \mathbb{R}^n :\ [w]^{\frac{1}{p}}_{A_p(d\sigma)} \, [M_\mu(f^p)]^{\frac{1}{p}} > \lambda\}\right) \\ &= \mu\left(\{x \in \mathbb{R}^n :\ M_\mu(f^p) > \frac{\lambda^p}{[w]_{A_p(d\sigma)}}\}\right) \\ &\leq \frac{D(\mu)\,[w]_{A_p(d\sigma)}}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p \, d\mu. \end{aligned} \tag{2.12}$$

This implies that $M_\sigma f$ is of *weak*-type (p,p) with respect to μ, with the *weak*-type norm at most $D(\mu)^{\frac{1}{p}} [w]^{\frac{1}{p}}_{A_p(d\sigma)}$. □

Remark 2.1.1. *We can actually avoid* $D(\mu)$ *constant, in (2.12), by using Theorem 1.10.2. The inequality (2.5) also implies the same inequality as (2.11) for the centered maximal functions. Thus we have*

$$M^c_\sigma f(x) \leq [w]^{\frac{1}{p}}_{A_p(d\sigma)} \left[M^c_\mu(f^p)(x)\right]^{\frac{1}{p}} \quad \textit{for a.e. } x. \tag{2.13}$$

Thus,

$$\begin{aligned}
\mu(\{x\in\mathbb{R}^n:\ M_\sigma f(x)>\lambda\}) &\le \mu(\{x\in\mathbb{R}^n:\ D(\sigma)\,M_\sigma^c f(x)>\lambda\})\\
&\le \mu\left(\{x\in\mathbb{R}^n:\ D(\sigma)[w]_{A_p(d\sigma)}^{\frac{1}{p}}\left[M_\mu^c(f^p)\right]^{\frac{1}{p}}>\lambda\}\right)\\
&= \mu\left(\{x\in\mathbb{R}^n:\ M_\mu^c(f^p)>\frac{\lambda^p}{D(\sigma)^p\,[w]_{A_p(d\sigma)}}\}\right)\\
&\le \frac{C(n)\,D(\sigma)^p\,[w]_{A_p(d\sigma)}}{\lambda^p}\int_{\mathbb{R}^n}|f(x)|^p\,d\mu,
\end{aligned}$$

where $C(n)$ is the dimensional constant as in Theorem 1.10.2.

The *weak*-norm here is exactly the same as in (2.4).

What we have obtained both for the centered and uncentered maximal function are the following *weak*-estimates.

If $d\mu = wd\sigma$, $w\in A_p(d\sigma)$, then

$$\mu(\{x\in\mathbb{R}^n:\ T\,f>\lambda\})\le\left(\frac{C(\sigma,\mu,p)\,[w]_{A_p(d\sigma)}^{\frac{1}{p}}}{\lambda}\int_{\mathbb{R}^n}|f(x)|^p\,d\mu\right)^p, \tag{2.14}$$

where $T=M_\sigma^c$ or $T=M_\sigma$, and the constant $C(\sigma,\mu,p)$ depends only on p, $D(\sigma)$ or $D(\mu)$. More precisely, for

$$\text{for }\ T=M_\sigma^c:\quad C_1(\sigma,\mu,p)=\min\{C(n)^{\frac{1}{p}},D(\mu)^{\frac{1}{p}}\}; \tag{2.15}$$

$$\text{for }\ T=M_\sigma:\quad C_2(\sigma,\mu,p)=\min\{C(n)^{\frac{1}{p}}D(\sigma),D(\mu)^{\frac{1}{p}}\}. \tag{2.16}$$

2.2 $RH_p(d\sigma)$ and Gehring's Estimates

In this subsection we are going to adapt Gehring's Lemma to the $d\sigma$ case, keeping track of $D(\sigma)$ and the size of ε. For that we need some preliminary lemmas.

Lemma 2.2.1. *Suppose that* $q \in (0, \infty)$ *and* $a \in (1, \infty)$,

$h : [1, \infty) \to [0, \infty)$ *is non-increasing, right-continuous with*

$$\lim_{t \to \infty} h(t) = 0$$

and that

$$-\int_t^\infty s^q \, dh(s) \le a t^q h(t). \tag{2.17}$$

for $t \in [1, \infty)$.

Then

$$-\int_1^\infty t^p \, dh(t) \le \frac{q}{aq - (a-1)p} \left(-\int_1^\infty t^q \, dh(t) \right), \tag{2.18}$$

for $p \in \left[q, q + \frac{q}{a-1}\right)$. *This inequality is sharp.*

This lemma can be found in [Ge] p.266, and will be used to prove the next lemma.

Lemma 2.2.2. *Suppose that* b, $q \in (1, \infty)$, Q *is an* n*-cube in* $\mathbb{R}^n$, $g : Q \to [0, \infty]$ *is* $L^q(d\sigma)$ *integrable in* Q *and*

$$\frac{1}{\sigma(Q')} \int_{Q'} g^q \, d\sigma \le b \left(\frac{1}{\sigma(Q')} \int_{Q'} g \, d\sigma \right)^q \tag{2.19}$$

for each parallel n*-cube* $Q' \subset Q$.

Then g *is* $L^p(d\sigma)$*-integrable in* Q *with*

$$\frac{1}{\sigma(Q)} \int_Q g^p \, d\sigma \le \frac{c}{q + c - p} \left(\frac{1}{\sigma(Q)} \int_Q g^q \, d\sigma \right)^{\frac{p}{q}} \tag{2.20}$$

for $p \in [q, q + c)$, *where* c *is a positive constant which depends only on* q, b, $D(\sigma)$ *and* n.

Proof. Without loss of generality we can assume

$$\int_Q g^q \, d\sigma = \sigma(Q). \tag{2.21}$$

This is because if g_o satisfies (2.19), then

$$g(x) := \frac{g_o(x)}{\frac{1}{\sigma(Q)}\int_Q g^q \, d\sigma}, \tag{2.22}$$

will satisfy both (2.19) and (2.21). If we have shown the Lemma for such a g, then substituting (2.22) into (2.20) will also imply (2.20) for g_o.

Therefore all we need to show is

$$\frac{1}{\sigma(Q)} \int_Q g^p \, d\sigma \leq \frac{c}{q+c-p}, \tag{2.23}$$

for a g that satisfies (2.21).

Define $E(t) = \{x \in Q : g(x) > t\}$. We can begin by showing

$$\int_{E(t)} g^q \, d\sigma \leq a t^{q-1} \int_{E(t)} g \, d\sigma \tag{2.24}$$

for $t \in [1, \infty)$, where a is a constant which depends only on q, b, $D(\sigma)$ and n.

Next for $t \in [1, \infty)$ choose $s > t$ such that $s^q := b\left(\frac{q}{q-1}t\right)^q$, and apply the Calderón-Zygmund decomposition, introduced in Section 1.9, at the level s^q, to obtain a disjoint sequence of parallel n-cubes $Q_j \subset Q$ such that

$$s^q < \frac{1}{\sigma(Q_j)} \int_{Q_j} g^q \, d\sigma \leq D(\sigma) s^q \tag{2.25}$$

for all j, and such that $g \leq s$ a.e. in $Q \backslash \bigcup_j Q_j$. This implies that the set $E(s)$ is contained, except for a subset of measure zero, in $\bigcup_j Q_j$. The cubes Q_j are disjoint, therefore with (2.25) we have

$$\int_{E(s)} g^q \, d\sigma \leq \sum_j \int_{Q_j} g^q \, d\sigma \leq s^q D(\sigma) \sum_j \sigma(Q_j). \tag{2.26}$$

The inequalities (2.25) and (2.19) imply that

$$b\left(\frac{q}{q-1}t\right)^q \leq b\left(\frac{1}{\sigma(Q_j)} \int_{Q_j} g \, d\sigma\right)^q,$$

therefore,

$$\frac{q}{q-1} t\sigma(Q_j) \le \int_{Q_j} g\, d\sigma \le \int_{Q_j \cap E(t)} g\, d\sigma + t\sigma(Q_j).$$

The second inequality holds because if $x \notin E(t)$, then $g(x) \le t$, and

$$\int_{Q_j \setminus E(t)} g\, d\sigma \le t\, \sigma(Q_j).$$

We now get,

$$\sigma(Q_j) \le \frac{q-1}{t} \int_{Q_j \cap E(t)} g\, d\sigma$$

for each j.

Combining this inequality with (2.26) yields

$$\int_{E(s)} g^q\, d\sigma \le D(\sigma)\, s^q \sum_j \frac{q-1}{t} \int_{Q_j \cap E(t)} g\, d\sigma \le D(\sigma)\, s^q\, \frac{q-1}{t} \int_{E(t)} g\, d\sigma, \tag{2.27}$$

If $x \in E(t) \setminus E(s)$, then $g(x) \le s$, therefore

$$\int_{E(t) \setminus E(s)} g^q\, d\sigma \le s^{q-1} \int_{E(t)} g\, d\sigma. \tag{2.28}$$

By (2.28) and (2.27) we obtain

$$\begin{aligned} \int_{E(t)} g^q\, d\sigma &= \int_{E(t)\setminus E(s)} g^q\, d\sigma + \int_{E(s)} g^q\, d\sigma \le \left[D(\sigma)\, s^q \frac{q-1}{t} + s^{q-1} \right] \int_{E(t)} g\, d\sigma \\ &= \left[D(\sigma) \left(\frac{s}{t}\right)^q (q-1) + \left(\frac{s}{t}\right)^{q-1} \right] t^{q-1} \int_{E(t)} g\, d\sigma \end{aligned}$$

This proves (2.24) with

$$\begin{aligned} a &= D(\sigma) \left(\frac{s}{t}\right)^q (q-1) + \left(\frac{s}{t}\right)^{q-1} \\ &= D(\sigma)\, b \left(\frac{q}{q-1}\right)^q (q-1) + \left(\frac{q}{q-1}\right)^{q-1} b^{\frac{q-1}{q}} \\ &\le \left(\frac{q}{q-1}\right)^{q-1} b\, q\, [D(\sigma) + 1] \le 6 D(\sigma)\, b\, q. \end{aligned}$$

The last inequality holds because $D(\sigma) + 1 < 2D(\sigma)$ and $\left(1 + \frac{1}{q-1}\right)^{q-1} \le e < 3$ for any $q > 1$.

Now, for $t \in [1,\infty)$, set $h(t) = \int_{E(t)} g\, d\sigma$. Then $h : [1,\infty) \to (0,\infty)$ is non-increasing, right-continuous and $\lim_{t\to\infty} h(t) = 0$.

Then, using (1.5) we get the following,

For any $r, t \in [1,\infty)$

$$\int_{E(t)} g^r \, d\sigma = \int_{E(t)} g^{r-1}\, g d\sigma = -\int_t^\infty s^{r-1} dh(s).$$

Thus inequality (2.24), can be rewritten in the form,

$$-\int_t^\infty s^{q-1} dh(s) \le a t^{q-1} h(t)$$

which shows that the function h satisfies the remaining hypothesis (2.17).

Then we can apply Lemma 2.2.1 with q and p , replaced by $q-1$ and $p-1$, to obtain

$$-\int_1^\infty t^{p-1} dh(t) \le \frac{q-1}{a(q-1)-(a-1)(p-1)} \left(-\int_1^\infty t^{q-1} dh(t) \right).$$

This is equivalent to

$$\int_{E(1)} g^p \, d\sigma \le \frac{q-1}{a(q-1)-(a-1)(p-1)} \int_{E(1)} g^q \, d\sigma$$

for $p \in [q, q+c)$, where

$$c = \frac{q-1}{a-1} > \frac{q-1}{6D(\sigma)\, b\, q}. \tag{2.29}$$

Since $g^p(x) \le g^q(x)$ in $Q \backslash E(1)$, and $\frac{a(q-1)}{a-1} = \frac{(a-1+1)(q-1)}{a-1} = q + c - 1$, therefore

$$\begin{aligned} \int_Q g^p \, d\sigma &\le \int_{E(1)} g^p \, d\sigma + \int_{Q\backslash E(1)} g^p \, d\sigma \\ &\le \frac{c}{q+c-p} \int_{E(1)} g^q \, d\sigma + \int_{Q\backslash E(1)} g^q \, d\sigma \\ &\le \frac{c}{q+c-p} \int_Q g^q \, d\sigma = \frac{c}{q+c-p}. \end{aligned}$$

for $p \in [q, q+c)$. This together with (2.23) yields (2.20). □

Corollary 2.2.1. *If $w \in RH_q(d\sigma)$, then $w \in RH_{q+\varepsilon}(d\sigma)$ with*

$$[w]_{RH_{q+\varepsilon}(d\sigma)} \leq 2[w]_{RH_q(d\sigma)}, \text{ for some } \varepsilon \sim \frac{1}{D(\sigma)[w]^q_{RH_q(d\sigma)}}.$$

If we take $g = w$, and choose $b = [w]^q_{RH_q(d\sigma)}$, then we can use Lemma 2.2.2.

