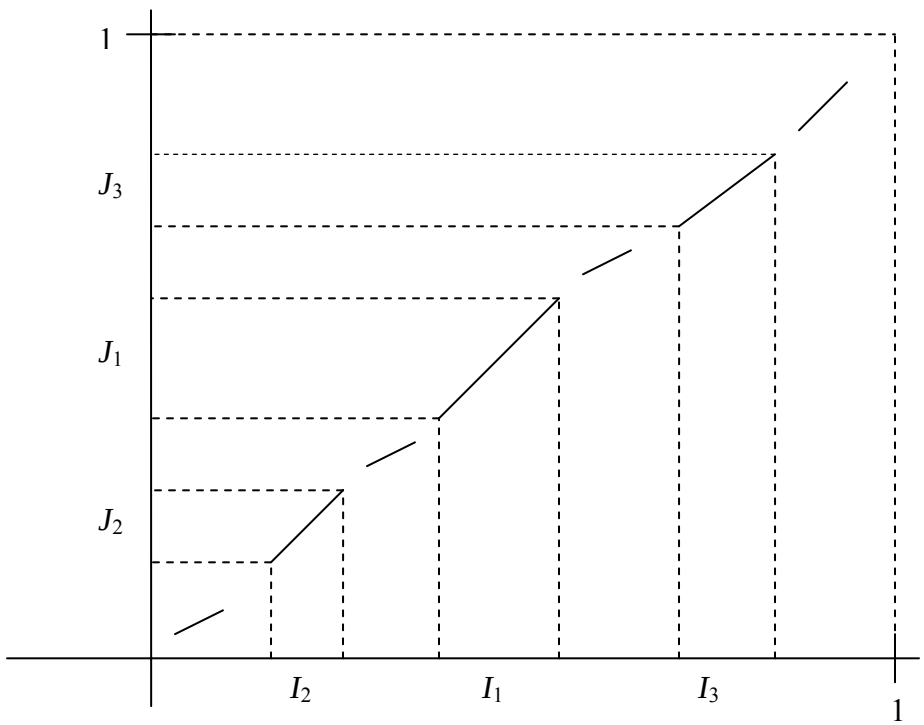
Math 563 Measure Theory Project 1 (Funky Functions Group)
Luis Zerón, Sergey Dyachenko

34. Let C_1 and C_2 be any two Cantor sets (constructed in Exercise 3). Show that there exists a function F: [0,1] $\to$ [0,1] with the following properties:

i) F is continuous and bijective.
ii) F is monotically increasing.
iii) F maps C_1 surjectively onto C_2.

[Hint: Copy the construction of the standard Cantor-Lebesgue function.]

Proof. Let I_1, I_2, … be the intervals removed from [0,1] in the construction of C_1 and let J_1, J_2, … be the intervals deleted from [0,1] in the construction of C_2, arranged in the same order. That is, I_1 and J_1 are the middle intervals removed in the first step; I_2 and J_2 are the "left middles" and I_3, J_3 are the "right middles" removed in the second step; and so on. Then map the interval I_n onto the interval J_n linearly and increasingly, for n=1,2,… (see graph below).



Note that $[0,1]\setminus C_1$ is dense in [0,1]. Indeed, take $x \in [0,1]$; if $x \in [0,1]\setminus C_1$ there is nothing to prove. If $x \in C_1$ then by Exercise 4 there is a sequence $\{x_n\}_{n\geq 1}$ such that $x_n \in [0,1]\setminus C_1$ and $x_n \to x$. Thus any neighborhood of x contains one element of $[0,1]\setminus C_1$ (in fact infinitely many) different from x. Hence x is a limit point of $[0,1]\setminus C_1$. This proves that $[0,1]\setminus C_1$ is dense in [0,1].
A similar argument shows that $[0,1]\setminus C_2$ is also dense in [0,1].

Hence F: $[0,1] \setminus C_1 \to [0,1] \setminus C_2$ is defined and strictly increasing on a dense subset of [0,1]. Since the range of F is also dense on [0,1], the domain of F can be extended to [0,1] so that F is increasing and continuous on [0,1] with range [0,1].

