The Gibbs' Phenomenon

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- $S_N f(x) := \sum_{|n|\leq N} \widehat{f}(n)e^{inx} := \sum_{|n|\leq N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}\, dx \right) e^{inx}$ denotes the partial Fourier sum for f at the point x

- Can use Fourier Sums to approximate $f : \mathbb{T} \to \mathbb{R}$ under the appropriate conditions

- **Def^n**: Pointwise Convergence of Functions:

  A series of functions $\{f_n\}_{n\in\mathbb{N}}$ *converges pointwise to* f *on the set* D *if* $\forall\epsilon > 0$ *and* $\forall x \in D$, $\exists N \in \mathbb{N}$ *such that* $|f_n(x) - f(x)| < \epsilon \forall n \geq N$.

- **Def^n**: Uniform Convergence of Functions:

  A series of functions $\{f_n\}_{n\in\mathbb{N}}$ *converges uniformly to* f *on the set* D *if* $\forall\epsilon > 0, \exists N \in \mathbb{N}$ *such that* $|f_n(x) - f(x)| < \epsilon \forall n \geq N$ *and* $\forall x \in D$.

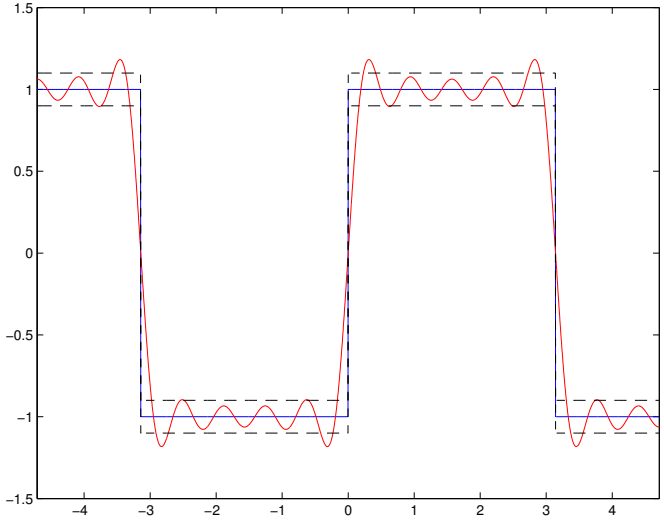

Figure : Pointwise Conv.

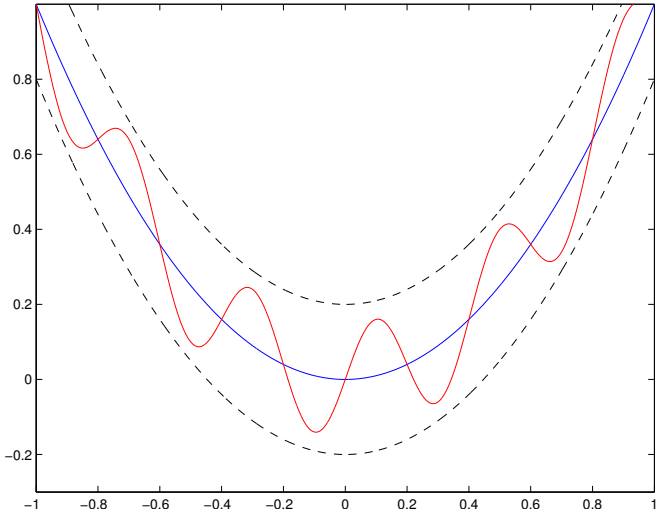

Figure : Uniform Conv.

- Some movies to illustrate the difference. (First Uniform then Pointwise)

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- Persistent "bump" in second illustrates Gibbs' Phenomenon.

Basic Background

- 1848: Property of overshooting discovered by Wilbraham
- 1899: Gibbs brings attention to behavior of Fourier Series (Gibbs observed same behavior as Wilbraham but by studying a different function)
- 1906: Maxime Brocher shows that the phenomenon occurs for general Fourier Series around a jump discontinuity

Key Players and Contributions

- Lord Kelvin: Constructed two machines while studying tide heights as a function of time, $h(t)$.
  - Machine capable of computing periodic function $h(t)$ using Fourier Coefficients
  - Constructed a "Harmonic Analyzer" capable of computing Fourier coefficients of past tide height functions, $h(t)$.
- A. A. Michelson:
  - Elaborated on Kelvin's device and constructed a machine "which would save considerable time and labour involved in calculations. . . of the resultant of a large number of simple harmonic motions."
  - Using first 80 coefficients of *sawtooth* wave, Michelson's machine closely approximated the *sawtooth* function except for two blips near the points of discontinuity

Key Players and Contributions Cont.