Thus, for any Q, by (2.20) and (2.19) we have

$$\begin{aligned} \left(\frac{1}{\sigma(Q)} \int_Q w^p \, d\sigma\right)^{\frac{1}{p}} &\leq \left(\frac{c}{q+c-p}\right)^{\frac{1}{p}} \left(\frac{1}{\sigma(Q)} \int_Q w^q \, d\sigma\right)^{\frac{1}{q}} \\ &\leq \left(\frac{c}{q+c-p}\right)^{\frac{1}{p}} [w]_{RH_q(d\sigma)} \left(\int_Q w \, d\sigma\right), \end{aligned} \tag{2.30}$$

which implies $w \in RH_{q+\varepsilon}(d\sigma)$, with $\varepsilon = p - q < c$. Furthermore, if we choose $\varepsilon = \frac{c}{2}$, then by (2.29) we obtain $\varepsilon \sim \frac{1}{D(\sigma)[w]^q_{RH_q(d\sigma)}}$ and $\left(\frac{c}{q+c-p}\right)^{\frac{1}{p}} \leq 2$, for any $p > 1$, which ends the proof.

2.3 Reverse Hölder Inequality for the $A_p(d\sigma)$

In this section we return to the properties of the $A_p(d\sigma)$ class. We analyze and adapt the proofs of the Reverse Hölder Inequality for the $A_p(d\sigma)$ weights, to determine the connection between the order of γ in Theorem 2.3.1 and the order of ε in Theorem 2.3.2. This will also help us understand better the classical Buckley's Theorem 1.6.3, in particular, his result stated without a proof in [Buc1], that the optimal ε appearing in the Coiffman-Fefferman Theorem should be $\varepsilon \sim [w]^{1-p'}_{A_p}$. This way we can avoid the constant $D(\mu)$, but $D(\sigma)$ appears instead as a consequence of the Calderón-Zygmund decomposition used in the proof.

Theorem 2.3.1 (Reverse Hölder Inequality)**.** *Let $w \in A_p(d\sigma)$ for some $1 \leq p < \infty$. Then there exist constants C and $\gamma > 0$ that depend only on the dimension n, on p,*

$D(\sigma)$ and on $[w]_{A_p(d\sigma)}$ such that for every cube Q we have

$$\left(\frac{1}{\sigma(Q)}\int_Q w(t)^{1+\gamma}\,d\sigma\right)^{\frac{1}{1+\gamma}} \leq \frac{C}{\sigma(Q)}\int_Q w(t)\,d\sigma \tag{2.31}$$

Proof. Let us fix a cube Q and set

$$\alpha_o = \frac{1}{\sigma(Q)}\int_Q w(t)\,d\sigma.$$

We also fix an $0 < \alpha < 1$. We define an increasing sequence of scalars

$$\alpha_o < \alpha_1 < \alpha_2 < ... < \alpha_k < ...$$

for $k \geq 0$ by setting

$$\alpha_{k+1} = (D(\sigma)\alpha^{-1})\alpha_k \quad \text{or} \quad \alpha_k = (D(\sigma)\alpha^{-1})^k\alpha_o,$$

and for each $s \geq 1$ we divide the cube Q into a mesh of 2^{ns} subcubes of side length equal to $2^{-s}l(Q)$. Among all these cubes, we select those with the property that the average of w over them is strictly greater than α_k and we isolate all maximal cubes with this property. In this way we obtain a collection $\{Q_{k,j}\}_j$ so that the following are satisfied:

1. $\alpha_k < \frac{1}{\sigma(Q_{k,j})}\int_{Q_{k,j}} w(t)\,d\sigma \leq D(\sigma)\alpha_k$

2. On $Q\backslash U_k$ we have $w \leq \alpha_k$ σ-a.e., where $U_k = \bigcup_j Q_{k,j}$; for each k, $\{Q_{k,j}\}_j$ are disjoint

3. Each $Q_{k+1,j}$ is contained in some $Q_{k,l}$

Property (i) is satisfied since the unique dyadic parent of $Q_{k,j}$ was not chosen in the selection procedure, while (ii) follows from the Lebesgue differentiation theorem (for doubling measures) using the fact that for almost all $x \notin U_k$ there exists a sequence of non selected cubes of decreasing lengths whose intersections is $\{x\}$. Property (iii) is satisfied since each $Q_{k,j}$ is the maximal subcube of Q with the property that the average of w over it is bigger than α_k. And since the average of w over $Q_{k+1,j}$ is also bigger than α_k, it follows that $Q_{k+1,j}$ must be contained in some maximal cube that has this property.

We will now compute the portion of $Q_{k,l}$ that is covered by cubes of the form $Q_{k+1,j}$ for some j. We have

$$\begin{aligned} D(\sigma)\alpha_k &\geq \frac{1}{\sigma(Q_{k,l})}\int_{Q_{k,l}\cap U_{k+1}} w(t)\,d\sigma \\ &= \frac{1}{\sigma(Q_{k,l})}\sum_{j\,:\,Q_{k+1,j}\subset Q_{k,l}} \sigma(Q_{k+1,j})\frac{1}{\sigma(Q_{k+1,j})}\int_{Q_{k+1,j}} w(t)\,d\sigma \\ &> \frac{\sigma(Q_{k,l}\cap U_{k+1})}{\sigma(Q_{k,l})}\alpha_{k+1} = \frac{\sigma(Q_{k,l}\cap U_{k+1})}{\sigma(Q_{k,l})}D(\sigma)\alpha^{-1}\alpha_k. \end{aligned}$$

It follows that $\sigma(Q_{k,l}\cap U_{k+1}) \leq \alpha\sigma(Q_{k,l})$. Thus, applying Lemma 2.1.3, we obtain

$$\frac{\mu(Q_{k,l}\cap U_{k+1})}{\mu(Q_{k,l})} < \beta = 1 - \frac{(1-\alpha)^p}{[w]_{A_p(d\sigma)}}$$

from which, summing over all l, we obtain

$$\mu(U_{k+1}) \leq \beta\mu(U_k).$$

The latter gives $\mu(U_k) \leq \beta^k\mu(U_o)$. We also have $\sigma(U_{k+1}) \leq \alpha\sigma(U_k)$; hence $\sigma(U_k) \to 0$ as $k \to \infty$. Therefore, the intersection of the $U_k's$ is a null set. We can therefore write

$$Q = (Q\backslash U_o) \cup (\bigcup_{k=0}^{\infty} U_k\backslash U_{k+1})$$

modulo a set of Lebesgue measure 0. Let us find a $\gamma > 0$ so that the Reverse Hölder

Inequality (2.3.1) holds. We have $w(x) \leq \alpha_k$ for almost all x in $Q \backslash U_k$ and therefore

$$\begin{aligned}
\int_Q w(t)^{\gamma+1}\, d\sigma &= \int_{Q\backslash U_o} w(t)^\gamma w(t)\, d\sigma + \sum_{k=0}^{\infty} \int_{U_k \backslash U_{k+1}} w(t)^\gamma w(t)\, d\sigma \\
&\leq \alpha_o^\gamma \mu(Q\backslash U_o) + \sum_{k=0}^{\infty} \alpha_{k+1}^\gamma \mu(U_k) \\
&\leq \alpha_o^\gamma \mu(Q\backslash U_o) + \sum_{k=0}^{\infty} \left((D(\sigma)\alpha^{-1})^{k+1}\alpha_o\right)^\gamma \beta^k \mu(U_o) \\
&\leq \alpha_o^\gamma \left(1 + (D(\sigma)\alpha^{-1})^\gamma \sum_{k=0}^{\infty} (D(\sigma)\alpha^{-1})^{\gamma k}\beta^k\right)(\mu(Q\backslash U_o) + \mu(U_o)) \\
&= \left(\frac{1}{\sigma(Q)}\int_Q w(t)\, d\sigma\right)^\gamma \left(1 + \frac{(D(\sigma)\alpha^{-1})^\gamma}{1-(D(\sigma)\alpha^{-1})^\gamma \beta}\right)\int_Q w(t)\, d\sigma
\end{aligned}$$

provided $\gamma > 0$ is chosen small enough so that $(D(\sigma)\alpha^{-1})^\gamma \beta < 1$. Keeping track of the constants, as in [Gr], we conclude the proof with

$$\gamma \;<\; \frac{-\log\beta}{\log D(\sigma) - \log\alpha} = \frac{\log\left([w]_{A_p(d\sigma)}\right) - \log\left([w]_{A_p(d\sigma)} - (1-\alpha)^p\right)}{\log D(\sigma) - \log\alpha}$$

and

$$C \;=\; \left(1 + \frac{(D(\sigma)\alpha^{-1})^\gamma}{1-(D(\sigma)\alpha^{-1})^\gamma \beta}\right)^{\frac{1}{\gamma+1}} = \left(1 + \frac{(D(\sigma)\alpha^{-1})^\gamma}{1-(D(\sigma)\alpha^{-1})^\gamma\left(1 - \frac{(1-\alpha)^p}{[w]_{A_p(d\sigma)}}\right)}\right)^{\frac{1}{\gamma+1}}$$

□

Note that since,

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + ...$$

if $|x| < 1$, and

$$\gamma < -\log\left(1 - \frac{(1-\alpha)^p}{[w]_{A_p(d\sigma)}}\right)\frac{1}{\log\frac{D(\sigma)}{\alpha}},$$

therefore if we set $\alpha = \frac{1}{D(\sigma)} < 1$, then

$$\gamma \sim \frac{1}{[w]_{A_p(d\sigma)}\log(D(\sigma))}. \tag{2.32}$$

The parameter α can be chosen to optimize the constant γ. However, we would also like to have a uniform bound on C in terms of $[w]_{A_p(d\sigma)}$. This will be much easier to see in the next Section.

2.3.1 C. Pérez' new proof

In this subsection we present a simpler proof of the Theorem 2.3.1 adapted to the $d\sigma$ case. It was inspired by Carlos Pérez' recent paper, (see [P1]) who was interested in obtaining similar and even more precise estimates for the A_p weights. The order of γ is the same as in (2.32) but the constant C here is much better. We need it to be uniformly bounded, and actually we can choose $C = 2$ (as observed in [Buc1] or [Wit1] p. 6). This fact will be crucial in Theorem 2.3.2, and then in the Interpolation Theorem 1.8.

Proof. Let

$$w_Q := \frac{1}{\sigma(Q)} \int_Q w \, d\sigma = \frac{\mu(Q)}{\sigma(Q)}.$$

By Proposition 1.2.1, with $\phi(x) = x^\gamma$, the underlying measure $d\mu = w d\sigma$,

$|f(x)| = w(x)$ and $X = Q$, we get

$$\begin{aligned} \frac{1}{\sigma(Q)} \int_Q w(x)^\gamma w(x) \, d\sigma &= \frac{\gamma}{\sigma(Q)} \int_0^\infty \lambda^{\gamma-1} \mu(\{x \in Q : \ w(x) > \lambda\}) \, d\lambda \\ &= \frac{\gamma}{\sigma(Q)} \int_0^{w_Q} \lambda^{\gamma-1} \mu(\{x \in Q : \ w(x) > \lambda\}) \, d\lambda \\ &\quad + \frac{\gamma}{\sigma(Q)} \int_{w_Q}^\infty \lambda^{\gamma-1} \mu(\{x \in Q : \ w(x) > \lambda\}) \, d\lambda \\ &= I + II. \end{aligned} \tag{2.33}$$

Clearly,

$$\begin{aligned} I &= \frac{\gamma}{\sigma(Q)} \int_0^{w_Q} \lambda^{\gamma-1} \mu(\{x \in Q : \ w(x) > \lambda\}) \, d\lambda \\ &\le \frac{\gamma}{\sigma(Q)} \int_0^{w_Q} \lambda^{\gamma-1} \mu(Q) \, d\lambda = w_Q \, \gamma \int_0^{w_Q} \lambda^{\gamma-1} \, d\lambda = (w_Q)^{\gamma+1}. \end{aligned}$$

Now for any Q define

$$E_Q = \{x \in Q : \; w(x) \leq \frac{1}{2^{p-1}\,[w]_{A_p(d\sigma)}} w_Q\}.$$

Then we claim

$$\sigma(E_Q) \leq \frac{\sigma(Q)}{2}. \tag{2.34}$$

By (2.6) for $E_Q \subset Q$ we have

$$\left(\frac{\sigma(E_Q)}{\sigma(Q)}\right)^p \leq [w]_{A_p(d\sigma)} \frac{\mu(E_Q)}{\mu(Q)}$$

Since by definition of E_Q

$$\mu(E_Q) = \int_{E_Q} w \, d\sigma \leq \int_{E_Q} \frac{w_Q}{2^{p-1}[w]_{A_p(d\sigma)}} \, d\sigma \leq \frac{w_Q}{2^{p-1}[w]_{A_p(d\sigma)}} \, \sigma(E_Q),$$

therefore

$$\left(\frac{\sigma(E_Q)}{\sigma(Q)}\right)^p \leq [w]_{A_p(d\sigma)} \frac{\mu(E_Q)}{\mu(Q)} \leq [w]_{A_p(d\sigma)} \frac{w_Q \; \sigma(E_Q)}{\mu(Q) \, 2^{p-1}\,[w]_{A_p(d\sigma)}} = \frac{\sigma(E_Q)}{2^{p-1}\,\sigma(Q)},$$

from which the claim follows.

Notice that (2.34) implies

$$\sigma(Q \backslash E_Q) > \frac{\sigma(Q)}{2}, \tag{2.35}$$

where $Q \backslash E_Q = \{x \in Q : \; w(x) > \frac{1}{2^{p-1}[w]_{A_p(d\sigma)}} w_Q\}$.

