Exercise 20
Show that there exist closed sets $A$ and $B$ s.t. $m(A) = m(B) = 0$ but

$$m(A + B) > 0$$

Proof
Let $A$ be the cantor set, we already know that $m(A) = 0$. Let $B = \frac{1}{2}A = \{\frac{x}{2}, x \in A\}$, then $m(B) = \frac{1}{2}m(A) = \frac{1}{2}0 = 0$, which means that $A$ and $B$ have measure zero. It is not hard to see that $A$ and $B$ are closed sets. Let $L$ the union of all open intervals that are excluded from $[0, 1]$ in the process of construction of the Cantor set, then $A = L^c \cap [0, 1]$. Since $L^c$ is closed, then $A$ is closed, which implies that $B$ is closed also.

Now let $A + B = \{z = x + y, x \in A, y \in B\}$, we have to show that $m(A + B) > 0$. Observe that $A + B$ is closed and the it is measurable. To show that $m(A + B) > 0$, we will prove that $[0, 1] \subset A + B$, which implies that $m(A + B) \geq m([0, 1]) \geq 1$.

Given $z \in [0, 1]$, we can write the ternary expansion of $z$.

$$z = 0.a_1a_2a_3a_4...,\quad a_i \in \{0, 1, 2\} \forall i.$$ Let

$$x = 0.b_1b_2b_3b_4...,\quad b_i = 0 \textrm{ if } a_i = 1 \textrm{ and } b_i = a_i \textrm{ if } a_i \neq 1$$ and

$$y = 0.c_1c_2c_3c_4...,\quad c_i = 1 \textrm{ if } a_i = 1 \textrm{ and } c_i = 0 \textrm{ if } a_i \neq 1,$$

where $z = x + y$. But observe that the ternary expansion of $x$ has just digits 0’s and 2’s, which implies that $x \in A$ and the ternary expansion of $y$ has just digits 0’s and 1’s, which implies that $2y \in A$, which imply $y \in B$. Therefore, $z \in A + B \Rightarrow [0, 1] \subset A + B \Rightarrow m(A + B) \geq 1$.

Exercise 32a
Let $\mathcal{N}$ denote the nonmeasurable subset of $I = [0, 1]$ constructed in section 1.3 of Stein and Shakarchi. If $E \subset \mathcal{N}$, $E \in \mathcal{M}$, then $m(E) = 0$.

Proof
Consider the union of the rational translates $E_k = E + r_k$ of $E$, where $\{r_k\}_{k=1}^{\infty} = \mathbb{Q} \cap [-1, 1]$.
Now take the exterior measure on both sides:

\[ m_* \left( \bigcup_{k=1}^{\infty} E_k \right) \leq m_* ([{-1, 2}]) \]

Since \( E \in \mathcal{M} \) and therefore \( E_k \in \mathcal{M} \), since \( E_k \cap E_l = \emptyset \ \forall k \neq l \) and since the measure is translational invariant, we have

\[ m_* \left( \bigcup_{k=1}^{\infty} E_k \right) = m \left( \bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} m(E_k) = \sum_{k=1}^{\infty} m(E) \leq 3 \]

\[ \Rightarrow m(E) = 0 \]

Exercise 32b

For all \( G \) s.t \( G \subset \mathbb{R} \), \( m_* (G) > 0 \) there exist \( E \subset G \), such that \( E \not\in \mathcal{M} \).

Proof

We are going to split up this claim into two. We are going to prove a "light version" of our claim, i.e. assume \( G \) to be bounded. But if we show that this assumption is made without loss of generality, we are done.

