34. Let \( C_1 \) and \( C_2 \) be any two Cantor sets (constructed in Exercise 3). Show that there exists a function \( F: [0,1] \to [0,1] \) with the following properties:

i) \( F \) is continuous and bijective.

ii) \( F \) is monotonically increasing.

iii) \( F \) maps \( C_1 \) surjectively onto \( C_2 \).

[ Hint: Copy the construction of the standard Cantor-Lebesgue function.]

**Proof.** Let \( I_1, I_2, \ldots \) be the intervals removed from \([0,1]\) in the construction of \( C_1 \) and let \( J_1, J_2, \ldots \) be the intervals deleted from \([0,1]\) in the construction of \( C_2 \), arranged in the same order. That is, \( I_1 \) and \( J_1 \) are the middle intervals removed in the first step; \( I_2 \) and \( J_2 \) are the “left middles” and \( I_3, J_3 \) are the “right middles” removed in the second step; and so on. Then map the interval \( I_n \) onto the interval \( J_n \) linearly and increasingly, for \( n=1,2,\ldots \) (see graph below).

Note that \([0,1] \setminus C_1\) is dense in \([0,1]\). Indeed, take \( x \in [0,1] \); if \( x \in [0,1] \setminus C_1 \) there is nothing to prove. If \( x \in C_1 \) then by Exercise 4 there is a sequence \( \{x_n\}_{n \geq 1} \) such that \( x_n \in [0,1] \setminus C_1 \) and \( x_n \to x \). Thus any neighborhood of \( x \) contains one element of \([0,1] \setminus C_1 \) (in fact infinitely many) different from \( x \). Hence \( x \) is a limit point of \([0,1] \setminus C_1 \). This proves that \([0,1] \setminus C_1\) is dense in \([0,1]\).

A similar argument shows that \([0,1] \setminus C_2\) is also dense in \([0,1]\).
Hence $F: [0,1] \setminus C_1 \to [0,1] \setminus C_2$ is defined and strictly increasing on a dense subset of $[0,1]$. Since the range of $F$ is also dense on $[0,1]$, the domain of $F$ can be extended to $[0,1]$ so that $F$ is increasing and continuous on $[0,1]$ with range $[0,1]$.

We extend the domain of definition of $F$ to all of $[0,1]$ by putting

$$F(0) = 0, \ F(1) = 1,$$

and

$$F(x_0) = \operatorname{Sup}\{ F(x) : x \in [0,1] \setminus C_1, x < x_0 \}. $$

We see that $F$ is now defined on all of $[0,1]$ since $F$ is bounded on $[0,1] \setminus C_1$.

**Property (ii).** $F$ is monotonically increasing.

Let $x_0, x_1 \in [0,1]$ such that $x_0 < x_1$. We must consider several cases:

a) Both $x_0$ and $x_1$ are in $[0,1] \setminus C_1$. Then, by construction, $F(x_0) < F(x_1)$.

b) $x_0$ is in $[0,1] \setminus C_1$ and $x_1$ is in $C_1$. Then

$$F(x_0) \leq \operatorname{Sup}\{ F(x) : x \in [0,1] \setminus C_1, x < x_1 \} = F(x_1)$$

c) $x_0$ is in $C_1$ and $x_1$ is in $[0,1] \setminus C_1$. Then $x_1$ is in some interval $I_k$ and therefore for all $x$ in $[0,1] \setminus C_1$ such that $x < x_0$ we have that $x < x_0 < x_1$ and $x$ is in some interval $J_l$ at the left of $I_k$ and thus, by construction, $F(x) < F(x_1)$ for all $x$ in $[0,1] \setminus C_2$ with $x < x_0$. But this implies that

$$\operatorname{Sup}\{ F(x) : x \in [0,1] \setminus C_1, x < x_0 \} \leq F(x_1), \ \text{i.e.,} \ F(x_0) \leq F(x_1). $$

Thus, in any case, $F$ is monotonically increasing. In fact $F$ is strictly increasing, since in cases (a) and (b) above, if $F(x_0) = F(x_1)$, then this value is in some interval $J_k$, but $F$ is injective there which implies that $x_0 = x_1$; but this contradicts that $x_0 < x_1$.