The second claim is the following:

For every $\lambda > w_Q$ we have

$$\mu(\{x \in Q : \; w(x) > \lambda\}) \leq 2D(\sigma)\,\lambda\,\sigma(\{x \in Q : \; w(x) > \frac{\lambda}{2^{p-1}\,[w]_{A_p(d\sigma)}}\}). \tag{2.36}$$

By Calderón-Zygmund decomposition of w at the level λ, we obtain a family of disjoint cubes $\{Q_i\}$ contained in Q and satisfying $\lambda < w_{Q_i} = \frac{\mu(Q_i)}{\sigma(Q_i)} \leq D(\sigma)\lambda$ for each i. Hence, except for a set of measure zero, we get

$$\{x \in Q : \; w(x) > \lambda\} \subset \{x \in Q : \; M^d_{\sigma,Q} w(x) > \lambda\} = \bigcup_i Q_i,$$

where $M^d_{\sigma,Q}$ is the dyadic maximal function, associated with measure σ and restricted to Q. Hence this together with (2.35) applied to each Q_i gives

$$\begin{aligned}
\mu(\{x\in Q:\ w(x)>\lambda\}) &\le \sum_i \mu(Q_i) \le D(\sigma)\,\lambda \sum_i \sigma(Q_i) \\
&\le 2D(\sigma)\lambda \sum_i \sigma(\{x\in Q_i:\ w(x) > \frac{w_{Q_i}}{2^{p-1}\,[w]_{A_p(d\sigma)}}\}) \\
&\le 2D(\sigma)\lambda\, \sigma(\{x\in Q:\ w(x) > \frac{\lambda}{2^{p-1}\,[w]_{A_p(d\sigma)}}\}),
\end{aligned}$$

since $w_{Q_i} > \lambda$. This proves the second claim.

Now we estimate the second integral in (2.33) denoted by II. By (2.36) and integration by substitution we get

$$\begin{aligned}
II &= \frac{\gamma}{\sigma(Q)} \int_{w_Q}^{\infty} \lambda^{\gamma}\, \mu(\{x\in Q:\ w(x)>\lambda\})\, \frac{d\lambda}{\lambda}, \\
&\le \frac{2D(\sigma)\,\gamma}{\sigma(Q)} \int_{w_Q}^{\infty} \lambda^{\gamma+1} \sigma(\{x\in Q:\ w(x) > \frac{\lambda}{2^{p-1}\,[w]_{A_p(d\sigma)}}\})\, \frac{d\lambda}{\lambda}, \\
&= \left(2^{p-1}\,[w]_{A_p(d\sigma)}\right)^{1+\gamma} \frac{2D(\sigma)\,\gamma}{\sigma(Q)} \int_{\frac{w_Q}{2^{p-1}\,[w]_{A_p(d\sigma)}}}^{\infty} \lambda^{\gamma+1}\, \sigma(\{x\in Q:\ w(x)>\lambda\})\, \frac{d\lambda}{\lambda}, \\
&\le \left(2^{p-1}\,[w]_{A_p(d\sigma)}\right)^{1+\gamma} \frac{2D(\sigma)\,\gamma}{\sigma(Q)} \int_0^{\infty} \lambda^{\gamma}\, \sigma(\{x\in Q:\ w(x)>\lambda\})\, d\lambda, \\
&= \left(2^{p-1}\,[w]_{A_p(d\sigma)}\right)^{1+\gamma} \frac{2D(\sigma)\,\gamma}{1+\gamma} \frac{1}{\sigma(Q)} \int_Q w(x)^{1+\gamma}\, d\sigma.
\end{aligned}$$

If we choose

$$\gamma = \frac{1}{2^{2p+1}\,D(\sigma)\,[w]_{A_p(d\sigma)}} < 1, \tag{2.37}$$

then $[w]^{\gamma}_{A_p(d\sigma)} \le [w]^{\frac{1}{[w]_{A_p(d\sigma)}}}_{A_p(d\sigma)} \le 2$ because the function

$$f(t) = t^{\frac{1}{t}} \le 2, \text{ for } t\ge 1.$$

Therefore,

$$\begin{aligned}
\left(2^{p-1}\,[w]_{A_p(d\sigma)}\right)^{1+\gamma} 2\,D(\sigma)\,\frac{\gamma}{1+\gamma} &\le 2^{p-1}[w]_{A_p(d\sigma)} 2^{(p-1)\gamma} [w]^{\gamma}_{A_p(d\sigma)} 2\,D(\sigma)\,\gamma \\
&\le 2^{2p}\,D(\sigma)\,[w]_{A_p(d\sigma)} \frac{1}{2^{2p+1}\,D(\sigma)\,[w]_{A_p(d\sigma)}} \le \frac{1}{2}.
\end{aligned}$$

Combing the estimates for I, II, with γ as in (2.37) we obtain

$$\frac{1}{\sigma(Q)}\int_Q w(x)^{\gamma+1}\,d\sigma \le (w_Q)^{\gamma+1}+\frac{1}{2}\left(\frac{1}{\sigma(Q)}\int_Q w(x)^{\gamma+1}\,d\sigma\right),$$

which is equivalent to

$$\left(\frac{1}{\sigma(Q)}\int_Q w(x)^{\gamma+1}\,d\sigma\right)^{\frac{1}{\gamma+1}} \le 2\left(\frac{1}{\sigma(Q)}\int_Q w\,d\sigma\right), \tag{2.38}$$

which ends the proof. □

2.3.2 Coiffman-Fefferman-Buckley Theorem $d\sigma$: a precise version

Now, we can prove a precise version of the Coiffman-Fefferman-Buckley Theorem $d\sigma$.

Theorem 2.3.2 (Coiffman-Fefferman-Buckley $d\sigma$). *Suppose that $w \in A_p(d\sigma)$, then $w \in A_{p-\varepsilon}(d\sigma)$ for some $\varepsilon > 0$. Moreover, ε can be chosen so that*

$$[w]_{A_{p-\varepsilon}(d\sigma)} \le 2^{p-1}[w]_{A_p(d\sigma)} \ \ \textit{with}\ \ \varepsilon \sim [w]^{1-p'}_{A_p(d\sigma)}\frac{1}{D(\sigma)}.$$

Proof. If $w \in A_p(d\sigma)$, then $\nu = w^{-\frac{1}{p-1}} \in A_{p'}(d\sigma)$, therefore by Theorem 2.3.1 $\nu \in RH_{1+\gamma}(d\sigma)$ for some $\gamma > 0$. More precisely, by (2.38) γ can be chosen so that

$$\frac{1}{\sigma(Q)}\int_Q \left(w^{-\frac{1}{p-1}}\right)^{1+\gamma} d\sigma \le \left(\frac{2}{\sigma(Q)}\int_Q w^{-\frac{1}{p-1}}\,d\sigma\right)^{1+\gamma}.$$

We can now conclude that

$$\begin{aligned}\left(\frac{1}{\sigma(Q)}\int_Q w\,d\sigma\right)\left(\frac{1}{\sigma(Q)}\int_Q w^{-\frac{1+\gamma}{p-1}}\,d\sigma\right)^{\frac{p-1}{1+\gamma}} &\le 2^{p-1}\left(\frac{1}{\sigma(Q)}\int_Q w\,d\sigma\right) \\ &\quad\times\left(\frac{1}{\sigma(Q)}\int_Q w^{-\frac{1}{p-1}}\,d\sigma\right)^{p-1} \\ &\le 2^{p-1}[w]_{A_p(d\sigma)}.\end{aligned} \tag{2.39}$$

If we choose $\varepsilon = (p-1)\frac{\gamma}{1+\gamma}$, then $\frac{p-1}{\gamma+1} = p - \varepsilon - 1$, and by (2.39) $w \in A_{p-\varepsilon}(d\sigma)$. Moreover, $[w]_{A_{p-\varepsilon}(d\sigma)} \le 2^{p-1}[w]_{A_p(d\sigma)}$.

Once again, since γ is small enough, we have

$$\frac{1}{1+\gamma} = 1 - \gamma + \gamma^2 - \gamma^3 + ...$$

Therefore,

$$\varepsilon \sim \frac{\gamma}{1+\gamma} \sim \gamma. \tag{2.40}$$

Taking into account (2.40), (2.37) and (1.15), we obtain

$$\varepsilon \sim \frac{1}{[\nu]_{A_{p'}(d\sigma)}} \frac{1}{D(\sigma)} = [w]^{1-p'}_{A_p(d\sigma)} \frac{1}{D(\sigma)} \tag{2.41}$$

□

2.4 Strong Estimates and Interpolation

In this section we are going to use the Interpolation Theorem 1.2.1 and self-improving properties of the $A_p(d\sigma)$ (or $RH_q(d\mu)$) weights to obtain *strong* and *sharp*-estimates for the operator M_σ. In view of the tautology (1.28), we can do it both on the $A_p(d\sigma)$ side, or on the $RH_{p'}(d\mu)$ side. Each time we get essentially the same power but with different doubling constants, $D(\sigma)$ and $D(\mu)$ respectively. In fact, the same argument can be applied to any sublinear operator T that satisfies the *weak*-inequalities (2.14).

2.4.1 Gehring's Estimates and Interpolation

In this subsection we use Interpolation and the Gehring's type estimates obtained in Section 2.2 to get the *strong*-estimates starting from the *weak*-ones. We can actually state the theorem for any sublinear operator T, and at the end we show the corresponding estimates for the operators M^c_σ and M_σ.

Theorem 2.4.1. *If $w \in A_p(d\sigma)$, $d\mu = wd\sigma$, and T is a sublinear operator of weak-type (p,p) i.e.*

$$\mu(\{x \in \mathbb{R}^n :\ Tf > \lambda\}) \le \left(\frac{C(\sigma,\mu,p)\,[w]_{A_p(d\sigma)}^{\frac{1}{p}}}{\lambda} \int_{\mathbb{R}^n} |f(x)|^p\, d\mu \right)^p, \tag{2.42}$$

then

$$\|Tf\|_{L^p(d\mu)} \le K_{p'} C(\sigma,\mu,p)\,(D(\mu))^{\frac{1}{p}}\,[w]_{A_p(d\sigma)}^{\frac{p'}{p}} \|f\|_{L^p(d\mu)},$$

where the constant $K_{p'}$ comes from the Interpolation Theorem 1.8 and $C(\sigma,\mu,p)$ is, as in (2.16), that is a product of functions which either do not depend on p or are $\frac{1}{p}$ powers of such functions.

Proof. Substituting $w := vu_o^{-1}$ into (2.3), since $d\sigma = u_o\,dx$, we obtain

$$\mu\{x \in \mathbb{R}^n :\ Tf > \lambda\} < C^p(\sigma,\mu,p)[w]_{A_p(d\sigma)} \left(\frac{\|f\|_{L^p(d\mu)}}{\lambda} \right)^p$$

which is, by the tautology (1.28), equivalent to

$$\mu(\{x \in \mathbb{R}^n :\ Tf > \lambda\}) < C^p(\sigma,\mu,p)[w^{-1}]^p_{RH_{p'}(wd\sigma)} \left(\frac{\|f\|_{L^p(d\mu)}}{\lambda} \right)^p$$

Suppose now $p > 1$. Then, by the Corollary, 2.2.1 $w^{-1} \in RH_{p'+\varepsilon}(wd\sigma)$ with comparable norm, where $\varepsilon \sim \frac{1}{[w^{-1}]^{p'}_{RH_{p'}(wd\sigma)} D(\mu)}$, trivially $w^{-1} \in RH_{p'-\varepsilon}(wd\sigma)$ and $[w^{-1}]_{RH_{p'-\varepsilon}(wd\sigma)} \le [w^{-1}]_{RH_{p'}(wd\sigma)}$.

We can now interpolate between p_0 and p_1, where
$p_0 := (p'+\varepsilon)' = \frac{p+\varepsilon(p-1)}{1+\varepsilon(p-1)} < p$, and $p_1 := (p'-\varepsilon)' = \frac{p-\varepsilon(p-1)}{1-\varepsilon(p-1)} > p$, if $\varepsilon < \frac{1}{p-1}$.
Also, $p_1 - p_0 = \frac{2\varepsilon(p-1)^2}{1-\varepsilon^2(p-1)^2}$. Since $\varepsilon < \frac{1}{p-1}$ we can expand the last expression in a geometric series

$$p_1 - p_0 = 2\varepsilon(p-1)^2 + 2\varepsilon^3(p-1)^4 + 2\varepsilon^5(p-1)^6 + ...$$

Thus $p_1 - p_0 \sim \varepsilon$. We choose t so that $\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}$.
By the Marcinkiewicz Interpolation Theorem 1.2.1 we obtain the *strong*-type result of the form

$$\|Tf\|_{L_p(d\mu)} \leq \frac{K_{p'} C(\sigma, \mu, p)[u_o v^{-1}]_{RH_{p'}(vdx)} \|f\|_{L_p(d\mu)}}{\left(\frac{1}{D(v)[u_o v^{-1}]^{p'}_{RH_{p'}(vdx)}} \right)^{\frac{1}{p}}}$$

$$\|Tf\|_{L_p(d\mu)} \leq K_{p'} C(\sigma, \mu, p)(D(v))^{\frac{1}{p}} [u_o v^{-1}]^{1+\frac{p'}{p}}_{RH_{p'}(vdx)} \|f\|_{L_p(d\mu)}$$

$$\|Tf\|_{L_p(d\mu)} \leq K_{p'} C(\sigma, \mu, p)(D(\mu))^{\frac{1}{p}} [u_o v^{-1}]^{p'}_{RH_{p'}(vdx)} \|f\|_{L_p(d\mu)}$$

This can be rewritten, using tautology (1.28), in the following form

$$\|Tf\|_{L_p(d\mu)} \leq K_{p'} C(\sigma, \mu, p)(D(\mu))^{\frac{1}{p}} [w]^{\frac{p'}{p}}_{A_p(d\sigma)} \|f\|_{L_p(d\mu)}, \tag{2.43}$$

which is exactly what we set out to prove. □

In particular, Theorem 2.4.1 and the *weak* estimates (2.14) imply the following estimates for maximal operators.