We extend the domain of definition of F to all of [0,1] by putting

$$F(0) = 0,\ F(1) = 1,$$

and

$$F(x_0) = \mathrm{Sup}\{\, F(x) \colon x \in [0,1] \setminus C_1,\ x < x_0 \}.$$

We see that F is now defined on all of [0,1] since F is bounded on $[0,1] \setminus C_1$.

Property (ii). F is monotonically increasing.

Let $x_0, x_1 \in [0,1]$ such that $x_0 < x_1$. We must consider several cases:

a) Both x_0 and x_1 are in $[0,1] \setminus C_1$. Then, by construction, $F(x_0) < F(x_1)$.

b) x_0 is in $[0,1] \setminus C_1$ and x_1 is in C_1. Then

$$F(x_0) \leq \mathrm{Sup}\{\, F(x) \colon x \in [0,1] \setminus C_1,\ x < x_1 \} = F(x_1)$$

c) x_0 is in C_1 and x_1 is in $[0,1] \setminus C_1$. Then x_1 is in some interval I_k and therefore for all x in $[0,1] \setminus C_1$ such that $x < x_0$ we have that $x < x_0 < x_1$ and x is in some interval I_j at the left of I_k and thus, by construction, $F(x) < F(x_1)$ for all x in $[0,1] \setminus C_2$ with $x < x_0$. But this implies that

$$\mathrm{Sup}\{\, F(x) \colon x \in [0,1] \setminus C_1,\ x < x_0 \} \leq F(x_1), \text{ i.e, } F(x_0) \leq F(x_1).$$

Thus, in any case, F is monotonically increasing. In fact F is strictly increasing, since in cases (a) and (b) above, if $F(x_0) = F(x_1)$, then this value is in some interval J_k, but F is injective there which implies that $x_0 = x_1$; but this contradicts that $x_0 < x_1$.

Property (i). F is continuous and bijective.

Let us show that F is continuous. Since F is monotonically increasing on [0,1], then by a well known theorem of real analysis (see for example Rudin's Principle of Mathematical Analysis, Theorem 4.29, p.95) we have that for any $x_0 \in (0,1)$,

$$F(x_0^+) := \lim_{x \to x_0^+} F(x) = \inf\{F(x) : x \in [0,1] \setminus C_1, x > x_0\}$$

$$F(x_0^-) := \lim_{x \to x_0^-} F(x) = \sup\{F(x) : x \in [0,1] \setminus C_1, x < x_0\}$$

We claim that

$$\sup\{F(x): x \in [0,1] \setminus C_1, x < x_0\} = \inf\{F(x): x \in [0,1] \setminus C_1, x > x_0\}.$$

Notice that if for any $x < x_0$ and any $t > x_0$, such that $x, t \in [0,1] \setminus C_1$, we have

$$F(x) \le F(t) \text{ and } F(x) \le \sup\{F(x): x \in [0,1] \setminus C_1, x < x_0\}$$

which implies that $\sup\{F(x): x \in [0,1] \setminus C_1, x < x_0\} \le F(t)$ for any $t > x_0$ and therefore

$$\begin{aligned}\sup\{F(x): x \in [0,1] \setminus C_1, x < x_0\} &\le \inf\{F(t): t \in [0,1] \setminus C_1, t > x_0\} \\ &= \inf\{F(x): x \in [0,1] \setminus C_1, x > x_0\}\end{aligned}$$

For the reverse inequality, suppose that $\sup\{F(x): x < x_0\} < \inf\{F(x): x > x_0\}$. Since $[0,1] \setminus C_2$ is dense in [0,1] then any sub-interval of [0,1] contains points of $[0,1] \setminus C_2$, i.e., values of F in $[0,1] \setminus C_2$. Then the sub-interval

$$(\sup\{F(x): x < x_0\}, \inf\{F(x): x > x_0\})$$

must contain values of F. Let $y = F(x_1)$ be one of such values.
If $x_1 > x_0$, then $y = F(x_1) \le \inf\{F(x): x > x_0\}$, a contradiction. If $x_1 < x_0$, then $y = F(x_1) \ge \sup\{F(x): x < x_0\}$, again a contradiction.