**Property (i).** $F$ is continuous and bijective.

Let us show that $F$ is continuous. Since $F$ is monotonically increasing on $[0,1]$, then by a well known theorem of real analysis (see for example Rudin’s Principle of Mathematical Analysis, Theorem 4.29, p.95) we have that for any $x_0 \in (0,1)$,

$$F(x_0^+) := \lim_{x \to x_0^+} F(x) = \inf\{ F(x) : x \in [0,1] \setminus C_1, x > x_0 \}$$

$$F(x_0^-) := \lim_{x \to x_0^-} F(x) = \sup\{ F(x) : x \in [0,1] \setminus C_1, x < x_0 \}$$
We claim that
\[ \sup \{ F(x) : x \in [0,1] \setminus C_1, x < x_0 \} = \inf \{ F(x) : x \in [0,1] \setminus C_1, x > x_0 \}. \]

Notice that if for any \( x < x_0 \) and any \( t > x_0 \), such that \( x, t \in [0,1] \setminus C_1 \), we have
\[ F(x) \leq F(t) \text{ and } F(x) \leq \sup \{ F(x) : x \in [0,1] \setminus C_1, x < x_0 \} \]
which implies that \( \sup \{ F(x) : x \in [0,1] \setminus C_1, x < x_0 \} \leq F(t) \) for any \( t > x_0 \) and therefore
\[ \sup \{ F(x) : x \in [0,1] \setminus C_1, x < x_0 \} \leq \inf \{ F(t) : t \in [0,1] \setminus C_1, t > x_0 \} \]
\[ = \inf \{ F(x) : x \in [0,1] \setminus C_1, x > x_0 \} \]

For the reverse inequality, suppose that \( \sup \{ F(x) : x < x_0 \} < \inf \{ F(x) : x > x_0 \} \). Since \([0,1] \setminus C_2\) is dense in \([0,1]\) then any sub-interval of \([0,1]\) contains points of \([0,1] \setminus C_2\), i.e., values of \( F \) in \([0,1] \setminus C_2\). Then the sub-interval
\[ (\sup \{ F(x) : x < x_0 \}, \inf \{ F(x) : x > x_0 \}) \]
must contain values of \( F \). Let \( y = F(x_1) \) be one of such values.
If \( x_1 > x_0 \), then \( y = F(x_1) \leq \inf \{ F(x) : x > x_0 \} \), a contradiction. If \( x_1 < x_0 \), then \( y = F(x_1) \geq \sup \{ F(x) : x < x_0 \} \), again a contradiction.

Thus we conclude that
\[ \sup \{ F(x) : x \in [0,1] \setminus C_1, x < x_0 \} = \inf \{ F(x) : x \in [0,1] \setminus C_1, x > x_0 \} \]
i.e., \( \lim_{x \to x_0} F(x) = \lim_{x \to x_0} F(x) = F(x_0) \).

This proves that \( F \) is continuous at \( x_0 \).

By a similar argument we can prove that \( F \) is continuous at 0 and at 1. Therefore \( F \) is continuous on \([0,1]\).

Let us now show that \( F \) is bijective. Since \( F \) is increasing and continuous on \([0,1]\) then \( F \) is injective. For the surjectivity, take any \( y \) in \([0,1]\). Since \( f \) is continuous on \([0,1]\), then by the intermediate value theorem there is an \( x \) in \([0,1]\) such that \( F(x) = y \). Hence \( F \) maps \([0,1]\) onto \([0,1]\) and therefore \( F : [0,1] \to [0,1] \) is bijective.
**Property (iii).** $F$ maps $C_1$ surjectively onto $C_2$.

Notice that:
- by construction $F: [0,1] \setminus C_1 \to [0,1] \setminus C_2$ is bijective
- by property (i), $F: [0,1] \to [0,1]$ is bijective.