Corollary 2.4.1. *If $w \in A_p(d\sigma)$, and $d\mu = wd\sigma$, then*

$$\|M^c_\sigma\|_{L_p(d\mu)} \leq K_{p'} C_1(\sigma, \mu, p) \, (D(\mu))^{\frac{1}{p}} [w]^{\frac{p'}{p}}_{A_p(d\sigma)} \|f\|_{L_p(d\mu)}, \tag{2.44}$$

and

$$\|M_\sigma\|_{L_p(d\mu)} \leq K_{p'} C_2(\sigma, \mu, p) \, (D(\mu))^{\frac{1}{p}} [w]^{\frac{p'}{p}}_{A_p(d\sigma)} \|f\|_{L_p(d\mu)}, \tag{2.45}$$

where as in (2.15) and (2.16),

$$C_1(\sigma, \mu, p) = \min\{C(n)^{\frac{1}{p}}, D(\mu)^{\frac{1}{p}}\},$$
$$C_2(\sigma, \mu, p) = \min\{C(n)^{\frac{1}{p}} D(\sigma), D(\mu)^{\frac{1}{p}}\}.$$

Corollary 2.4.2. *In particular, if we specify* $d\sigma = wdx$ *and* $d\mu = w^{-1}d\sigma = dx$, *then* $D(\mu) = 3^n$, $[w^{-1}]^{\frac{p'}{p}}_{A_p(wdx)} = [w]^{p'}_{RH_{p'}}$ *and we get the following estimates for* $M_\sigma = M_w$,

$$\|M_w f\|_{L^p} \leq C_{n,p}\,[w]^{p'}_{RH_{p'}}\,\|f\|_{L_p}. \tag{2.46}$$

This generalizes results in [Per3] to $p \neq 2$. *Similar results have been found in [P] and [Mo]. Furthermore, the power* p' *is sharp (see [Mo]).*

2.4.2 Coifman-Fefferman-Buckley $d\sigma$ and Interpolation

This subsection is analogous to Subsection 2.4.3. Here again we use Interpolation and the Coifman-Fefferman-Buckley Theorem $d\sigma$, obtained in Section 2.3, to get the *strong*-estimates starting from the *weak*-ones. We first state the theorem for any sublinear operator T, and at the end we show the corresponding estimates for the operators M^c_σ and M_σ

Theorem 2.4.2. *If* $w \in A_p(d\sigma)$, $d\mu = wd\sigma$, *and* T *is a sublinear operator of weak-type* (p,p) *i.e.*

$$\mu(\{x \in \mathbb{R}^n : \, T\,f > \lambda\}) \leq \left(\frac{C(\sigma,\mu,p)\,[w]^{\frac{1}{p}}_{A_p(d\sigma)}}{\lambda}\int_{\mathbb{R}^n} |f(x)|^p\,d\mu\right)^p, \tag{2.47}$$

then

$$\|Tf\|_{L^p(d\mu)} \leq K_p C(\sigma,\mu,p)\,(D(\sigma))^{\frac{1}{p}}\,[w]^{\frac{p'}{p}}_{A_p(d\sigma)}\|f\|_{L^p(d\mu)}, \tag{2.48}$$

where the constant K_p *comes from the Interpolation Theorem 1.8 and* $C(\sigma,\mu,p)$ *is, as in (2.16), a product of functions which either do not depend on* p *or are* $\frac{1}{p}$ *powers of such functions..*

Proof. We can interpolate again between the corresponding *weak*-estimates for $p_0 := p-\varepsilon$ and $p_1 := p+\varepsilon$ to obtain another version of Buckley's type estimate. Clearly, if

$w \in A_p(d\sigma)$, then $w \in A_{p+\varepsilon}(d\sigma)$ and $[w]_{A_{p+\varepsilon}(d\sigma)} \leq [w]_{A_p(d\sigma)}$. Moreover, by Theorem 2.3.2 ε can be chosen so that

$$[w]_{A_{p-\varepsilon}(d\sigma)} \leq 2^{p-1}[w]_{A_p(d\sigma)} \text{ , with } \varepsilon \sim [w]^{1-p'}_{A_p(d\sigma)} \frac{1}{D(\sigma)}.$$

Thus, by the Marcinkiewicz Interpolation Theorem 1.2.1 we obtain the *strong* -type result of the form

$$\|Tf\|_{L_p(d\mu)} \leq \frac{K_p\, C(\sigma,\mu,p)[w]^{\frac{1}{p}}_{A_p(d\sigma)}\|f\|_{L_p(d\mu)}}{\left([w]^{1-p'}_{A_p(d\sigma)} \frac{1}{D(\sigma)}\right)^{\frac{1}{p}}}$$

$$\|Tf\|_{L_p(d\mu)} \leq K_p\, C(\sigma,\mu,p) D(\sigma)^{\frac{1}{p}} [w]^{\frac{p'}{p}}_{A_p(d\sigma)}\|f\|_{L_p(d\mu)} \tag{2.49}$$

□

In particular, Theorem 2.4.1 and the *weak*-estimates (2.14) imply the following estimates for maximal operators. This is an analogue of Corollary 2.4.1.

Corollary 2.4.3. *If $w \in A_p(d\sigma)$, and $d\mu = w d\sigma$, then*

$$\|M^c_\sigma f\|_{L_p(d\mu)} \leq K_p\, C_1(\sigma,\mu,p)\,(D(\sigma))^{\frac{1}{p}} [w]^{\frac{p'}{p}}_{A_p(d\sigma)}\|f\|_{L_p(d\mu)}, \tag{2.50}$$

and

$$\|M_\sigma f\|_{L_p(d\mu)} \leq K_p\, C_2(\sigma,\mu,p)\ (D(\sigma))^{\frac{1}{p}} [w]^{\frac{p'}{p}}_{A_p(d\sigma)}\|f\|_{L_p(d\mu)}, \tag{2.51}$$

where as in (2.15) and (2.16),

$$C_1(\sigma,\mu,p) = \min\{C(n)^{\frac{1}{p}}, D(\mu)^{\frac{1}{p}}\},$$
$$C_2(\sigma,\mu,p) = \min\{C(n)^{\frac{1}{p}} D(\sigma), D(\mu)^{\frac{1}{p}}\}.$$

2.5 A. Lerner's approach

In this section we adapt a very recent result, due to A. Lerner to the case $d\sigma$, (see [Le1]). We obtain yet another proof of the boundedness of M_σ, and *sharp* estimates, but without using interpolation. More precisely we will prove.

Theorem 2.5.1. *If* $w \in A_p(d\sigma)$, $d\mu = wd\sigma$, *then*

$$\|M_\sigma f\|_{L^p(d\mu)} \le C(n)\,(D(\sigma))^{\frac{2p-1}{p-1}}\,[w]_{A_p(d\sigma)}^{\frac{p'}{p}}\|f\|_{L^p(d\mu)}. \tag{2.52}$$

Proof. Let $\nu := w^{-\frac{1}{p-1}}$ and define

$$A_p(Q) := \frac{\mu(Q)}{\sigma(Q)}\left(\frac{\upsilon(3Q)}{\sigma(Q)}\right)^{p-1}, \text{ where } \upsilon(Q) := \int_Q \nu\, d\sigma. \tag{2.53}$$

Then, using the doubling of σ and the definition of $[w]_{A_p(d\sigma)}$ we get

$$\begin{aligned}
[A_p(Q)]^{\frac{1}{p-1}} &= \left(\frac{\mu(Q)}{\sigma(Q)}\right)^{\frac{1}{p-1}}\left(\frac{\upsilon(3Q)}{\sigma(Q)}\right) \\
&\le \left(D(\sigma)\frac{\mu(Q)}{\sigma(3Q)}\right)^{\frac{1}{p-1}}\left(D(\sigma)\frac{\upsilon(3Q)}{\sigma(3Q)}\right) \\
&= [D(\sigma)]^{\frac{p}{p-1}}\left(\frac{\mu(Q)}{\sigma(3Q)}\right)^{\frac{1}{p-1}}\left(\frac{\upsilon(3Q)}{\sigma(3Q)}\right) \\
&\le [D(\sigma)]^{\frac{p}{p-1}}\left(\frac{\mu(3Q)}{\sigma(3Q)}\left(\frac{\upsilon(3Q)}{\sigma(3Q)}\right)^{p-1}\right)^{\frac{1}{p-1}} \\
&\le [D(\sigma)]^{\frac{p}{p-1}}\,[w]_{A_p(d\sigma)}^{\frac{1}{p-1}}.
\end{aligned} \tag{2.54}$$

Now, using (2.54) yields

$$\begin{aligned}
\frac{1}{\sigma(Q)}\int_Q |f|\,d\sigma &= [A_p(Q)]^{\frac{1}{p-1}}\left(\frac{\sigma(Q)}{\mu(Q)}\left(\frac{1}{\upsilon(3Q)}\int_Q |f|\,d\sigma\right)^{p-1}\right)^{\frac{1}{p-1}}\\
&\le [D(\sigma)]^{\frac{p}{p-1}}\,[w]_{A_p(d\sigma)}^{\frac{1}{p-1}}\left(\frac{\sigma(Q)}{\mu(Q)}\left(\frac{1}{\upsilon(3Q)}\int_Q |f|\,d\sigma\right)^{p-1}\right)^{\frac{1}{p-1}}\\
&= [D(\sigma)]^{\frac{p}{p-1}}\,[w]_{A_p(d\sigma)}^{\frac{1}{p-1}}\left(\frac{\sigma(Q)}{\mu(Q)}\left(\frac{1}{\upsilon(3Q)}\int_Q (|f|\nu^{-1})\nu\,d\sigma\right)^{p-1}\right)^{\frac{1}{p-1}}\\
&\le [D(\sigma)]^{\frac{p}{p-1}}\,[w]_{A_p(d\sigma)}^{\frac{1}{p-1}}\left(\frac{1}{\mu(Q)}\int_Q M_\upsilon^c(f\nu^{-1})^{p-1}d\sigma\right)^{\frac{1}{p-1}}, \qquad (2.55)\\
&= [D(\sigma)]^{\frac{p}{p-1}}\,[w]_{A_p(d\sigma)}^{\frac{1}{p-1}}\left(\frac{1}{\mu(Q)}\int_Q (M_\upsilon^c(f\nu^{-1})^{p-1}w^{-1})w\,d\sigma\right)^{\frac{1}{p-1}},\\
&= [D(\sigma)]^{\frac{p}{p-1}}\,[w]_{A_p(d\sigma)}^{\frac{1}{p-1}}\left(\frac{1}{\mu(Q)}\int_Q (M_\upsilon^c(f\nu^{-1})^{p-1}w^{-1})\,d\mu\right)^{\frac{1}{p-1}} \qquad (2.56)
\end{aligned}$$

where M_υ^c denotes the weighted centered maximal function associated with measure $d\upsilon = \nu\,d\sigma$.

Note that the inequality (2.55) comes from the following geometrical observation: for any $x \in Q$, if Q_x denotes the cube congruent to Q and centered at x, then $Q \subset 2Q_x \subset 3Q$. Hence, using this and the definition of the maximal function, for any $x \in Q$

$$M_\upsilon^c(f\nu^{-1})(x) \ge \frac{1}{\upsilon(2Q_x)}\int_{Q_x} (|f|\nu^{-1})\nu\,d\sigma \ge \frac{1}{\upsilon(3Q)}\int_Q |f|\,d\sigma.$$

Let us recall (1.9) which says

$$M_\sigma^c f(x) \le M_\sigma f(x) \le D(\sigma)\,M_\sigma^c f(x). \qquad (2.57)$$

Thus, if we take the supremum in (2.56) over all cubes Q centered at x we get

$$M_\sigma^c f(x) \le [D(\sigma)]^{\frac{p}{p-1}}\,[w]_{A_p(d\sigma)}^{\frac{1}{p-1}}\left(M_\mu^c(M_\upsilon^c(f\nu^{-1})^{p-1})w^{-1})(x)\right)^{\frac{1}{p-1}}. \qquad (2.58)$$

Finally, using (2.58) and (2.57) yields

$$M_\sigma f(x) \le [D(\sigma)]^{\frac{2p-1}{p-1}}\,[w]_{A_p(d\sigma)}^{\frac{p'}{p}}\left(M_\mu^c(M_\upsilon^c(f\nu^{-1})^{p-1})w^{-1})(x)\right)^{\frac{1}{p-1}}. \qquad (2.59)$$

By Theorem 1.10.2 both $\|M_\mu^c\|_{L^{p'}(d\mu)}$ and $\|M_v^c\|_{L^p(\nu d\sigma)}$ are finite with constant uniformly in w. Therefore,

$$\|M_\sigma f\|_{L^p(d\mu)} \leq C(n)\,(D(\sigma))^{\frac{2p-1}{p-1}}\,[w]_{A_p(d\sigma)}^{\frac{p'}{p}}\|f\|_{L^p(d\mu)}. \tag{2.60}$$

□

2.6 Strong estimates for M_σ revisited

In this Section we summarize the estimates obtained for uncentered maximal function M_σ, associated with measure σ. Taking into account the estimates (2.45), (2.51) and (2.52), we conclude with

$$\|M_\sigma f(x)\|_{L^p(d\mu)} \leq K_{\sigma,\mu,p}\,[w]_{A_p(d\sigma)}^{\frac{p'}{p}}\,\|f\|_{L^p(d\mu)}, \tag{2.61}$$

where the constant $K_{\sigma,\mu,p}$ may depend on the doubling constants $D(\sigma)$ or $D(\mu)$, and the constants K_p and $K_{p'}$ come from the Interpolation Theorem 1.8. More precisely,

$$K_{\sigma,\mu,p} = \min\{K_{p'}\,C_2(\sigma,\mu,p)(D(\mu))^{\frac{1}{p}}\,; K_p\,C_2(\sigma,\mu,p)(D(\sigma))^{\frac{1}{p}};\; C(n)[D(\sigma)]^{\frac{2p-1}{p-1}}\}, \tag{2.62}$$

where $C_2(\sigma,\mu,p) = \min\{C(n)^{\frac{1}{p}}D(\sigma), D(\mu)^{\frac{1}{p}}\}$, $K_p = 2^{\frac{p+1}{p}}$ and $K_{p'} = \frac{2^{\frac{p+1}{p}}}{(p-1)^{\frac{2}{p}}}$.

