Outer Jordan Content

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1 Definition

The **outer Jordan content** $J_*(E)$ of a set E in $\mathbb{R}$ is defined by

$$J_*(E) = \inf \sum_{j=1}^{N} |I_j|$$

where the inf is taked over every *finite* covering $E \subset \bigcup_{j=1}^{N} I_j$, by intervals I_j. Compare this with the **exterior measure** $m_*(E)$ of a set E where the infimum is taken over all *countable* coverings $E \subset \bigcup_{j=1}^{\infty} I_j$.

The **inner Jordan content** $J^*(E)$ of a set E in $\mathbb{R}$ is defined by

$$J^*(E) = \sup \sum_{j=1}^{N} |I_J|$$

where the sup is taken over every finite set $E \supset \bigcup_{j=1}^{N} I_j$. A set E is said to be **Jordan measurable** if

$$J_*(E) = J^*(E)$$

.

2 History

The history of the Jordan content (or Peano-Jordan measure) is a small part of a much larger story about math in the late 19th century. The ideas of inner and outer Jordan content were developed independently by the Italian mathematician Guiseppe Peano and the French mathematician Camille Jordan, as generalizations of the concepts of upper and lower Riemann integrals.

Both were dissatisfied with Riemann integration theory when applied in higher dimensions. At the time, it was believed that for rectangles in a partition meeting the boundary of an arbitrary set, the sum of the areas of such

rectangles could be made arbitrarily small. However, this is false, even in the one-dimensional case, as there exist nowhere dense sets which cannot be enclosed by intervals of arbitrarily small length, such as the Smith-Volterra-Cantor set (sometimes called the fat Cantor set.) Peano, in particular, was able to demonstrate a curve which fills the unit square.

Motivated by Cantor's work in set theory, Jordan and Peano set out to provide a more general adaptation of the Riemann integral, and were the first to apply set theory to analysis and integration. Jordan introduced the concept of a Jordan measurable set in 1892. As Jordan was a highly regarded mathematician, his work and endorsement of the application of Cantorian set theory to analysis generated much interest from other prominent mathematicians of the day. Among those interested were Emile Borel and Henri Lebesgue.

Borel took an alternate route, from that of Peano and Jordan, to the question of measurability and stated properties that a measure on sets should have. His ideas, however, introduced some unnecessary conventions in the theory.

Lebesgue later rectified Borel's theory when he consolidated Borel's ideas with those of Jordan to create the notion of Lebesgue measure and the Lebesgue integral.

3 Exercise 14

3.1 Prove that $J_*(E) = J_*(\bar{E})$ for every set E.

First direction ($\leq$):

Let $E \subset \mathbb{R}$. Since $E \subset \bar{E}$, if $\varphi = \{I_j\}_{j=1}^N$ covers $\bar{E}$, then φ covers E as well. Thus , $J_*(E) \leq J_*(\bar{E})$.

Second direction ($\geq$):

To see $J_*(E) \geq J_*(\bar{E})$, it will suffice to show that if $\varphi = \bigcup_{j=1}^N I_j$ covers E, then $\bar{\varphi} = \bigcup_{j=1}^N \bar{I}_j$ covers $\bar{E}$. This is sufficient because, by the definition of $|I_j|$ for $I_j \subset R$, $|I_j| = |\bar{I}_j|$.

Let φ cover E. Suppose $\bar{\varphi}$ does not cover $\bar{E}$, then $\exists x \in \bar{E}$, such that $x \notin \bar{\varphi}$. By definition, $\bar{E} = E \cup E'$ where E' is the set of all limit points of E. Thus $\forall \varepsilon > 0$, $(x - \varepsilon, x + \varepsilon) \cap E \neq \emptyset$, but $E \subset \bar{\varphi}$ which implies $(x - \varepsilon, x + \varepsilon) \cap \bar{\varphi} \neq \emptyset$. Therefore x is a limit point of $\bar{\varphi}$.