Thus we conclude that

$$\sup\{F(x): x \in [0,1] \setminus C_1, x < x_0\} = \inf\{F(x): x \in [0,1] \setminus C_1, x > x_0\}$$

i.e., $\lim_{x \to x_0^-} F(x) = \lim_{x \to x_0^+} F(x) = F(x_0)$.

This proves that F is continuous at x_0.

By a similar argument we can prove that F is continuous at 0 and at 1. Therefore F is continuous on [0,1].

Let us now show that F is bijective. Since F is increasing and continuous on [0,1] then F is injective. For the surjectivity, take any y in [0,1]. Since f is continuous on [0,1], then by the intermediate value theorem there is an x in [0,1] such that $F(x) = y$. Hence F maps [0,1] onto [0,1] and therefore F: $[0,1] \to [0,1]$ is bijective.

Property (iii). F maps C_1 surjectively onto C_2.

Notice that:

- by construction F: $[0,1] \setminus C_1 \to [0,1] \setminus C_2$ is bijective
- by property (i), F: $[0,1] \to [0,1]$ is bijective.

Therefore $F: C_1 \to C_2$ is bijective, i.e., F maps C_1 surjectively onto C_2.

Remark. Since F is a continuous one-one mapping of the compact metric space [0,1] onto the metric space [0,1], then the inverse mapping F^{-1} defined by $F^{-1}(F(x)) = x$ is a continuous mapping (see for example Rudin's Principles of Mathematical Analysis, Theorem 4.17, p.90). This implies that any two Cantor sets are homeomorphic.

35. Give an example of a measurable function f and a continuous function Φ so that $f \circ \Phi$ is non-measurable.

[Hint: Let $\Phi : C_1 \to C_2$ as in Exercise 34, with $m(C_1) > 0$ and $m(C_2) = 0$.
Let $N \subset C_1$ be non-measurable, and take $f = \chi_{\Phi(N)}$.]

Use the construction in the hint to show that there exists a Lebesgue measurable set that is not a Borel set.

Proof. Following the hint, let $\Phi : C_1 \to C_2$ as in Exercise 34, with $m(C_1) > 0$ and $m(C_2) = 0$. Since $m(C_1) > 0$, there exists a non-measurable set $N \subset C_1$. Now take

$$f(x) = \chi_{\Phi(N)}(x) = \begin{cases} 1, & x \in \Phi(N) \\ 0, & x \notin \Phi(N) \end{cases}.$$

Since $\Phi(N) \subset C_2$ (by 34.(iii)) and $m(C_2) = 0$, then $m(\Phi(N)) = 0$, so $\Phi(N)$ is measurable and therefore f is measurable.

Now consider $f \circ \Phi$. Notice that $(f \circ \Phi)^{-1}((0,+\infty)) = N$ is non-measurable and therefore $f \circ \Phi$ is non.measurable.

** $(f \circ \Phi)^{-1}((0,+\infty)) = N$. Indeed:

$$x \in N \Rightarrow \Phi(x) \in \Phi(N) \Rightarrow f(\Phi(x)) = 1 \in (0,+\infty) \Rightarrow x \in (f \circ \Phi)^{-1}((0,+\infty))$$

$$x \in (f \circ \Phi)^{-1}((0,+\infty)) \Rightarrow (f \circ \Phi)(x) \in (0,+\infty) \Rightarrow f(\Phi(x)) \in (0,+\infty)$$

$$\Rightarrow f(\Phi(x)) = 1 \Rightarrow \Phi(x) \in \Phi(N) \Rightarrow x \in \Phi^{-1}(\Phi(N)) = N$$

**

For the second statement, let $\Phi : C_1 \to C_2$ as in Exercise 34, with $m(C_1) = 0$ and $m(C_2) > 0$.

Since $m(C_2) > 0$, there exists a non-measurable set $V \subset C_2$. Then $A = \Phi^{-1}(V) \subset C_1$ and since $m(C_1) = 0$ then $m(A) = 0$ and therefore A is measurable.