Chapter 3

Sharp Extrapolation Theorems $d\sigma$.

In this chapter we use the estimates obtained for the Maximal Function M_σ in Chapter 2 to build Sharp Extrapolation Theorems $d\sigma$, where $d\sigma = u_o dx$, for some $u_o \in A_\infty$. We begin with the classical Rubio de Francia algorithm and follow closely [DrGrPerPet] to obtain a similar theorem, tracking down the dependence on the $[w]_{A_p(d\sigma)}$ characteristic constant. Furthermore, we use the fact observed in [CrMPe1] that the proofs of extrapolations theorems depend not on the properties of the operators, but rather on duality, the structure of the A_p weights, and norm inequalities for the Hardy-Littlewood maximal operator. Therefore, we can eliminate the superfluous operators and replace inequality (1.9) with

$$\int g(x)^r w(x)\, dx \leq C \int f(x)^r w(x)\, dx, \tag{3.1}$$

and concentrate on $\mathcal{F}$, a family of ordered pairs (g, f) of non-negative measurable functions such that the left-hand side of (3.1) is finite. As a consequence of adopting this approach we can apply this theorem directly to some known inverse estimates, presented in Chapter 4. Furthermore, vector-valued inequalities and many other results follow from extrapolation.

In order to prove these theorems we will need some technical lemmas. The lemmas and their proofs follows closely the corresponding ones in the case $d\sigma = dx$ in [DrGrPerPet].

3.1 Weight Lemmas

Careful examination of the extrapolation theorem justifies the need for lemmas discussed in this Section. We will use the *strong*-estimates, obtained in Chapter 2, for M_σ in $L^q(\nu_o d\eta)$, for $d\eta = \nu_o d\sigma$ and $\nu_o \in A_q(d\sigma)$. More precisely, in two cases when $q = p$ and $d\eta = d\mu$; or when $q = p'$ and $d\eta = w^{1-p'}d\sigma$. Let us recall the constant $K_{\sigma,\eta,q}$ obtained there is of the form

$$K_{\sigma,\eta,q} = \min\{K_{q'}\, C_2(\sigma,\eta,q)(D(\eta))^{\frac{1}{q}}; K_q\, C_2(\sigma,\eta,q)(D(\sigma))^{\frac{1}{q}}; C(n)[D(\sigma)]^{\frac{2q-1}{q-1}}\}, \quad (3.2)$$

$$\text{where } C_2(\sigma,\eta,q) = \min\{C(n)^{\frac{1}{q}} D(\sigma), D(\eta)^{\frac{1}{q}}\},\ K_q = 2^{\frac{q+1}{q}} \quad \text{and} \quad K_{q'} = \frac{2^{\frac{q+1}{q}}}{(q-1)^{\frac{2}{q}}}.$$

Lemma 3.1.1. *Take $p, s > 1$, $w \in A_p(d\sigma)$ and $u \in L^s(d\mu)$, $d\mu = w d\sigma$. Let*

$$S_\sigma(u) := \left(w^{-1} M_\sigma(|u|^{\frac{s}{p'}} w)\right)^{\frac{p'}{s}}$$

(a) *Then S_σ is bounded in $L^s(d\mu)$. Moreover,*

$$\|S_\sigma\|_{L^s(d\mu)\to L^s(d\mu)} \leq C(p')^{\frac{p'}{s}} K_{\sigma,\eta,p'}^{\frac{1}{s}} [w]_{A_p(d\sigma)}^{\frac{p'}{s}},$$

where $d\eta = w^{1-p'}d\sigma$ and the constant $K_{\sigma,\eta,p'}$ as in (3.2).

(b) *Let p, s be such that $r := \frac{p}{s'} \in [1,\infty)$. Take a nonnegative function $u \in L^s(d\mu)$. If $r > 1$, then the pair $(uw, S_\sigma(u)w) \in A_r(d\sigma)$*

Furthermore,

$$\sup_Q \left(\frac{1}{\sigma(Q)}\int_Q uw\, d\sigma\right)\left(\frac{1}{\sigma(Q)}\int_Q (S_\sigma(u)w)^{-\frac{1}{r-1}}\, d\sigma\right)^{r-1} \leq [w]_{A_p(d\sigma)}^{1-\frac{p}{s'}}. \quad (3.3)$$

If $r = 1$, the $A_1(d\sigma)$ condition on the pair $(uw,\, S_\sigma(u)w)$ also holds and translates into

$$M_\sigma(uw) \leq S_\sigma(u)w. \tag{3.4}$$

Proof. **(a)** Estimating directly the norm we obtain

$$\begin{aligned}
\|S_\sigma(u)\|_{L^s(d\mu)} &= \left(\int [w^{-1}M_\sigma(|u|^{\frac{s}{p'}}w)]^{p'} w\, d\sigma\right)^{\frac{1}{s}} = \left(\int [M_\sigma(|u|^{\frac{s}{p'}}w)]^{p'} w^{1-p'} d\sigma\right)^{\frac{1}{s}} \\
&\leq \|M_\sigma(|u|^{\frac{s}{p'}}w)\|_{L^{p'}(w^{1-p'}d\sigma)}^{\frac{p'}{s}} \\
&\leq \|M_\sigma\|_{L^{p'}(w^{1-p'}d\sigma)\to L^{p'}(w^{1-p'}d\sigma)}^{\frac{p'}{s}} \||u|^{\frac{s}{p'}}w\|_{L^{p'}(w^{1-p'}d\sigma)} \\
&\leq \|M_\sigma\|_{L^{p'}(w^{1-p'}d\sigma)\to L^{p'}(w^{1-p'}d\sigma)}^{\frac{p'}{s}} \|u\|_{L^s(d\mu)}.
\end{aligned}$$

Now, it only remains to insert the estimate from Section 2.6 and to recall (1.15) to obtain

$$\begin{aligned}
\|M_\sigma\|_{L^{p'}(w^{1-p'}d\sigma)\to L^{p'}(w^{1-p'}d\sigma)} &\leq C(p')K_{\sigma,\eta,p'}^{\frac{1}{p'}}[w^{1-p'}]_{A_{p'}(d\sigma)}^{\frac{(p')'}{p'}} \\
&= C(p')K_{\sigma,\eta,p'}^{\frac{1}{p'}}\left([w]_{A_p(d\sigma)}^{\frac{p'}{p}}\right)^{\frac{p}{p'}} \\
&= C(p')K_{\sigma,\eta,p'}^{\frac{1}{p'}}[w]_{A_p(d\sigma)}.
\end{aligned}$$

Thus, $\|S_\sigma\|_{L^s(d\mu)\to L^s(d\mu)} \leq C(p')^{\frac{p'}{s}}K_{\sigma,\eta,p'}^{\frac{1}{s}}[w]_{A_p(d\sigma)}^{\frac{p'}{s}}$, as claimed.

(ii) If $s = p'$, we have $r = 1$, then $S_\sigma(u)w = M_\sigma(uw)$. Automatically the two weight $A_1(d\sigma)$ condition, $M_\sigma(uw) \leq S_\sigma(u)w$, holds by (3.4).

If $s > p' > 1$, then $p > s' > 1$ and $r > 1$.

Note that

$$(r-1) = \frac{p-s'}{s'} = \frac{p-\frac{s}{s-1}}{\frac{s}{s-1}} = \frac{(p-1)s-p}{s} = \frac{p-1}{s}\left(s - \frac{p}{p-1}\right) = (p-1)\left(1-\frac{p'}{s}\right),$$

and in particular

$$-\frac{1}{p-1} = -\frac{1}{r-1}\left(1-\frac{p'}{s}\right). \tag{3.5}$$

By definition of the maximal function, if $x \in Q$,

$$m_Q^\sigma\left(u^{\frac{s}{p'}} w\right) \leq M_\sigma(u^{\frac{s}{p'}} w)(x) = \sup_{Q \ni x} m_Q^\sigma(u^{\frac{s}{p'}} w), \tag{3.6}$$

where $m_Q^\sigma(f)$ denotes the mean, with respect to measure $d\sigma$, of the function f over cube Q. Consequently, we can estimate $m_Q^\sigma\left((S_\sigma(u)w)^{\frac{-1}{r-1}}\right)$

$$\begin{aligned} m_Q^\sigma\left((S_\sigma(u)w)^{\frac{-1}{r-1}}\right) &= m_Q^\sigma\left([(w^{-1}M_\sigma(u^{\frac{s}{p'}} w))^{\frac{p'}{s}} w]^{\frac{-1}{r-1}}\right) \\ &= m_Q^\sigma\left([M_\sigma(u^{\frac{s}{p'}} w)]^{\frac{p'}{s}(\frac{-1}{r-1})} w^{(\frac{-1}{r-1})(1-\frac{p'}{s})}\right) \\ &= m_Q^\sigma\left([M_\sigma(u^{\frac{s}{p'}} w)]^{\frac{p'}{s}(\frac{-1}{r-1})} w^{(\frac{-1}{p-1})}\right) \end{aligned} \tag{3.7}$$

But if we raise (3.6) to a negative power we get

$$\left(M_\sigma(u^{\frac{s}{p'}} w)\right)^{(-\frac{1}{r-1})\frac{p'}{s}} \leq \left(m_Q^\sigma(u^{\frac{s}{p'}} w)\right)^{(-\frac{1}{r-1})\frac{p'}{s}}. \tag{3.8}$$

Hence, by (3.7), (3.8) and (3.5)

$$\begin{aligned} \left[m_Q^\sigma\left((S_\sigma(u)w)^{\frac{-1}{r-1}}\right)\right]^{r-1} &\leq \left[m_Q^\sigma(u^{\frac{s}{p'}} w)\right]^{-\frac{p'}{s}} \left[m_Q^\sigma(w^{\frac{-1}{p-1}})\right]^{r-1} \\ &\leq \left[m_Q^\sigma(u^{\frac{s}{p'}} w)\right]^{-\frac{p'}{s}} \left[m_Q^\sigma(w^{\frac{-1}{p-1}})\right]^{(p-1)(1-\frac{p'}{s})} \end{aligned} \tag{3.9}$$

Using Hölder's inequality with exponents $q = \frac{s}{p'} > 1$ and $q' = \frac{s}{s-p'}$, we get the estimate

$$\begin{aligned} m_Q^\sigma(u\, w) &= m_Q^\sigma\left(u\, w^{\frac{p'}{s}}\, w^{1-\frac{p'}{s}}\right) \\ &\leq \left(m_Q^\sigma(u^{\frac{s}{p'}}\, w)\right)^{\frac{p'}{s}} (m_Q w)^{1-\frac{p'}{s}} \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10) we get

$$m_Q^\sigma(u\, w)\, m_Q^\sigma\left((S_\sigma(u)w)^{\frac{-1}{r-1}}\right)^{r-1} \leq \left[m_Q^\sigma(w)\, m_Q^\sigma\left(w^{\frac{-1}{p-1}}\right)^{p-1}\right]^{1-\frac{p'}{s}} \leq [w]_{A_p(d\sigma)}^{1-\frac{p'}{s}}$$

Taking supremum on the left-hand-side, over all cubes Q with sides parallel to the axes, we obtain the desired inequality (3.3). □

Lemma 3.1.2. *Let p, s, r and w be as in the Lemma 3.1.1(b). Then for each $u \geq 0$, $u \in L^s(d\mu)$, there exists $v \in L^s(d\mu)$ such that*

(a) $u(x) \leq v(x)$ *a.e. and* $\|v\|_{L^s(d\mu)} \leq 2\|u\|_{L^s(d\mu)}$;

(b) $vw \in A_r(d\sigma)$, *and* $[vw]_{A_r(d\sigma)} \leq 2C(p')^{\frac{p'}{s}} K^{\frac{1}{s}}_{\sigma,\eta,p'}[w]_{A_p(d\sigma)}$ *where* $d\eta = w^{1-p'}d\sigma$ *and the constant* $K_{\sigma,\eta,p'}$ *as in (3.2).*

Proof. Define v via the following convergent Neumann series:

$$v = \sum_{n=0}^{\infty} \frac{S^n_\sigma u}{2^n \|S_\sigma\|^n} = u + \frac{S_\sigma(u)}{2\|S_\sigma\|} + ...,$$

where $\|S_\sigma\| := \|S_\sigma\|_{L^s(d\mu)\to L^s(d\mu)}$.