However, since $\bar{\varphi}$ is the finite union of closed intervals, $\bar{\varphi}$ is closed and must contain all of its limit points, so $x \in \bar{\varphi}$, which is a contradiction. Therefore, if $E \subset \varphi$, then $\bar{E} \subset \bar{\varphi}$, which implies $J_*(E) \geq J_*(\bar{E})$.
Thus $J_*(E) = J_*(\bar{E})$. $\square$

3.2 Exhibit a countable subset $E \subset [0,1]$ such that $J_*(E) = 1$ while $m_*(E) = 0$.

Let $E = \mathbb{Q} \bigcap [0,1]$. Since E is dense in $[0,1]$, $J_*(E) = J_*(\overline{E}) = J_*([0,1]) = 1$. However for each rational q_j, $m_*(\{q_j\}) = 0$ means $\sum_{j=1}^{\infty} m_*(\{q_j\}) = 0$.

Moreover, there exists an infinite sequence of sets $E_n = (\mathbb{Q} \bigcap [0,1]) \backslash S_n$ where S_n is countably infinite, such that $J_*(E_n) = 1$ while $m_*(E_n) = 0$. Let $S_n = \{\frac{n}{p} : p = n \cdots \infty\}$. Between any two rationals $\frac{n}{p}$, $\frac{n}{p+1} \in E_n$, there is a subset of rationals, $Q_p = E_n \bigcap (\frac{n}{p}, \frac{n}{p+1})$, that is dense in the interval $[\frac{n}{p}, \frac{n}{p+1}]$ which means E_n is dense in $[0,1]$. Hence $J_*(E_n) = 1$ and $m_*(E_n) = 0$.

4 Properties of outer Jordan content

In this section we compare the properties of the exterior measure $m_*(E)$ with the outer Jordan content $J_*(E)$.

As stated by Stein and Shakarchi [4], the properties of $m_*(E)$ are:

1. Monotonicity
2. Countable Sub-additivity
3. Approximation of $m*(E)$ by $m*(\mathcal{O})$ $E \subset \mathcal{O}$ open
4. Additivity of separated sets
5. Additivity of the countable union of almost disjoint cubes

4.1 Monotonicity

If $E \subseteq F$, then $J_*(E) \leq J_*(F)$.
Proof:
If a finite collection of intervals, $\{I_j\}_{j=1}^{N}$, covers F, then $E \subset F \subset \bigcup_{j=1}^{N} I_j$. This implies that

$$J_*(E) = \inf \sum_{j=1}^{N} |I_j| \leq \inf \sum_{j=1}^{M} |J_j| = J_*(F)$$

the infimums are taken over $E \subset \cup_{j=1}^{N} I_j$ and $F \subset \cup_{j=1}^{M} J_j$, respectively. So the outer Jordan content is monotonic with respect to containment. $\square$

4.2 Not countably subadditive

If $E = \bigcup_{j=1}^{\infty} E_j$ then, in general, $J_*(E) \nleq \sum_{j=1}^{\infty} J_*(E_j)$.
Proof:
Let $E = \mathbb{Q} \bigcap [0,1]$. Since $\mathbb{Q}$ is dense in $[0,1]$, $J_*(E) = J_*(\overline{E}) = J_*([0,1]) = 1$. However for each rational q_j, $J_*(\{q_j\}) = 0$ which means $\sum_{j=1}^{\infty} J_*(\{q_j\}) = 0$,

where q_j is an ordering of $\mathbb{Q}$. $\square$

However, *finite* subadditivity does hold.

4.3 Approximation by open sets

If $E \subset \mathbb{R}$, then $J_*(E) = \inf J_*(\mathcal{O})$, where the infimum is taken over all open sets $\mathcal{O}$ such that $E \subset \mathcal{O}$.
Proof:

Since J_* is monotonic, $J_*(E) \leq \inf(J_*(\mathcal{O}))$. It remains to show that $\inf(J_*(\mathcal{O})) \leq J_*(E)$.