Therefore $F: C_1 \to C_2$ is bijective, i.e., $F$ maps $C_1$ surjectively onto $C_2$.

**Remark.** Since $F$ is a continuous one-one mapping of the compact metric space $[0,1]$ onto the metric space $[0,1]$, then the inverse mapping $F^{-1}$ defined by $F^{-1}(F(x)) = x$ is a continuous mapping (see for example Rudin’s Principles of Mathematical Analysis, Theorem 4.17, p.90). This implies that any two Cantor sets are homeomorphic.

**35.** Give an example of a measurable function $f$ and a continuous function $\Phi$ so that $f \circ \Phi$ is non-measurable.

[Hint: Let $\Phi: C_1 \to C_2$ as in Exercise 34, with $m(C_1) > 0$ and $m(C_2) = 0$.

Let $N \subset C_1$ be non-measurable, and take $f = \chi_{\Phi(N)}$.

Use the construction in the hint to show that there exists a Lebesgue measurable set that is not a Borel set.

**Proof.** Following the hint, let $\Phi: C_1 \to C_2$ as in Exercise 34, with $m(C_1) > 0$ and $m(C_2) = 0$. Since $m(C_1) > 0$, there exists a non-measurable set $N \subset C_1$. Now take

$$f(x) = \chi_{\Phi(N)}(x) = \begin{cases} 1, & x \in \Phi(N) \\ 0, & x \notin \Phi(N) \end{cases}.$$  

Since $\Phi(N) \subset C_2$ (by 34.(iii)) and $m(C_2) = 0$, then $m(\Phi(N)) = 0$, so $\Phi(N)$ is measurable and therefore $f$ is measurable.

Now consider $f \circ \Phi$. Notice that $(f \circ \Phi)^{-1}((0,\infty)) = N$ is non-measurable and therefore $f \circ \Phi$ is non-measurable.

**** $(f \circ \Phi)^{-1}((0,\infty)) = N$. Indeed:

\[ x \in N \Rightarrow \Phi(x) \in \Phi(N) \Rightarrow f(\Phi(x)) = 1 \in (0,\infty) \Rightarrow x \in (f \circ \Phi)^{-1}((0,\infty)) \]

\[ x \in (f \circ \Phi)^{-1}((0,\infty)) \Rightarrow (f \circ \Phi)(x) \in (0,\infty) \Rightarrow f(\Phi(x)) \in (0,\infty) \]

\[ \Rightarrow f(\Phi(x)) = 1 \Rightarrow \Phi(x) \in \Phi(N) \Rightarrow x \in \Phi^{-1}(\Phi(N)) = N \]

**"**
For the second statement, let $\Phi : C_1 \to C_2$ as in Exercise 34, with $m(C_1) = 0$ and $m(C_2) > 0$.

Since $m(C_2) > 0$, there exists a non-measurable set $V \subset C_2$. Then $A = \Phi^{-1}(V) \subset C_1$ and since $m(C_1) = 0$ then $m(A) = 0$ and therefore $A$ is measurable.

Now suppose that $A$ is a Borel set. Then $\Phi(A) = \Phi(\Phi^{-1}(V)) = V$ is also a Borel set since $\Phi$ is a continuous one-one function and continuous one-one functions map Borel sets onto Borel sets (see proof of this below). But if $V$ is a Borel set, then $V$ is measurable, a contradiction.

**Theorem.** If $f$ is a one-one continuous mapping of $R$ onto $R$, then $f$ maps Borel sets onto Borel sets.

**Proof.** Recall that if $f$ is a one-one map of $X$ onto $Y$, then for $A, B \subset X$,

$$f(A \cap B) = f(A) \cap f(B) \text{ and } f(A \setminus B) = f(A) \setminus f(B).$$