Since

$$\|v\|_{L^s(d\mu)} \leq \sum_{n=0}^{\infty} \frac{\|S^n_\sigma u\|_{L^s(d\mu)}}{2^n \|S_\sigma\|^n} \leq \sum_{n=0}^{\infty} \frac{\|S_\sigma\| \, \|u\|_{L^s(d\mu)}}{2^n \|S_\sigma\|^n} = 2\|u\|_{L^s(d\mu)},$$

thus **(a)** is clearly satisfied.

(b) It follows from the definition of v and the sublinearity of S_σ that

$$\begin{aligned} S_\sigma v &= S_\sigma \left(\sum_{n=0}^{\infty} \frac{S^n_\sigma u}{2^n \|S_\sigma\|^n} \right) \leq \sum_{n=0}^{\infty} \frac{S^{n+1}_\sigma u}{2^n \|S_\sigma\|^n} \frac{2\,\|S_\sigma\|}{2\,\|S_\sigma\|} \\ &\leq 2\,\|S_\sigma\| \sum_{n=0}^{\infty} \frac{S^{n+1}_\sigma u}{2^{n+1}\,\|S_\sigma\|^{n+1}} \leq 2\|S_\sigma\|v, \end{aligned}$$

thus $S_\sigma(v) \leq 2\|S_\sigma\|v$. If we take a negative power, we will get

$$v^{-1} \leq 2\|S_\sigma\|\,(S_\sigma v)^{-1}. \tag{3.11}$$

Suppose $r > 1$. By the previous lemma, since $v \in L^s(d\mu)$ and $p > s' > 1$, the pair $(vw, S_\sigma(v)w) \in A_r(d\sigma)$ with the $A_r(d\sigma)$-constant bounded by $[w]^{1-\frac{p'}{s}}_{A_p(d\sigma)}$.

Also, since

$$\|S_\sigma\|_{L^s(d\mu)} \leq C(p')^{\frac{p'}{s}} K_{\sigma,\eta,p'}^{\frac{1}{s}} [w]_{A_p(d\sigma)}^{\frac{p'}{s}},$$

using (3.11) and (3.3) we can estimate $[vw]_{A_r(d\sigma)}$:

$$\begin{aligned} m_Q^\sigma(vw)\, m_Q^\sigma\left((vw)^{-\frac{1}{r-1}}\right)^{r-1} &\leq m_Q^\sigma(vw)\, m_Q^\sigma\left((S_\sigma(v)w)^{-\frac{1}{r-1}}\right)^{r-1} 2\|S_\sigma\| \\ &\leq [w]_{A_p(d\sigma)}^{1-\frac{p'}{s}}\, 2C(p')^{\frac{p'}{s}} K_{\sigma,\eta,p'}^{\frac{1}{s}} [w]_{A_p(d\sigma)}^{\frac{p'}{s}} \\ &= 2C(p')^{\frac{p'}{s}} K_{\sigma,\eta,p'}^{\frac{1}{s}}\, [w]_{A_p(d\sigma)}. \end{aligned}$$

Taking supremum on the left hand side, over all cubes Q with sides parallel to the axes, we obtain the desired estimate for $[vw]_{A_r(d\sigma)}$, $r > 1$.

When $r = 1$, then $s = p'$ and $\|S_\sigma\| \leq C(p')K_{\sigma,\eta,p'}^{\frac{1}{p'}}[w]_{A_p(d\sigma)}$, futhermore

$$M_\sigma(vw) \leq S_\sigma(v)w \leq 2\|S_\sigma\|vw \leq 2C(p')K_{\sigma,\eta,p'}^{\frac{1}{p'}}[w]_{A_p(d\sigma)}\, vw$$

We conclude that $[vw]_{A_1(d\sigma)} \leq 2C(p')K_{\sigma,\eta,p'}^{\frac{1}{p'}}[w]_{A_p(d\sigma)}$. □

Lemma 3.1.3. *Fix r satisfying $1 \leq r < \infty$, $d\mu = wd\sigma$.*

(a) *Let $1 \leq r < p < \infty$ and $r := \frac{p}{s'}$. Let $w \in A_p(d\sigma)$, then for every $u \geq 0$, $u \in L^s(d\mu)$, there exists $v \geq 0$, $v \in L^s(d\mu)$, such that $u(x) \leq v(x)$ a.e. and $\|v\|_{L^s(d\mu)} \leq 2\|u\|_{L^s(d\mu)}$.*
Moreover, $vw \in A_r(d\sigma)$ and $[vw]_{A_r(d\sigma)} \leq 2C(p')^{\frac{p-r}{p-1}} K_{\sigma,\eta,p'}^{\frac{p-r}{p}}[w]_{A_p(d\sigma)}$,
where $d\eta = w^{1-p'}d\sigma$ and the constant $K_{\sigma,\eta,p'}$ as in (3.2).

(b) *Let $1 < p < r$ and define $s := \frac{p}{r-p} > 0$. Let $w \in A_p(d\sigma)$, then for every $u \geq 0$, $u \in L^s(d\mu)$, there exists $v \geq 0$, $v \in L^s(d\mu)$ such that, $u(x) \leq v(x)$, a.e. and $\|v\|_{L^s(d\mu)} \leq 2^{r-1}\|u\|_{L^s(d\mu)}$.*
Furthermore, $v^{-1}w \in A_r(d\sigma)$ and

$$[v^{-1}w]_{A_r(d\sigma)} \leq 2^{r-1}\left(C(p)^{r-p}K_{\sigma,\mu,p}^{\frac{r-p}{p}}[w]_{A_p(d\sigma)}\right)^{\frac{r-1}{p-1}},$$

where $d\mu = wd\sigma$, the constant $K_{\sigma,\mu,p}$ as in Section 2.6, and $C(p)$ denotes the constants in (2.49).

Proof. **(a)** Clearly $r \geq 1$ implies $s' \leq p$, and we can now use Lemma 3.1.2 after observing that $\frac{p'}{s} = \frac{p-r}{p-1}$.
(b) Take p, r and s as in the formulation of the lemma. (Notice that everything that is being said still holds if $0 < s < 1$). Now the dual exponents satisfy the opposite inequality, $r' < p'$, and if we define $s^* := (\frac{p'}{r'})' > 1$, then
$s^* = \frac{\frac{p'}{r'}}{\frac{p'}{r'}-1} = \frac{p'}{p'-r'} = \frac{\frac{p}{p-1}}{\frac{p}{p-1}-\frac{r}{r-1}} = \frac{\frac{p}{p-1}}{\frac{p(r-1)-r(p-1)}{(p-1)(r-1)}} = \frac{p(r-1)}{r-p} = s(r-1).$

We apply the previous case with p', r' and $w^{1-p'} \in A_{p'}(d\sigma)$ instead of p, r and $w \in A_p(d\sigma)$, respectively. If $u \geq 0$, $u \in L^s(d\mu)$, then $u_0 = u^{\frac{s}{s*}} w^{\frac{p'}{s*}} \in L^{s*}(w^{1-p'}d\sigma)$ and by (a) there exists $v_0 \in L^{s*}(w^{1-p'}d\sigma)$ such that

$$u_0 \leq v_0 \text{ a.e.}, \qquad \|v_0\|_{L^{s*}(w^{1-p'}d\sigma)} \leq 2\|u_0\|_{L^{s*}(w^{1-p'}d\sigma)}, \tag{3.12}$$

and

$$v_0 w^{1-p'} \in A_{r'}(d\sigma), \qquad [v_0 w^{1-p'}]_{A_{r'}(d\sigma)} \leq 2C(p)^{\frac{p'-r'}{p'-1}} K_{\sigma,\mu,p}^{\frac{p'-r'}{p'}} [w^{1-p'}]_{A_{p'}(d\sigma)} \tag{3.13}$$
$$= 2C(p)^{\frac{r-p}{p-1}} K_{\sigma,\mu,p}^{\frac{r-p}{p(p-1)}} [w]_{A_p(d\sigma)}^{\frac{1}{p-1}}.$$

Define v so that $v_0 = v^{\frac{s}{s^*}} w^{\frac{p'}{s*}}$, that is, $v = v_0^{\frac{s*}{s}} w^{-\frac{p'}{s}}$. Then clearly
$u(x) \leq v(x)$ *a.e.*, and 3.12 implies $\|v\|_{L^s(d\mu)} \leq 2^{r-1}\|u\|_{L^s(d\mu)}$.

Since $\frac{s^*}{s} = r-1$ and $1 + \frac{p'}{s} = 1 + \frac{p}{p-1}\frac{r-p}{p} = \frac{p-1-p+r}{p-1} = (1-r)(1-p')$, then by (1.15), $v^{-1}w = v_0^{\frac{-s^*}{s}} w^{1+\frac{p'}{s}} = (v_0 w^{1-p'})^{1-r} \in A_r(d\sigma)$.

Thus, (3.13) implies

$$[v^{-1}w]_{A_r(d\sigma)} = [(v_0 w^{1-p'})^{1-r}]_{A_r(d\sigma)}$$
$$= [v_0 w^{1-p'}]_{A_{r'}(d\sigma)}^{\frac{1}{r'-1}} \leq 2^{r-1} \left(C(p)^{r-p} K_{\sigma,\mu,p}^{\frac{r-p}{p}} [w]_{A_p(d\sigma)} \right)^{\frac{r-1}{p-1}}.$$

□

3.2 Sharp Extrapolation Theorem $d\sigma$.

Now we are ready to present and prove the Sharp Extrapolation Theorem $d\sigma$, in an operator-free form, as mentioned at the at the beginning of this chapter.

Theorem 3.2.1. *Given a family $\mathcal{F}$, suppose there is $1 \leq r < \infty$ and functions $(f, g) \in \mathcal{F}$ such that $g \in L^r(ud\sigma)$ for all weights $u \in A_r(d\sigma)$ and*

$$\|g\|_{L^r(ud\sigma)} \leq C\|f\|_{L^r(ud\sigma)}, \text{ for any } (f, g) \in \mathcal{F}, \tag{3.14}$$

where the constant C depends only on $[u]_{A_r(d\sigma)}$, and possibly on $D(\sigma)$ but not on (g, f).

Then for all $1 < p < \infty$, $(f, g) \in \mathcal{F}$

$$\|g\|_{L^p(wd\sigma)} \leq K_{\sigma,w,p}\, \|f\|_{L^p(wd\sigma)}, \tag{3.15}$$

with $K_{\sigma,w,p}$ depending possibly only on $[w]_{A_p(d\sigma)}$, p and $D(\sigma)$.

More precisely, suppose for each $B > 1$ and $(f, g) \in \mathcal{F}$ there is a constant $N_r(B) > 0$ such that

$$\|g\|_{L^r(ud\sigma)} \leq N_r(B)\|f\|_{L^r(ud\sigma)}, \tag{3.16}$$

for all $u \in A_r(d\sigma)$, with $[u]_{A_r(d\sigma)} \leq B$. Then for any $1 < p < \infty$, and $B > 1$ there is $N_p(B) > 0$ such that for all weights $w \in A_p(d\sigma)$ with $[w]_{A_p(d\sigma)} \leq B$

$$\|g\|_{L^p(wd\sigma)} \leq N_p(B)\|f\|_{L^p(wd\sigma)}. \tag{3.17}$$

Also, if we assume that $N_r(B)$ is the smallest constant satisfying (3.16), then for any $1 < p < \infty$ and all $B > 1$ one can choose $N_p(B)$ in (3.17) so that

$$N_p(B) \leq \begin{cases} 2^{\frac{1}{r}} N_r\left(2C(p')^{\frac{p-r}{p-1}} K_{\sigma,\eta,p'}^{\frac{p-r}{p-1}} B\right) & \text{if } p > r, \\ 2^{\frac{r-1}{r}} N_r\left(2^{r-1}(C(p)^{p-r} K_{\sigma,\mu,p}^{\frac{r-p}{p}} B)^{\frac{r-1}{p-1}}\right) & \text{if } p < r. \end{cases}$$

Here $C(p)$ is the constant appearing in (2.49).

Theorem 3.2.1 is a consequence of Lemma 3.1.3.

Proof. Case 1: Assume $1 \leq r < p$, $w \in A_p(d\sigma)$, $d\mu = wd\sigma$, and $\frac{1}{s} = 1 - \frac{r}{p}$, i.e. $s' = \frac{p}{r}$. Then

$$\|g\|^r_{L^p(d\mu)} = \left(\int |g(x)|^p w(x)\, d\sigma\right)^{\frac{r}{p}} = \||g|^r\|_{L^{s'}(d\mu)} = \sup_{\substack{u\geq 0\\ \|u\|_{L^s(d\mu)}=1}} \int |g(x)|^r u(x)w(x)\, d\sigma,$$

where the last equality holds by duality. For each such u, by Lemma 3.1.3a, there is $v \in L^s(d\mu)$ such that $u \leq v$, a.e. , and $\|v\|_{L^s(d\mu)} \leq 2\|u\|_{L^s(d\mu)} = 2$.