Fix $\epsilon > 0$, and choose cubes Q_j, $E \subset \bigcup_{j=1}^N Q_j$, such that

$$\sum_{j=1}^N |Q_j| \leq J_*(E) + \frac{\epsilon}{2}.$$

Define an open cube, $Q_j^0 \supset Q_j$ such that $|Q_j^0| \leq |Q_j| + \frac{\epsilon}{2^{j+1}}$. Then $\mathcal{O} = \bigcup_{j=1}^N Q_j^0$ is open, and

$$\begin{aligned} J_*(\mathcal{O}) &\leq \sum_{j=1}^N J_*(Q_j^0) && (1)\\ &= \sum_{j=1}^N |Q_j^0| && (2)\\ &\leq \sum_{j=1}^N |Q_j| + \frac{\epsilon}{2^{j+1}} && (3)\\ &\leq \sum_{j=1}^N |Q_j| + \frac{\epsilon}{2} && (4)\\ &\leq J_*(E) + \epsilon. && (5) \end{aligned}$$

Therefore, $\inf J_*(\mathcal{O}) \leq J_*(E)$. $\square$

Note, however, that $J_*(E)$ cannot be approximated by $m_*(\mathcal{O})$. This can be shown by the set $E = \mathbb{Q} \bigcap [0, 1]$.

4.4 Additivity of separated sets

If $E = E_1 \cup E_2$, and $d(E_1, E_2) > 0$, then $J_*(E) = J_*(E_1) + J_*(E_2)$.
Proof:

First we show that $J_*(E) \leq J_*(E_1) + J_*(E_2)$. Assume $\exists \varphi = \{I_j\}_{j=1}^N$ such that $E_1 \subset \cup_{j=1}^N I_j$ and $\exists \Theta = \{J_j\}_{j=1}^M$ such that $E_2 \subset \cup_{j=1}^M J_j$. As $E = E_1 \cup E_2$,

$E \subset \Theta \cup \varphi$. By definition of the outer Jordan content we have

$$J_*(E_1) \geq \sum_{j=1}^{N} |I_j| - \frac{\varepsilon}{2}$$

$$J_*(E_2) \geq \sum_{j=1}^{M} |J_j| - \frac{\varepsilon}{2}.$$

By summing both sides of the inequalities we have

$$J_*(E_1) + J_*(E_2) \geq \sum_{j=1}^{N} |I_j| + \sum_{j=1}^{M} |J_j| - \varepsilon.$$

Hence,

$$J_*(E_1) + J_*(E_2) \geq J_*(E) - \varepsilon.$$

Since ε can be chosen arbitrarily small, we have shown $J_*(E) \leq J_*(E_1) + J_*(E_2)$.

For the other direction of the inequality, select $\delta > 0$ such that $d(E_1, E_2) > \delta$. Take a covering $E \subset \cup_{j=1}^{N} I_j$ with

$$\sum_{j=1}^{N} |I_j| \leq J_*(E) + \varepsilon.$$

Then divide the intervals I_j into subintervals with the length less than δ. Each new interval can intersect at most one of the two sets E_1, E_2. Let the sets J_1 and J_2 contain the indices of the intervals in I_j which intersect E_1 and E_2, respectively. Then $J_1 \cap J_2$ is empty and we have

$$E_1 \subset \cup_{j \in J_1} I_j$$

$$E_2 \subset \cup_{j \in J_2} J_j.$$

Therefore,

$$J_*(E_1) + J_*(E_2) \leq \sum_{j \in J_1} |I_j| + \sum_{j \in J_2} |I_j| \leq \sum_{j=1}^{N} |I_j|$$

which means,

$$J_*(E_1) + J_*(E_2) \leq J_*(E) + \varepsilon.$$

Since ε can be arbitrarily small, the proof is complete. □

4.5 Countable unions of of almost disjoint cubes are not additive

If a set E is the countable union of almost disjoint cubes $E = \bigcup_{j=1}^{\infty} Q_j$, then in general, $J_*(E) \neq \sum_{j=1}^{\infty} |Q_j|$.

Proof:

Consider $E = \mathbb{Q} \bigcap [0, 1]$. □

References

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[3] Grattan-Guinness and H. J. M. Bos, *From the Calculus to Set Theory, 1630-1910: An Introductory History* (Princeton, New Jersey: Princeton University Press, 1980), 164-180.

[4] Elias Stein, Rami Shakarchi *Real Analysis: Measure Theory, Integration, and Hilbert Spaces* (Princeton, New Jersy: Princeton University Press, 2005), 13-15, 41.