Now set

$$\mathcal{U} = \{ A \subset R : f(A) \in \mathcal{B} \},$$

where $\mathcal{B}$ denotes the collection of Borel sets. We claim that $\mathcal{U}$ is a $\sigma$-algebra. Indeed, if $A \in \mathcal{U}$, then $A^c \in \mathcal{U}$, because $f(A^c) = f(R \setminus A) = R \setminus f(A)$. Also, if $\{A_n\}$ is a sequence of sets in $\mathcal{U}$, then $f(\bigcup_{n=1}^\infty A_n) = \bigcup_{n=1}^\infty f(A_n)$, which shows that $\bigcup_{n=1}^\infty A_n \in \mathcal{U}$. Now since $f$ is strictly monotonic (because it is one-one and continuous), we have that

$$f([a,b]) = [f(a), f(b)] \text{ or } f([a,b]) = [f(b), f(a)].$$

Hence $\mathcal{U}$ contains all closed intervals and therefore all Borel sets.
Exercise 36

This exercise provides an example of a measurable function $f$ on $[0, 1]$ such that every function $g$ equivalent to $f$ (in the sense that $f$ and $g$ differ only on a set of measure zero) is discontinuous at every point.

(a) Construct a measurable set $E \subset [0, 1]$ such that for any non-empty open sub-interval $I$ in $[0, 1]$, both sets $E \cap I$ and $E^c \cap I$ have positive measure.

(b) Show that $f = \chi_E$ has the property that whenever $g(x) = f(x)$ a.e $x$, then $g$ must be discontinuous at every point in $[0, 1]$.

Solution to part (a)

Let $\{u_k\}_{k=1}^\infty$ be a sequence of all intervals with rational endpoints. We will denote $u_1 = (\alpha_1, \beta_1)$, where $\alpha_1, \beta_1$ are rational numbers in $[0, 1]$. Because $u_1$ is an open interval, we can construct a Cantor-like set $A_1$ of positive measure entirely inside $u_1$. Since Cantor (and Cantor-like) sets are nowhere dense, we can always find an open interval $(a, b)$ inside $u_1$, such that $(a, b)$ is disjoint from $A_1$. Therefore we can construct another Cantor-like set $A_2$, such that $A_2 \subset (a, b) \subset u_1$.

Now let’s take an open interval $u_2 = (\alpha_2, \beta_2)$. We can construct Cantor-like set $A_3$ entirely inside $u_2$ such that $A_3$ is disjoint from all previously constructed sets and, by the same argument we construct $A_4$ inside $u_2$ such that it is disjoint from all previous Cantor-like sets. Notice that sets $A_k$ and $A_j$ satisfy the following conditions:

(i) $A_k$ and $A_j$ are disjoint and nowhere dense,

(ii) $A_{2k-1} \subset u_k$ and $A_{2k} \subset u_k$.

Repeating this process indefinitely we get a sequence $A_k$ that has the properties:

(i) $A_{2k-1}$ and $A_{2k}$ are disjoint and nowhere dense $\forall k \in \mathbb{N},$
(ii) $A_{2k-1} \subset u_k$ and $A_{2k} \subset u_k$.

Let’s take $E = \bigcup_{k=1}^\infty A_{2k}$ then $\forall I \exists k : u_k \subseteq I$.

\[
E \cap I = (\bigcup_{k=1}^\infty A_{2k}) \cap I \supset A_{2k} \cap I = A_{2k}
\]  \hspace{1cm} (1)

\[
m(E \cap I) \geq m(A_{2k}) > 0
\]  \hspace{1cm} (2)

To show the second inequality let’s notice that $\bigcup_{k=1}^\infty A_{2k-1}$ is not a subset of $E$. 

1
This means that $\bigcup_{k=1}^{\infty} A_{2k-1} \subset E^c$.

$$E^c \cap I \supset (\bigcup_{k=1}^{\infty} A_{2k-1}) \cap I = A_{2k-1}$$

$$m(E^c \cap I) \geq m(A_{2k-1}) > 0$$

This concludes the proof of part (a).

**Solution to part (b)**

To show that the claim is valid, we will need to use the following Theorem (without proof):

**The Baire Category Theorem:** Let $X$ be a complete metric space,
(a) If $\{U_n\}_{n=1}^{\infty}$ is a sequence of open dense subsets of $X$, then $\bigcap_{n=1}^{\infty} U_n$ is dense in $X$,
(b) $X$ is not a countable union of nowhere dense sets.