Furthermore, $vw \in A_r(d\sigma)$ and $[vw]_{A_r(d\sigma)} \leq 2C(p')^{\frac{p-r}{p-1}} K^{\frac{p-r}{p}}_{\sigma,\eta,p'}[w]_{A_p(d\sigma)}$.
Then,

$$\int |g(x)|^r u(x)w(x)\, d\sigma \leq \int |g(x)|^r v(x)w(x)\, d\sigma \leq \|g\|^r_{L^r(vwd\sigma)}. \tag{3.18}$$

Now, by the hypothesis,

$$\|g\|^r_{L^r(vwd\sigma)} \leq N^r_r([vw]_{A_r(d\sigma)})\|f\|^r_{L^r(vwd\sigma)}. \tag{3.19}$$

Moreover, by Hölder's inequality we obtain

$$\begin{aligned} \|f\|_{L^r(vwd\sigma)} &= \int |f(x)|^r v(x)w(x)^{\frac{r}{p}} w(x)^{1-\frac{r}{p}}\, d\sigma \\ &\leq \left(\int |f(x)|^p d\mu\right)^{\frac{r}{p}} \left(\int v(x)^s d\mu\right)^{\frac{1}{s}} \leq 2\|f\|^r_{L^p(d\mu)} \end{aligned} \tag{3.20}$$

Since, by definition, N_r is an increasing function and

$$[vw]_{A_r(d\sigma)} \leq 2C(p')^{\frac{p-r}{p-1}} K^{\frac{p-r}{p}}_{\sigma,\eta,p'}[w]_{A_p(d\sigma)},$$

thus combining (3.18), (3.19) and (3.20) we obtain

$$\int |g(x)|^r u(x)w(x)\, d\sigma \leq 2N^r_r\left(2C(p')^{\frac{p-r}{p-1}} K^{\frac{p-r}{p}}_{\sigma,\eta,p'}[w]_{A_p(d\sigma)}\right)\|f\|^r_{L^p(d\mu)}.$$

Taking the supremum over all admissible u we obtain the desired inequality,

$$\|g\|_{L^p(d\mu)} \leq 2^{\frac{1}{r}} N_r \left(2C(p')^{\frac{p-r}{p-1}} K(\sigma,\, w^{1-p'} d\sigma)^{\frac{p-r}{p}} [w]_{A_p(d\sigma)}\right) \|f\|_{L^p(d\mu)}$$

In particular, if $\|g\|_{L^r(\omega d\sigma)} \leq C_\sigma\, [\omega]_{A_r(d\sigma)} \|f\|_{L^r(\omega d\sigma)}$, where C_σ may depend only on $D(\sigma)$, then for $p > r$ we get

$$\|g\|_{L^p(wd\sigma)} \leq C2^{\frac{r+1}{r}} C(p')^{\frac{p-r}{p-1}} K_{\sigma,\eta,p'}^{\frac{p-r}{p}} [w]_{A_p(d\sigma)} \|f\|_{L^p(wd\sigma)}.$$

Case 2:

Assume $1 < p < r$ and define $s := \frac{p}{r-p}$. For $f \in L^p(d\mu)$ define $u = |f|^{r-p}$.

Then $u \in L^s(d\mu)$ and $\|u\|_{L^s(d\mu)} = \|f\|_{L^p(d\mu)}^{r-p}$. By Lemma 3.1.3b there exists a function v such that

$$u \;\leq\; v \quad \text{a.e.,} \tag{3.21}$$

$$\|v\|_{L^s(d\mu)} \;\leq\; 2^{r-1}\|u\|_{L^s(d\mu)} = 2^{r-1}\|f\|_{L^p(d\mu)}^{r-p}. \tag{3.22}$$

Furthermore $v^{-1}w \in A_r(d\sigma)$, and

$$[v^{-1}w]_{A_r(d\sigma)} \leq 2^{r-1} \left(C(p)^{r-p} K_{\sigma,\mu,p}^{\frac{r-p}{p}} [w]_{A_p(d\sigma)}\right)^{\frac{r-1}{p-1}}$$

Now, using Hölder's inequality, with $q = \frac{r}{p} > 1$, $q' = \frac{r}{r-p}$ and $\frac{q'}{q} = s$, we obtain

$$\begin{aligned}
\|g\|_{L^p(d\mu)}^r &= \left(\int |g(x)|^p v(x)^{-\frac{p}{r}} v(x)^{\frac{p}{r}} w(x)\, d\sigma\right)^{\frac{r}{p}} \\
&\leq \left(\int \left(v(x)^{\frac{p}{r}}\right)^{q'} w(x)\, d\sigma\right)^{\frac{r}{p\,q'}} \int \left(|g(x)|^p v(x)^{-\frac{p}{r}}\right)^{\frac{r}{p}} w(x)\, d\sigma \\
&= \|v\|_{L^s(d\mu)} \int |g(x)|^r v(x)^{-1} w(x)\, d\sigma
\end{aligned}$$

By construction $v(x)^{-1}w(x) \in A_r(d\sigma)$, and we can use the hypothesis to get

$$\int |g(x)|^r v(x)^{-1} w(x)\, d\sigma \leq N_r^r \left([v^{-1}w]_{A_r(d\sigma)}\right) \int |f(x)|^r v^{-1}(x) w(x)\, d\sigma \tag{3.23}$$

Furthermore, since by (3.21) $v^{-1}(x) \leq |f(x)|^{p-r}$ a.e. we get

$$\int |f(x)|^r v^{-1}(x) w(x)\, d\sigma \leq \int |f(x)|^r |f(x)|^{p-r}\, d\mu = \|f\|^r_{L^p(d\mu)} \tag{3.24}$$

Thus combing (3.22), (3.23) and (3.24) yields

$$\|g\|^r_{L^p(d\mu)} \leq 2^{r-1} N_r^r \left(2^{r-1} \left(C(p)^{r-p} K_{\sigma,\mu,p}^{\frac{r-p}{p}} [w]_{A_p(d\sigma)} \right)^{\frac{r-1}{p-1}} \right) \|f\|^r_{L^p(d\mu)}.$$

We conclude that

$$\|g\|_{L^p(d\mu)} \leq 2^{\frac{r-1}{r}} N_r \left(2^{r-1} \left(C(p)^{r-p} K_{\sigma,\mu,p}^{\frac{r-p}{p}} [w]_{A_p(d\sigma)} \right)^{\frac{r-1}{p-1}} \right) \|f\|_{L^p(d\mu)}.$$

In particular, if we know that $\|g\|_{L^r(\omega d\sigma)} \leq C_\sigma [\omega]_{A_r(d\sigma)} \|f\|_{L^r(d\mu)}$, for all $\omega \in A_r(d\sigma)$ and all $(g, f) \in \mathcal{F}$, where C_σ may depend on the doubling constant $D(\sigma)$, then for $1 < p < r$ we have

$$\|g\|_{L^p(d\mu)} \leq C_\sigma 2^{\frac{r^2-1}{r}} C(p)^{\frac{(r-p)(r-1)}{p-1}} [w]^{\frac{r-1}{p-1}}_{A_p(d\sigma)} K_{\sigma,\mu,p}^{\frac{(r-p)(r-1)}{p(p-1)}} \|f\|_{L^r(d\mu)}.$$

Specializing further, when $r = 2$ and $\|g\|_{L^2(d\mu)} \leq C_\sigma\, [w]_{A_2(d\sigma)} \|f\|_{L^2(d\mu)}$, then

$$\|g\|_{L^p(wd\sigma)} \leq C_\sigma(p) [w]^\alpha_{A_p(d\sigma)} \|f\|_{L^p(wd\sigma)},$$

where $\alpha = \max\{1, \frac{p'}{p}\}$ and

$$C_\sigma(p) = 2\sqrt{2} C_\sigma \times \begin{cases} C(p')^{\frac{p-2}{p-1}} K_{\sigma,\eta,p'}^{\frac{p-2}{p}} & \text{if } p \geq 2, \\ C(p)^{\frac{2-p}{p-1}} K_{\sigma,\mu,p}^{\frac{2-p}{p(p-1)}} & \text{if } 1 < p \leq 2. \end{cases}$$

□

3.3 Lerner's type extrapolation dσ

In this section we use Lemma 3.1.1 to generalize Lerner's Theorem, (see [Le1]). First, we use ordered pairs (f, g) instead of (f, Tf). We also have initial two weight estimates from $L^{p_o}(v_o\,d\sigma)$ to $L^{p_o}(w_o\,d\sigma)$ for any $1 < p_o < \infty$, not just $p_o = 2$ and $d\sigma = dx$.

Theorem 3.3.1. *Let $p > p_0 > 1$. Suppose*

$$\|g\|_{L^{p_o}(w_o\,d\sigma)} \leq C[v_o^{-\frac{p_o'}{p_o}}]^{\alpha}_{A_\infty(d\sigma)}[w_o, v_o]^{\beta}_{A_{p_o}(d\sigma)}\|f\|_{L^{p_o}(v_o\,d\sigma)}$$

for all weights $(w_o, v_o) \in A_{p_o}(d\sigma)$, and some constant C depending possibly on $D(\sigma)$. Then for any $w \in A_p(d\sigma)$,

$$\|g\|_{L^p(w\,d\sigma)} \leq C_\sigma\,[w]_{A_p(d\sigma)}^{\frac{p-p_o+\alpha+\beta(p_o-1)}{p_o(p-1)}}\|f\|_{L^{p_o}(w\,d\sigma)},$$

where $C_\sigma = C\,K_{\sigma,\eta,p'}^{\frac{p-p_o}{p\,p_o}}$, and $K_{\sigma,\eta,p'}$ as in (3.2).

Proof. Let $s' := \frac{p}{p_o} > 1$. Take an arbitrary function $u \geq 0$ with $\|u\|_{L^s(wd\sigma)} = 1$, and set

$$S_\sigma(u) = \left\{w^{-1}M_\sigma(u^{\frac{p-1}{p-p_o}}w)\right\}^{\frac{p-p_o}{p-1}}.$$

By Lemma 3.1.1a we have

$$\|S_\sigma(u)\|_{L^s(wd\sigma)} \leq C(p')^{\frac{p-p_o}{p-1}}K_{\sigma,\eta,p'}^{\frac{p-p_0}{p}}[w]_{A_p(d\sigma)}^{\frac{p-p_o}{p-1}}, \tag{3.25}$$

where the constant $K_{\sigma,\eta,p'}$ as in (3.2). Furthermore, by Lemma 3.1.1b when $r := \frac{p}{s'} = p_o$ we get

$$[uw, S_\sigma(u)w]_{A_{p_o}(d\sigma)} \leq [w]_{A_p(d\sigma)}^{\frac{p_o-1}{p-1}}. \tag{3.26}$$

Now, by duality

$$\|g\|^{p_o}_{L^{p_o}(w\,d\sigma)} = \left(\int_{\mathbb{R}^n} g^{p_o\,s'}wd\sigma\right)^{s'} = \||g|^{p_o}\|_{L^{s'}(w\,d\sigma)} = \sup_{\substack{u\geq 0\\ \|u\|_{L^s(wd\sigma)}=1}}\int |g|^{p_o}uw\,d\sigma$$

Applying the assumption with $v_o := S_\sigma(u)w$ and $w_o := uw$ we obtain

$$\int_{\mathbb{R}^n}|g|^{p_o}uw\,d\sigma \quad\leq\quad C[(S_\sigma(u)w)^{-\frac{p_o'}{p_o}}]^{\alpha}_{A_\infty(d\sigma)}[uw, S_\sigma(u)w]^{\beta}_{A_{p_o}(d\sigma)}\|f\|^{p_o}_{L^{p_o}(S_\sigma(u)wd\sigma)},$$

By Hölder inequality

$$\begin{aligned}\|f\|^{p_o}_{L^{p_o}(S_\sigma(u)wd\sigma)} &= \int |f|^{p_o} S_\sigma(u) w\, d\sigma \\ &\leq \left(\int (|f|^{p_o})^{\frac{p}{p_o}} w\, d\sigma\right)^{\frac{p_o}{p}} \left(\int (S_\sigma(u))^{\frac{p}{p-p_o}} w\, d\sigma\right)^{\frac{p-p_o}{p}} \\ &= \|f\|_{L^{p_o}(wd\sigma)} \|S_\sigma(u)\|_{L^s(wd\sigma)}\end{aligned}$$

In order to estimate $[(S_\sigma(u)w)^{-\frac{p_o'}{p_o}}]_{A_\infty(d\sigma)}$ it suffices to show $(S_\sigma(u)w)^{-\frac{p_o'}{p_o}} \in A_r(d\sigma)$ for some $r > 1$ and use (1.25).

