7) Consider the curve $\Gamma = \{y = f(x)\}$ in $\mathbb{R}^2$, $x \in [0, 1]$ and $f \in C^2([0, 1])$. Then show that

(i) $m(\Gamma + \Gamma) > 0$

$\iff$

(ii) $\Gamma + \Gamma$ contains an open set

$\iff$

(iii) $f$ is not linear

(i) $\implies$ (iii)

This will be done by contrapositive. Assume that $f$ is linear. Then

$$
\Gamma + \Gamma = \{(x + y, ax + b + ay + b) : x, y \in [0, 1]\}
$$

$$
= \{(x + y, a(x + y) + 2b) : x, y \in [0, 1]\}
$$

$$
= \{(z, az + 2b) : z \in [0, 2]\}
$$

$$
= \{y = ax + 2b : x \in [0, 2]\}
$$

and since $\{y = ax + 2b : x \in [0, 2]\}$ is a continuous function in $\mathbb{R}^2$; we have by Exercise 37 (see the solution enclosed) that it has measure zero. So we have that

$f$ is linear $\implies$ $m(\Gamma + \Gamma) = 0$

so by contrapositive have

$m(\Gamma + \Gamma) > 0 \implies f$ is not linear

(ii) $\implies$ (i)

Let $\Gamma + \Gamma$ contain an open set $U$. Since $U$ is open it contains an open cube $Q$ with positive volume. It is assumed that $\Gamma + \Gamma$ is measurable. Then by monotonicity

$$
Q \subset \Gamma + \Gamma \implies m(\Gamma + \Gamma) \geq m(Q) > 0
$$

(iii) $\implies$ (ii)

First if we choose $g$ such that $g(x) = f(x) - f(a)$ for some $a \in [0, 1]$ (a is chosen according to certain conditions described below). Then (from Jaime) have that the shift by $-f(a)$ does not affect the differentiability or non-linearity properties of $g$ inherited from $f$. It is easy to see that:

$$
\Gamma' + \Gamma' = \{(x + y, g(x) + g(y)) : x, y \in [0, 1]\}
$$

$$
= \{(x + y, f(x) + f(y) - 2f(a)) : x, y \in [0, 1]\}
$$

$$
= \{(x + y, f(x) + f(y)) + (0, -2f(a)) : x, y \in [0, 1]\}
$$

$$
= \Gamma + (0, -2f(a))
$$
for any $a \in [0, 1]$ we choose. Since the measure of a set is translation-invariant, $m(\Gamma + \Gamma) = m(\Gamma' + \Gamma')$. We have that each element of the set $\Gamma' + \Gamma'$ is an ordered pair of the form $(x+y, g(x)+g(y))$. Let $z = x + y$ and substitute to obtain $(z, g(x)+g(z-x))$ where $x \in [0, 1], (z-x) \in [0, 1]$. Since $g$ is not linear and $g''$ is continuous there exists some interval, $J \subseteq [0,1]$, so that WLOG we may assume that $g''(x) > 0, \forall x \in J$. So now choose the $a$ and choose $0 < b < a$ such that $[a-b, a+2b] \subseteq I$ then for $x \in [a, a+b]$ and $z \in [2a, 2a+b]$ have $(z-x) \in [a-b, a+2b]$.

Then we can approximate $g$ as follows

$$g(x) = g(a) + g'(a)(x-a) + R$$

where there exists an $\xi_x$ between $a$ and $x$ (or $x$ and $a$ if $x < a$) such that

$$R = \frac{g''(\xi_x)}{2}(x-a)^2$$

by Taylor’s theorem. Because we are in the interval, $J$, that we carefully defined above the following holds

$$\frac{g''(\xi_x)}{2} > 0$$

and

$$\frac{g''(\xi_{z-x})}{2} > 0$$

since $\xi_x, \xi_{z-x} \in [a-b, a+2b]$. The Taylor expansion implies that we have elements of $\Gamma' + \Gamma'$ in the form

$$\left( z, g'(a)(x-a) + \frac{g''(\xi_x)}{2}(x-a)^2 + g'(a)(z-x-a) + \frac{g''(\xi_{z-x})}{2}(z-x-a)^2 \right)$$

$$= \left( z, g'(a)(z-2a) + \frac{g''(\xi_x)}{2}(x-a)^2 + \frac{g''(\xi_{z-x})}{2}(z-x-a)^2 \right)$$

Clearly the $y$-coordinate is a (uniformly) continuous function of both $x$ and $z$; $g'(a)$ is a fixed value, by hypothesis $g''(\xi_x)$ and $g''(\xi_{z-x})$ are both continuous and positive for any $x, (z-x) \in [a-b, a+2b]$ and $(x-a)^2, (z-2a), (z-x-a)^2$ are all (uniformly) continuous since we are in a compact interval.

Define:

$$P_1 := (2a, 0), \quad x = a, \ z = 2a$$

$$P_2 := \left( 2a, \frac{g''(\xi_{a+b})}{2}(b)^2 + \frac{g''(\xi_{a-b})}{2}(b)^2 \right), \quad x = a+b, \ z = 2a.$$  

Notice that for point $P_2$ the $y$ coordinate is strictly positive and since the function in the $y$ coordinate is continuous every value from $P_1$ to $P_2$ is attained. Let

$$C_1 = \frac{g''(\xi_{a+b})}{2}(b)^2 + \frac{g''(\xi_{a-b})}{2}(b)^2$$

Again using uniform continuity of the $y$-coordinate we can then choose $\epsilon > 0$ such that

$$\left| g'(a)(z-2a) + \frac{g''(\xi_{z-a})}{2}(z-2a)^2 \right| < \frac{C_1}{8}, \forall z \in [2a, 2a+\epsilon], \ x = a$$

$$\left| C_1 - g'(a)(z-2a) - \frac{g''(\xi_{a+b})}{2}b^2 - \frac{g''(\xi_{z-a-b})}{2}(z-2a-b)^2 \right| < \frac{C_1}{8}, \forall z \in [2a, 2a+\epsilon], \ x = a+b$$

(0.1)

(0.2)
Then for each $z \in [2a, 2a+\epsilon]$ we can again vary $x$ from $a$ to $a+b$. Because the $y$ coordinate is a continuous function of $x$ we then have for each fixed $z$ every value between the two points

$$P3_z = \left(z, g'(a)(z-2a) + \frac{g''(\xi_{z-a})(z-2a)^2}{2}\right), \quad x = a$$

$$P4_z = \left(z, g'(a)(z-2a) + \frac{g''(\xi_{a+b})(z-2a+b)^2}{2} + \frac{g''(\xi_{z-a-b})(z-2a-b)^2}{2}\right), \quad x = a+b$$

is attained.

Because $\epsilon$ was chosen carefully so that

$$|(z,0) - P3_z| < \frac{C_1}{8}$$
$$|(z,C_1) - P4_z| < \frac{C_1}{8}$$

we have that for each $z$ the $y$ values from $\frac{3}{8}C_1$ to $\frac{5}{8}C_1$ are certainly attained, i.e.

$$P4_z - P3_z \geq \frac{6}{8}C_1 = \frac{3}{4}C_1 > \frac{1}{4}C_1, \quad \forall \ z \in [2a, 2a+\epsilon]$$

So we have that

$$\left[2a, 2a+\epsilon\right] \times \left[\frac{3}{8}C_1, \frac{5}{8}C_1\right]$$

is contained in $\Gamma' + \Gamma'$. 