Let $f(x) = \chi_E$ and let $g(x) = f(x)$ almost everywhere. By definition of continuity $g(x)$ is continuous if and only if:

$$\lim_{\tilde{x} \to x} g(\tilde{x}) = g(x)$$

Without loss of generality, let’s assume that $x \in I$, where $I$ is an open interval in $[0,1]$. Let $F$ be the set where $f(x)$ is different from $g(x)$, so that $m(F) = 0$. We see that $x$ has no other option, but to satisfy one of the following:

1. $x \in (E \cap I \cap F^c)$
2. $x \in (E^c \cap I \cap F^c)$
3. $x \in F$.

As we already proved $m(E \cap I) > 0$ and $m(E^c \cap I) > 0$.

Using the result of part (a) we know that $m(E \cap I) > 0$ and $m(E^c \cap I) > 0$ therefore $g(x) = f(x)$ on $E \cap I \cap F^c$ and $E^c \cap I \cap F^c$. 

2
First, let’s show that if case (1) is true, then \( g \) is discontinuous:

Suppose \( x \in E \cap F^c \).

Because \( E^c = (\bigcup A_{2k})^c = \bigcap A_{2k}^c \), where \( A_{2k}^c \) are dense in \([0,1]\), according to Baire Cathegory Theorem, \( E^c \) is dense in \([0,1]\) and \( E^c \cap F^c \) is also dense in \([0,1]\).

Because \( E^c \cap F^c \) is dense we can find a sequence \( \{x_n\}_{n=1}^\infty \in E^c \cap F^c \) converging to \( x \), then \( g(x_n) = f(x_n) = \chi_{E}(x_n) \equiv 0 \), but \( g(x) = f(x) = \chi_{E}(x) = 1 \). This implies discontinuity of \( g \).

Now, let’s show that if case (2) is true, then \( g \) is discontinuous, too:

Suppose \( x \in (E^c \cap I) \cap F^c \). Then \( g(x) = f(x) = \chi_{E}(x) \equiv 0 \).

Now let’s find a sequence \( \{x_n\}_{n=1}^\infty \) convergent to \( x \), such that \( g(x_n) \) does not converge to \( g(x) \equiv 0 \).

Among the sequence of intervals with rational endpoints \( \{u_k\} \), let’s choose a subsequence of shrinking, nested intervals \( \{u_{k_j}\} \), so that:

\[
\alpha_{k_j} < \alpha_{k_{j+1}} \leq x \leq \beta_{k_{j+1}} < \beta_{k_j},
\]

(6)

this can be done, because as we know that, any number can be approximated arbitrarily close by a rational number.

Constructing Cantor-like sets \( A_{2k_j} \) in each of the open intervals \( \{u_{k_j}\} \) as suggested in part (a), and by choosing a sequence \( \{x_{k_j}\}_{j}^\infty \in A_{2k_j} \subset u_{k_j} \), we obtain a sequence, converging to \( x \), that is entirely inside \( E \cap F^c \).

Note \( g(x_{k_j}) = \chi_{E}(x) \equiv 1 \ \forall j \) and so \( g(x_{k_j}) \) does not converge to \( g(x) \equiv 0 \), this implies that \( g \) is discontinuous.

Therefore \( g(x) = f(x) \) a.e. \( x \) is discontinuous almost everywhere.

Using case (1) and case (2) we notice for case (3) the following fact:

Suppose \( x \in F \), then in any neighborhood of \( x \) we have points from both \( E \) and \( E^c \) and thus, the sequence constructed by taking points from each of the sets \( E \) and \( E^c \) such that \( x_{2n-1} \in E \) and \( x_{2n} \in E^c \) we construct a sequence convergent to \( x \), that has no limit. This contradicts the definition of continuity.

The conclusion for these three cases gives us the result that \( g \) has to be discontinuous everywhere.