Now suppose that A is a Borel set. Then $\Phi(A) = \Phi(\Phi^{-1}(V)) = V$ is also a Borel set since Φ is a continuous one-one function and continuous one-one functions map Borel sets onto Borel sets (see proof of this below). But if V is a Borel set, then V is measurable, a contradiction.

**

Theorem. If *f* is a one-one continuous mapping of R onto R, then *f* maps Borel sets onto Borel sets.

Proof. Recall that if *f* is a one-one map of X onto Y, then for $A, B \subset X$,

$f(A \cap B) = f(A) \cap f(B)$ and $f(A \setminus B) = f(A) \setminus f(B)$.

Now set

$$\mathcal{U} = \{ A \subset R : f(A) \in \mathcal{B} \},$$

where $\mathcal{B}$ denotes the collection of Borel sets. We claim that $\mathcal{U}$ is a σ-algebra. Indeed, if $A \in \mathcal{U}$, then $A^c \in \mathcal{U}$, because $f(A^c) = f(R \setminus A) = R \setminus f(A)$. Also, if $\{A_n\}$ is a sequence of sets in $\mathcal{U}$, then $f(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f(A_n)$, which shows that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{U}$. Now since f is strictly monotonic (because it is one-one and continuous), we have that

$$f([a,b]) = [f(a),f(b)] \text{ or } f([a,b]) = [f(b),f(a)].$$

Hence $\mathcal{U}$ contains all closed intervals and therefore all Borel sets.

Exercise 36

This exercise provides an example of a measurable function f on $[0,1]$ such that every function g equivalent to f (in the sense that f and g differ only on a set of measure zero) is discontinuous at every point.

(a) Construct a measurable set $E \subset [0,1]$ such that for any non-empty open sub-interval I in $[0,1]$, both sets $E \bigcap I$ and $E^c \bigcap I$ have positive measure.

(b) Show that $f = \chi_E$ has the property that whenever $g(x) = f(x)$ a.e x, then g must be discontinuous at every point in $[0,1]$.

Solution to part (a)

Let $\{u_k\}_{k=1}^{\infty}$ be a sequence of all intervals with rational endpoints
We will denote $u_1 = (\alpha_1, \beta_1)$, where α_1, β_1 are rational numbers in $[0,1]$. Because u_1 is an open interval, we can construct a Cantor-like set A_1 of positive measure entirely inside u_1. Since Cantor (and Cantor-like) sets are nowhere dense, we can always find an open interval (a,b) inside u_1, such that (a,b) is disjoint from A_1. Therefore we can construct another Cantor-like set A_2, such that $A_2 \subset (a,b) \subset u_1$.
Now let's take an open interval $u_2 = (\alpha_2, \beta_2)$. We can construct Cantor-like set A_3 entirely inside u_2 such that A_3 is disjoint from all previosly constructed sets and, by the same argument we construct A_4 inside u_2 such that it is disjoint from all previous Cantor-like sets. Notice that sets A_k and A_j satisfy the following conditions:

$(i) A_k$ and A_j are disjoint and nowhere dense,

$(ii) A_{2k-1} \subset u_k$ and $A_{2k} \subset u_k$.

Repeating this process indefinitely we get a sequence A_k that has the properties:

(i) A_{2k-1} and A_{2k} are disjoint and nowhere dense $\forall k \in \mathbb{N}$,

(ii) $A_{2k-1} \subset u_k$ and $A_{2k} \subset u_k$.

Let's take $E = \bigcup\limits_{k=1}^{\infty} A_{2k}$ then $\forall I \ \exists k : u_k \subseteq I$.

$$E \bigcap I = (\bigcup_{k=1}^{\infty} A_{2k}) \bigcap I \supset A_{2k} \bigcap I = A_{2k} \tag{1}$$

$$m(E \bigcap I) \geq m(A_{2k}) > 0 \tag{2}$$

To show the second inequality let's notice that $\bigcup\limits_{k=1}^{\infty} A_{2k-1}$ is not a subset of E.

This means that $\bigcup_{k=1}^{\infty} A_{2k-1} \subset E^c$.

$$E^c \bigcap I \supset (\bigcup_{k=1}^{\infty} A_{2k-1}) \bigcap I \supset A_{2k-1} \bigcap I = A_{2k-1} \tag{3}$$

$$m(E^c \bigcap I) \geq m(A_{2k-1}) > 0 \tag{4}$$

This concludes the proof of part (a).