- Wilbraham
  - 1848: Wilbraham investigated the equation:

$$y = \cos(x) + \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} + ... + \frac{\cos((2n-1)x)}{2n-1} + ...$$

  - Discovered that "at a distance of $\frac{\pi}{4}$ alternately above and below it, joined by perpendiculars which are themselves part of the locus" the equation overshoots $\frac{\pi}{4}$ by a distance of: $\frac{1}{2}\left|\int_{\pi}^{\infty} \frac{\sin(x)}{x}\, dx\right|$
  - Also suggested that similar analysis of $y = 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - ... + (-1)^{n+1}\frac{\sin(nx)}{n}\right)$ (An equation explored later on) would lead to an analogous result
  - These discoveries gained little attention at the time

Key Players and Contributions Cont.

- J. Willard Gibbs
  - 1898: Published and article in *Nature* investigating the behavior of the function given by:

$$y = 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \ldots + (-1)^{n+1}\frac{\sin(nx)}{n}\right)$$

  - Gibbs observed in this first article that the limiting behavior of the function (*sawtooth*) had "vertical portions, which are bisected the axis of X, [extending] beyond the points where they meet the inclined portions, their total lengths being express by four times the definite integral $\int_0^{\pi} \frac{\sin(u)}{u}\, du$."

Key Players and Contributions Cont.

- Brocher
  - 1906: In an article in *Annals of Mathematics*, Brocher demonstrated that Gibbs' Phenomenon will be observed in any Fourier Series of a function f with a jump discontinuity saying that the limiting curve of the approximating curves has a vertical line that "has to be produced beyond these points by an amount that bears a devinite ratio to the magnitude of the jump."
  - Before Brocher, Gibbs' Phenomenon had only been observed in specific series without any generalization of the concept

3. Gibbs' Phenomenon: An Example

Consider the *square wave* function given by:

$$g(x) := \begin{cases} -1 & \text{if } -\pi \leq x < 0 \\ 1 & \text{if } 0 \leq x < \pi \end{cases}$$

<u>Goal:</u> Analytically prove that Gibbs' Phenomenon occurs at jump discontinuities of this 2π-periodic function

Using the definition $a_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$, basic Calculus yields

$$S_N f(x) = \sum_{n=-N}^{N} a_n e^{in\theta} = \frac{1}{\pi} \sum_{\substack{|n| \le N \\ n \text{ is odd}}} \left(\frac{2}{n}\right) \frac{e^{in\theta}}{i}$$

$$= \frac{4}{\pi} \sum_{m=1}^{M} \frac{\sin((2m-1)\theta)}{2m-1}$$

where M is the largest integer such that $2M - 1 \le N$.

Here we notice that

$$\frac{\sin((2m-1)\theta)}{2m-1} = \int_0^x \cos((2m-1)\theta)\, d\theta$$

which yields the following identity:

$$\begin{aligned} S_{2M-1}g(x) &= \frac{4}{\pi}\sum_{m=1}^{M} \frac{\sin((2m-1)x)}{2m-1} \\ &= \frac{4}{\pi}\sum_{m=1}^{M} \int_0^x \cos((2m-1)\theta)\, d\theta \\ &= \frac{4}{\pi}\int_0^x \sum_{m=1}^{M} \cos((2m-1)\theta)\, d\theta \end{aligned}$$

Now using the identity

$$\sum_{m=1}^{M} \cos((2m-1)x) = \frac{\sin(2Mx)}{2\sin(x)}$$

we get the following:

$$\begin{aligned} S_{2M-1}g(x) &= \frac{4}{\pi}\int_0^x \sum_{m=1}^{M} \cos((2m-1)\theta)\, d\theta \\ &= \frac{4}{\pi}\int_0^x \frac{\sin(2M\theta)}{2\sin(\theta)}\, d\theta \end{aligned}$$

At this point, using basic analysis of the derivative of $S_{2M-1}g(x)$, we see that there are local maximum and minimum values at

$$x_{M,+} = \frac{\pi}{2M} \qquad \textit{and} \qquad x_{M,-} = -\frac{\pi}{2M}$$