Since $\left(1 - \frac{p-p_o}{p-1}\right)(-\frac{p_o'}{p_o}) = -\frac{1}{p-1}$ we have

$$\begin{aligned}\left(\frac{1}{\sigma(Q)}\int_Q (S_\sigma(u)w)^{-\frac{p_o'}{p_o}}\, d\sigma\right) &= \frac{1}{\sigma(Q)}\int_Q \left[\left(w^{-1} M_\sigma\left(u^{\frac{p-1}{p_o-1}}\right)\right)^{\frac{p-1}{p_o-1}} w\, d\sigma\right]^{-\frac{p_o'}{p_o}} \\ &= \frac{1}{\sigma(Q)}\int_Q w^{(1-\frac{p-p_o}{p-1})(-\frac{p_o'}{p_o})} M_\sigma\left(u^{\frac{p-1}{p_o-1}}\right)^{\frac{p-p_o}{p-1}(-\frac{p_o'}{p_o})} d\sigma\end{aligned}$$

$$\leq \operatorname*{essinf}_Q \left[M_\sigma\left(u^{\frac{p-1}{p-p_o}} w\right)(x)\right]^{\frac{p-p_o}{p-1}(-\frac{1}{p_o-1})} \left(\frac{1}{\sigma(Q)}\int_Q w^{-\frac{1}{p-1}}\, d\sigma\right)$$

Next, by Hölder inequality, with the exponents $q := (p-1)(r-1) > 0$ and $q' = \frac{q}{q-1} = \frac{(p-1)(r-1)}{(p-1)(r-1)-1}$:

$$\begin{aligned}&\left(\frac{1}{\sigma(Q)}\int_Q (S_\sigma(u)w)^{\frac{p_o'}{p_o}\frac{1}{r-1}}\, d\sigma\right)^{r-1} \\ &= \left\{\frac{1}{\sigma(Q)}\int_Q \left[w\left(w^{-1} M_\sigma(u^{\frac{p-1}{p-p_o}} w)^{\frac{p-p_o}{p-1}}\right)^{\frac{p_o'}{p_o}}\right]^{\frac{1}{r-1}} d\sigma\right\}^{r-1} \\ &= \left(\frac{1}{\sigma(Q)}\int_Q w^{\frac{1}{(p-1)(r-1)}} M_\sigma(u^{\frac{p-p_o}{p-1}} w)^{\frac{p-p_o}{p-1}\frac{p_o'}{p_o}\frac{1}{r-1}}\, d\sigma\right)^{r-1} \\ &\leq \left[\frac{1}{\sigma(Q)}\int_Q \left(w^{\frac{1}{(p-1)(r-1)}}\right)^{(p-1)(r-1)} d\sigma\right]^{\frac{r-1}{(p-1)(r-1)}} \\ &\qquad \times \left(\frac{1}{\sigma(Q)}\int_Q M_\sigma(u^{\frac{p-1}{p-p_o}} w)^{\frac{p-p_o}{(p_o-1)[(p-1)(r-1)-1]}}\, d\sigma\right)^{\frac{(p-1)(r-1)-1}{(p-1)}}\end{aligned}$$

$$\leq \left(\int_Q w\,d\sigma\right)^{\frac{1}{p-1}} \left(\frac{1}{\sigma(Q)}\int_Q M_\sigma(u^{\frac{p-1}{p-p_o}}w)^{\frac{p-p_o}{(p_o-1)[(p-1)(r-1)-1]}}\,d\sigma\right)^{\frac{(p-1)(r-1)-1}{(p-1)}}$$

If we choose $r > \frac{p_o}{p_o-1}$, then $\frac{p-p_o}{(p_o-1)[(p-1)(r-1)-1]} < 1$, and hence by Lemma 1.4.1

$$\left(\int_Q M_\sigma\left(u^{\frac{p-1}{p-p_o}}w\right)^{\frac{p-p_o}{(p_o-1)[(p-1)(r-1)-1]}}\,d\sigma\right)^{\frac{(p-1)(r-1)-1}{p-1}} \leq C_{p,p_o,n}\operatorname*{essinf}_Q\left[M_\sigma\left(u^{\frac{p-1}{p-p_o}}w\right)(x)\right]^{\frac{p-p_o}{(p_o-1)(p-1)}}$$

Combing three latter estimates yields

$$\left(\frac{1}{\sigma(Q)}\int_Q (S_\sigma(u)w)^{-\frac{p_o'}{p_o}}\,d\sigma\right)\left(\frac{1}{\sigma(Q)}\int_Q (S_\sigma(u)w)^{\frac{p_o'}{p_o}\frac{1}{r-1}}\,d\sigma\right)^{r-1} \leq C\left(\frac{1}{\sigma(Q)}\int_Q w^{-\frac{1}{p-1}}\,d\sigma\right)\left(\frac{1}{\sigma(Q)}\int_Q w\,d\sigma\right) \leq C[w]_{A_p(d\sigma)}^{\frac{1}{p-1}}.$$

Therefore by (1.25),

$$[(S_\sigma(u)w)^{-\frac{p_o'}{p_o}}]^\alpha_{A_\infty(d\sigma)} \leq [(S_\sigma(u)w)^{-\frac{p_o'}{p_o}}]^\alpha_{A_r(d\sigma)} \leq C[w]_{A_p(d\sigma)}^{\frac{\alpha}{p-1}}$$

Unifying the latter estimate with (3.25) and (3.26) gives

$$\int_{\mathbb{R}^n} |g|^{p_o}uw\,d\sigma \leq C(p')^{\frac{p-p_o}{p-1}}K_{\sigma,\eta,p'}^{\frac{p-p_o}{p}}[w]_{A_p}^{\frac{p-p_o+\alpha+\beta(p_o-1)}{p-1}}\|f\|^{p_o}_{L^{p_o}(wd\sigma)}$$

Taking the supremum over all $u \geq 0$ with $\|u\|_{L^s(w)} = 1$, we obtain

$$\|g\|^{p_o}_{L^p(wd\sigma)} \leq CK_{\sigma,\eta,p'}^{\frac{p-p_o}{p}}[w]_{A_p(d\sigma)}^{\frac{p-p_o+\alpha+\beta(p_o-1)}{p-1}}\|f\|^{p_o}_{L^{p_o}(wd\sigma)}.$$

□

The extrapolation theorems require some initial estimates. O. Beznosova [B1] was able to obtain the following estimates for the dyadic square function S_σ^d with $A_q^d(d\sigma)$ condition instead of $A_\infty^d(d\sigma)$.

Theorem 3.3.2 (Beznosowa)**.** *Let v and w be weights, such that $v \in A_q^d(d\sigma)$ for some $q > 1$ and the pair $(v, w) \in A_2^d(d\sigma)$, then the dyadic square function S_σ^d is bounded from $L^2(v^{-1})$ to $L^2(wd\sigma)$. Furthermore, for all $f \in L^2(v^{-1}d\sigma)$,*

$$\|S_\sigma^d\|^2_{L^2(wd\sigma)} \leq C\frac{2q-1}{q}[D^d(d\sigma)]^{q+1}\,[v]_{A_q^d(d\sigma)}\,[v,w]_{A_2^d(d\sigma)}\|f\|_{L^2(v^{-1}d\sigma)}. \tag{3.27}$$

When analyzing the proof of Theorem 3.3.1 we realized that we only needed

$(w_o, v_o) \in A_{p0(d\sigma)}$, and $r > \frac{p_o}{p_o-1}$, and it suffices to assume that the initial two-weight estimates are of the form

$$\|g\|_{L^{p_o}(d\sigma)} \leq C\,[v_o]_{A_r(d\sigma)}\,[w_o, v_o]_{A_{p_o}(d\sigma)}\|f\|_{L^{p_o}(v_o\,d\sigma)}.$$

In fact, we have a variant of the Theorem 3.3.1 that can be used in this case.

Theorem 3.3.3. *Let $p > p_0 > 1$. Suppose*

$$\|g\|_{L^{p_o}(w_o\,d\sigma)} \leq C[v_o^{-\frac{p_o'}{p_o}}]^{\alpha}_{A_r(d\sigma)}[w_o, v_o]^{\beta}_{A_{p_o}(d\sigma)}\|f\|_{L^{p_o}(v_o\,d\sigma)}$$

for all weights $(w_o, v_o) \in A_{p_o}(d\sigma)$, some constant C depending possibly on $D(\sigma)$; and some positive exponents α, β.
Then for any $w \in A_p(d\sigma)$,

$$\|g\|_{L^p(w\,d\sigma)} \leq C_\sigma\,[w]_{A_p(d\sigma)}^{\frac{p-p_o+\alpha+\beta(p_o-1)}{p_o(p-1)}}\|f\|_{L^{p_o}(w\,d\sigma)},$$

where $C_\sigma = C\,K_{\sigma,\eta,p'}^{\frac{p-p_o}{p\,p_o}}$, and the constant $K_{\sigma,\eta,p'}$ as in (3.2).

Now, we can apply this version of the theorem to Beznosowa estimates (3.27) when $p_o = 2$, $\alpha = \beta = \frac{1}{2}$ with $w_o = w$ and $v_o = v^{-1}$.

Corollary 3.3.1. *S_σ^d is bounded in $L^p(wd\sigma)$, for $w \in A_p^d(d\sigma)$, $1 < p < \infty$ and*

$$\|S_\sigma^d\|_{L^p(w\,d\sigma)} \leq C_\sigma\,[w]_{A_p(d\sigma)}^{\max\{1,\frac{p}{2}\}\frac{1}{p-1}}\|f\|_{L^{p_o}(w\,d\sigma)}.$$

Note that for $1 < p < 2$, this is just the *sharp* extrapolation theorem $d\sigma$. For $d\sigma = dx$ and $C_\sigma = C$ this exactly the result obtained by Lerner [Le1] for $p > 2$.

Chapter 4

Applications

In this chapter we present some examples and applications which motivated this work. In the paper [Per1] sharp bounds on the L^2 norms for the *Haar multiplier*

$$T_w f(x) = \sum_{I\in D} \frac{w(x)}{m_I}\langle f,\, h_I\rangle h_I(x)$$

were obtained. First, the bound on the norm of the dyadic square function in the weighted space $L^2(w)$ was *lifted* to the case $d\sigma$, where $d\sigma = udx$ for some $u \in A_\infty$, and S_σ is defined using averages with respect to $d\sigma$ instead of dx, similarly $A_2^d(d\sigma)$, and $[v]_{A_2^d(d\sigma)} = \sup_{I\in D}\left(\frac{1}{\sigma(I)}\int_I v d\sigma\right)\left(\frac{1}{\sigma(I)}\int_I v^{-1} d\sigma\right)$. Using

$$\|S_\sigma^d\|_{L^2(vd\sigma)\to L^2(vd\sigma)} \le C[v]_{A_2(d\sigma)},$$

and then taking $d\sigma = wdx$ and $v = w^{-1}$ allowed to push the weight into the operator, which is equivalent [Per4] to

$$\|T_w\|_{L^2} = \|T_w^*\|_{L^2} \sim \|S_w\|_{L^2} = \|S_w\|_{L^2(w^{-1}wdx)} \le D(w)\,[w^{-1}]_{A_2(wdx)} = D(w)\,[w]_{RH_2}^2.$$

T_w are known [Per3] to be bounded if and only if $w \in RH_p^d$.

We are interested in obtaining *sharp* L^p estimates for T_w of the form:

$$\|T_w f\|_{L^p} \le C[w]_{RH_p}^{\alpha_p} \|f\|_{L^p}.$$

Sharp extrapolation $d\sigma$ and Lerner type $d\sigma$ theorems provide the first approximation to α_p. Since S_σ is bounded in $L^2(vd\sigma)$ for any $v \in A_2(d\sigma)$ with a bound that depends linearly on $[v]_{A_2(d\sigma)}$ [Per4], then by extrapolation S_σ is bounded in $L^{p'}(\nu d\sigma)$ for any $\nu \in A_{p'}(d\sigma)$.
Also,

Corollary 4.0.2. *Given $\sigma \in \mathcal{D}$, $\nu \in A_{p'}(d\sigma)$*

$$\|S_\sigma f\|_{L^{p'}(\nu d\sigma)} \le C_\sigma \, [\nu]_{A_{p'}(d\sigma)}^{\alpha_{p'}} \|f\|_{L^{p'}(\nu d\sigma)},$$

where the exponent $\alpha'_p = \max\{\frac{1}{p'-1}, 1\}$ is sharp only for $p' \le 2$ and C_σ

If we choose $\nu := w^{-1}$, then we also will obtain that the adjoint operator T_w^* is bounded in $L^{p'}(dx)$ for any $w^{-1} \in A_{p'}(d\sigma) = A_{p'}(wdx)$ if and only if $w \in RH_p(dx)$. Recalling that $[w]_{RH_p(dx)} = [w^{-1}]_{A_{p'}(wdx)}^{\frac{1}{p'}}$, we obtain the boundedness in L^p for Haar multiplier.

Corollary 4.0.3.

$$\|T_w\|_{L^p} \le C_\sigma [w]_{RH_p(dx)}^{p'\alpha_{p'}},$$

where C_σ depends only on the doubling constant of the measure σ.

As we mentioned at the beginning of Chapter 3, the operator-free form of the extrapolation theorems is more convenient and general. The Theorem 3.2.1 can be thus applied directly to extrapolate some known inverse estimates for the dyadic square function S_σ^d. They were established in [PetPo], and [Per2] for the measure $d\sigma$.

Given $\sigma \in \mathcal{D}$ and $v \in A_2^d(d\sigma)$ the following holds

$$\|f\|_{L^2(vd\sigma)} \le C_\sigma [v]_{A_2(d\sigma)}^{\frac{1}{2}} \|S_\sigma^d f\|_{L^2(vd\sigma)}. \tag{4.1}$$

Therefore by the Theorem 3.2.1 we get

Corollary 4.0.4. *Let $\sigma \in \mathcal{D}$, $w \in A_p(d\sigma)$, then*

$$\|f\|_{L^p(wd\sigma)} \leq K_\sigma [w]_{A_p(d\sigma)}^{\frac{\alpha_p}{2}} \|S_\sigma^d f\|_{L^p(wd\sigma)}, \tag{4.2}$$

where $\alpha_p = \max\{\frac{1}{p-1}, 1\}$ and the constants C_σ, K_σ may depend on the $D(\sigma)$.

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