Solution to part (b)

To show that the claim is valid, we will need to use the following Theorem(without proof):

The Baire Cathegory Theorem: Let X be a complete metric space,

(a) If $\{U_n\}_1^{\infty}$ is a sequence of open dense subsets of X, then $\bigcap_1^{\infty} U_n$ is dense in X,

(b) X is not a countable union of nowhere dense sets.

Let $f(x) = \chi_E$ and let $g(x) = f(x)$ almost everywhere. By definition of continuity $g(x)$ is continuous if and only if:

$$\lim_{\tilde{x} \to x} g(\tilde{x}) = g(x) \tag{5}$$

Without loss of generality, let's assume that $x \in I$, where I is an open interval in $[0, 1]$. Let F be the set where $f(x)$ is different from $g(x)$, so that $m(F) = 0$. We see that x has no other option, but to satisfy one of the following:

(1) $x \in (E \bigcap I \bigcap F^c)$
(2) $x \in (E^c \bigcap I \bigcap F^c)$
(3) $x \in F$.
As we already proved $m(E \bigcap I) > 0$ and $m(E^c \bigcap I) > 0$.

Using the result of part (a) we know that $m(E \bigcap I) > 0$ and $m(E^c \bigcap I) > 0$ therefore $g(x) = f(x)$ on $E \bigcap I \bigcap F^c$ and $E^c \bigcap I \bigcap F^c$.

First, let's show that if case (1) is true, then g is discontinuos:
Suppose $x \in E \bigcap F^c$.

Because $E^c = (\bigcup A_{2k})^c = \bigcap A_{2k}^c$, where A_{2k}^c are dense in $[0,1]$, according to Baire Cathegory Theorem, E^c is dense in $[0,1]$ and $E^c \bigcap F^c$ is also dense in $[0,1]$

Because $E^c \bigcap F^c$ is dense we can find a sequence $\{x_n\}_1^\infty \in E^c \bigcap F^c$ converging to x, then $g(x_n) = f(x_n) = \chi_E(x_n) \equiv 0$, but $g(x) = f(x) = \chi_E(x) = 1$. This implies discontinuity of g.

Now, let's show that if case (2) is true, then g is discontinuos, too:
Suppose $x \in (E^c \bigcap I \bigcap F^c)$. Then $g(x) = f(x) = \chi_E(x) \equiv 0$.

Now let's find a sequence $\{x_n\}_{n=1}^\infty$ convergent to x, such that $g(x_n)$ does not converge to $g(x) \equiv 0$.

Among the sequence of intervals with rational endpoints $\{u_k\}$, let's choose a subsequence of shrinking, nested intervals $\{u_{k_j}\}$, so that:

$$\alpha_{k_j} < \alpha_{k_{j+1}} \leq x \leq \beta_{k_{j+1}} < \beta_{k_j}, \tag{6}$$

this can be done, because as we know that, any number can be approximated arbitrarily close by a rational number.
Constructing Cantor-like sets A_{2k_j} in each of the open intervals $\{u_{k_j}\}$ as suggested in part (a), and by choosing a sequence $\{x_{k_j}\}_j^\infty \in A_{2k_j} \subset u_{k_j}$, we obtain a sequence, converging to x, that is entirely inside $E \bigcap F^c$.

Note $g(x_{k_j}) = \chi_E(x) \equiv 1 \ \forall j$ and so $g(x_{k_j})$ does not converge to $g(x) \equiv 0$, this implies that g is discontinuous.

Therefore $g(x) = f(x)$ a.e. x is discontinuous almost everywhere.

Using case (1) and case (2) we notice for case (3) the following fact:
Suppose $x \in F$, then in any neiborhood of x we have points from both E and E^c and thus, the sequence constructed by taking points from each of the sets E and E^c such that $x_{2n-1} \in E$ and $x_{2n} \in E^c$ we construct a sequence convergent to x, that has no limit. This contradicts the definition of continuity.

The conclusion for these three cases gives us the result that g has to be discontinuous everywhere.