Here, using the substitution $u = 2M\theta$ and the fact that for $u \approx 0$, $\sin(u) \approx u$, we get that for large M

$$S_{2M-1}g(x_{m,+}) \approx \frac{2}{\pi}\int_0^{\pi} \frac{\sin(\theta)}{\theta}\, d\theta$$

$$and$$

$$S_{2M-1}g(x_{m,-}) \approx -\frac{2}{\pi}\int_0^{\pi} \frac{\sin(\theta)}{\theta}\, d\theta$$

Further, the value $\frac{2}{\pi}\int\limits_0^{\pi} \frac{\sin(\theta)}{\theta}\, d\theta \approx 1.17898$. Have shown that $\forall M >> 1$, $M \in \mathbb{N}$, $\exists x = x_{M,+}$ such that $S_{2M-1}g(x_{m,+}) \approx 1.17898$.
$\Rightarrow$ is always a "blip" near the jump discontinuity, namely at $x = \frac{\pi}{2M}$, where the value $S_{2M-1}g(x)$ is not near 1!

- Have shown analytically that a "blip" persists for all $S_{2M-1}g$ near the jump discontinuity.

- Demonstrated again by square wave

- Have shown analytically that a "blip" persists for all $S_{2M-1}g$ near the jump discontinuity.
- Demonstrated again by square wave
- We can also see the same behavior in the 2π-periodic *sawtooth* function given by $f(x) = x$ on the interval $[-\pi, \pi)$.

For both of the *square wave* and *sawtooth* functions, we have shown that a "blip" of constant size persists

$\Rightarrow$ while $S_M g \to g$ pointwise, we now know that $S_M g \not\to g$ uniformly since for small enough neighborhoods, the "blip" is always outside the boundaries. (revisit square wave animation)

Kernels And Convolutions

- Kernels:
  - **Def^n:** (Good Kernel) A family $\{K_n\}_{n\in\mathbb{N}}$ real-valued integrable functions on the circle is a *family of good kernels* if the following hold:

(i) $\forall n \in N,\ \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} K_n(\theta)\,d\theta = 1.$

(ii) $\exists M > 0$ such that $\forall n \in \mathbb{N},\ \int\limits_{-\pi}^{\pi} |K_n(\theta)|\,d\theta \leq M.$

(iii) $\forall \delta > 0,\ \int\limits_{\delta\leq|\theta|<\pi} |K_n(\theta)|\,d\theta \to 0$ as $n \to \infty.$

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    $$\Rightarrow \int_{-\pi}^{\pi} D_N(\theta)\,d\theta = \sum_{|n|\leq N}\int_{-\pi}^{\pi} e^{in\theta}\,d\theta$$
  - Fejér: $F_N(\theta) = [D_0(\theta) + D_1(\theta) + ... + D_{N-1}(\theta)]/N$
  - Recall: Dirichlet Kernel is *NOT* a good kernel and Fejér Kernel is.

Kernels And Convolutions Cont.

- Convolution:
  - **Def^n:** The convolution of two functions, f and g is given by:

$$(f * g)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(\theta - y)\, dy \tag{1}$$

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  - From a previous homework, we also know that $F_M * f = \sigma_M f$ $\Rightarrow$ $F_M * f$ is of sort an averaging of the Dirichlet kernels, or more importantly, the partial Fourier sums.

The Problem is Quite General

- Brocher showed Gibbs' Phenomenon occurs with any 2π-periodic piecewise-continuous function with a jump discontinuity
- Overshoot is proportional to height of jump:
  - Square Wave: Overshoot ≈ 0.17898
  - $\Rightarrow\ 0.17898 = k(2)$
  - $\Rightarrow\ k = 0.08949$
  - $\Rightarrow$ Overshoot/Undershoot given by $\approx 0.08949(f(x_d^+) - f(x_d^-)$ where x_d is point of discontinuity
  - Suggests an overshoot of $\approx 0.08949(2\pi) \approx 0.56228$ for a maximum height of ≈ 3.70228
- Thus, need a way to resolve this for modeling of periodic phenomena to be practical
- Using Fejér Kernel averages Dirichlet kernels and could therefore be a way of eliminating this phenomenon
- animation experiment

5. Conclusion

- Gibbs' Phenomenon observed in all 2π-periodic functions with a jump discontinuity
- In some cases, it is quite basic to show it's persistence analytically
- Causes nastiness in approximations of functions by Fourier series, but is easily remedied with *averaging* using Fejér Kernel