This is the w.l.o.g.-claim:
\[ \forall G \subset \mathbb{R}, \ m_* (G) > 0 \ \exists \ \hat{G} \subseteq G, 0 < m_* (\hat{G}) < \infty, \hat{G} \text{ bounded.} \]

We write \( G \) as a countable union of bounded sets:
\[ G = \bigcup_{j=-\infty}^{+\infty} (G \cap [j, j+1)) := \bigcup_{j=-\infty}^{+\infty} G_j, \]

since \( \bigcup_{j=-\infty}^{+\infty} [j, j+1) = \mathbb{R} \). Taking the countable subadditivity of the exterior measure into account, this leads to:

\[ m_* (G) \leq \sum_{j=-\infty}^{+\infty} m_* (G_j) \quad (1) \]

All \( G_j \) are bounded subsets of \( G \). So we only have to prove that one of them has exterior measure \( > 0 \). We are going to do this by assuming the contrary:

Assume: \( m_* (G_j) = 0 \ \forall j. \)

Because of (1) this yields \( m_* (G) = 0 \). So we have reached contradiction and therefore at least one of the \( G_j \) must have \( m_* (G_j) > 0 \).
Now we claim the "light version" of 32 b). This is always without loss of
generality, since we always can find a bounded $\tilde{G} \subset G$ if $G$ itself is not bounded.
If a nonmeasurable set is contained in $\tilde{G}$, then it must be contained in $G$ too.

For every $\tilde{G} \subset \mathbb{R}$, bounded, $m_*(\tilde{G}) > 0 \ni E \subset \tilde{G}$, such that $E \notin \mathcal{M}$.

Consider the equivalence classes defined in section 1.3 of Stein, Shakarchi.
According to the axiom of choice, for a collection of nonempty subsets \{E$_\alpha$\} of $\tilde{G}$ \exists a choice function $f_{ch} : \alpha \to x_\alpha$, where $x_\alpha \in E_\alpha$. Let the subsets be $E_\alpha = \{ x : x \in \tilde{G}, x - \alpha \in \mathbb{Q} \}$ and define $\mathcal{E} = \{ x_\alpha \}$.

Since $\tilde{G}$ is bounded, $\exists I = [a, b]$, such that $\tilde{G} \subset I$. Consider now the rational translates $E_k = \mathcal{E} + r_k$ of $\mathcal{E}$, where $(r_k)_{k=1}^\infty = \mathbb{Q} \cap [-(b-a), b-a]$. So now we have:

$$\tilde{G} \subset \bigcup_{j=1}^\infty E_j \subset [2a-b, 2b-a]$$

Now we take the exterior measure:

$$0 < m_*(\tilde{G}) \leq m_*\left( \bigcup_{j=1}^\infty E_j \right) \leq 3(b-a)$$

Now assume $E \in \mathcal{M} \Rightarrow E_j \in \mathcal{M}$. Since the measure is translation invariant and the $E_j$ are disjoint, we get

$$0 < \sum_{j=1}^\infty m(E_j) = \sum_{j=1}^\infty m(\mathcal{E}) \leq 3(b-a).$$

The inequality sign on the left hand side contradicts the one on the right hand side, since the first implies $m(\mathcal{E}) > 0$ and the latter $m(\mathcal{E}) = 0$. Therefore, $\mathcal{E} \notin \mathcal{M}$.

**Exercise 33**
Show that $\mathcal{N}^c = I - \mathcal{N}$ satisfies $m_*(\mathcal{N}^c) = 1$ and

$$m_*(\mathcal{N}^c) + m_*(\mathcal{N}) \neq m_*(\mathcal{N}^c \bigcup \mathcal{N})$$

**Proof**
Assume $m_*(\mathcal{N}^c) < 1$ then given $\epsilon > 0 \exists$ an open set $U$ s.t. $U \subset I$, $\mathcal{N}^c \subset U$ and $m_*(U) < 1 - \epsilon$. If $\mathcal{N}^c \subset U$ then $U^c \subset \mathcal{N}$ and since $U$ is measurable then $U^c$ is also measurable. So, by the previous exercise, $m(U^c) = 0$.

However, $I = U \bigcup U^c \Rightarrow 1 = m(I) = m(U) + m(U^c) < 1$, absurd. Therefore $m_*(\mathcal{N}^c) = 1$. Since $\mathcal{N}$ is nonmeasurable the $m_*(\mathcal{N}) > 0$. Then

$$1 = m_*(I) = m_*\left( \{N\} \bigcup \{N\}^c \right) \neq m_*(\{N\}^c) + m_*(\mathcal{N}) > 1 + 0 = 1.$$