1 Borel-Cantelli Lemma

Exercise 16 is the introduction of the Borel-Cantelli Lemma using Lebesque measure. An approach using probability measure will be introduced later in the course.

Definition 1.1. Let \( \{E_1, E_2, E_3, \ldots\} \) be a sequence of sets. From this, we can define a decreasing sequence of sets \( B_1 \supseteq B_2 \supseteq B_3 \ldots \) as follows:

\[
B_n = \bigcup_{k=n}^{\infty} E_k
\]

Let \( E = \{x: x \in E_k \text{ for infinitely many } k\} \) be defined to mean

\[
E = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k := \limsup_{n \to \infty} (E_n)
\]

Lemma 1.1. Suppose \( \{E_k\}_{k=1}^{\infty} \) is a countable family of measurable subsets of \( \mathbb{R}^d \) and that

\[
\sum_{k=1}^{\infty} m(E_k) < \infty
\]

and

\[
E = \limsup_{k \to \infty} (E_k).
\]

Then \( E \) is measurable and \( m(E) = 0 \).

Proof. To show that \( E \) is measurable, note that

\[
B_n = \bigcup_{k=n}^{\infty} E_k.
\]
Each set $B_i$ is the countable union of measurable sets and is therefore measurable. Furthermore,

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \bigcap_{n=1}^{\infty} B_n.$$ 

The set $E$ is the countable intersection of measurable sets and is therefore measurable.

It remains to be shown that $m(E) = 0$. Note that

$$\sum_{k=1}^{\infty} m(E_k) = M < \infty.$$ 

Define the two sequences

$$s_n = \sum_{k=1}^{n-1} m(E_k)$$

and

$$t_n = \sum_{k=n}^{\infty} m(E_k).$$

Note that $m(E_k) \geq 0$ for all $k$. Therefore $\{s_n\}$ is a monotone increasing sequence and $\{t_n\}$ is a monotone decreasing sequence with the following properties:

$$\lim_{n \to \infty} s_n = M$$

and the "tail"

$$\lim_{n \to \infty} t_n = 0.$$ 

In other words,

$$\forall \epsilon > 0, \exists N \text{ such that } n > N \Rightarrow t_n < \epsilon$$

By the properties of measurability,

$$E = \bigcap_{n=1}^{\infty} B_n$$

$$m(E) = m\left(\bigcap_{n=1}^{\infty} B_n\right) \leq m(B_n) \forall n \in \mathbb{N}$$

$$B_n = \bigcup_{k=n}^{\infty} E_k$$
\[ m(B_n) = m \left( \bigcup_{k=n}^{\infty} E_k \right) \leq \sum_{k=n}^{\infty} m(E_k) = t_n. \]

\[ 0 \leq m(E) \leq m(B_n) \leq t_n \leq \epsilon \]

\[ \Rightarrow m(E) = 0 \]

2 Exercise 17

From Stein and Sakarchi we have: Let \( \{f_n\} \) be a sequence of measurable functions on \([0, 1]\) with \( |f_n(x)| < \infty \) a.e. \( x \). Then \( \exists \) a sequence \( c_n \) of positive real numbers such that:

\[ \frac{f_n(x)}{c_n} \to 0 \text{ a.e. } x \]

Proof. First we show that \( \exists k > 0 \) where for the set

\[ A^n_k = \{ x : |f_n(x)| > k \} \text{ then} \]

\[ m(A^n_k) < \epsilon \]

Assume otherwise, then for some fixed \( n \), \( \exists \epsilon > 0 \) such that \( \forall k > 0, \)

\[ m\{ x : |f(x)| > k \} \geq \epsilon \]

then when we let \( k \to \infty \), we have

\[ \lim_{k \to \infty} m\{ x : |f_n(x)| > k \} \geq \epsilon, \]

which forms a decreasing sequence. We define the sequence as follows

\[ A_* = \{ x : |f_n(x)| = \infty \} \]

\[ A_{k+1} = \{ x : |f_n(x)| > k + 1 \} \]

\[ A_k = \{ x : |f_n(x)| > k \} \]

The sequence is then decreasing so that \( A_* \subset \ldots A_{k+1} \subseteq A_k \ldots \)

Then \( m(A_1) \geq m(A_2) \geq m(A_3) \ldots \) and further define

\[ \lim_{n \to \infty} m(A_n) = M > \epsilon > 0. \]
Then by Corollary 3.3 from our text which states: for a decreasing sequence where \( E_k \) decreases to \( E \) and the \( m(E_k) < \infty \) for some \( k \), then

\[
m(E) = \lim_{N \to \infty} m(E_N)
\]

We know this applies to our sequence because we are given that \( f_n(x) \) is a countable family of measurable functions and \( x \in [0, 1] \). Furthermore, since \( A_k \downarrow A_* \) and by the measurability of each \( A_k \) and by the boundedness of the measure of the sequence, that is, \( m(A_k) < \infty \), we can apply Corollary 3.3 which gives us:

\[
\lim_{k \to \infty} m(A_k) = m(A_*)
\]

When we combine these results with what we were given, by the definition of almost everywhere in our text; \(|f_n(x)| < \infty\ a.e.\ x\), implies

\[
m(A_*) = m \{ x : |f_n(x)| > \infty \} = 0
\]

this gives us a contradiction. Our premise was that the measure of the set had to be \( \geq \epsilon \) for \( \epsilon > 0 \), however, we have shown that:

\[
m \{ x : |f_n(x)| > k \} < \epsilon
\]

Note: \( \Rightarrow m(E_n) \leq \epsilon \Rightarrow m(\bigcap_{n \to \infty} E_n) \leq \epsilon \) We can now define \( \epsilon = \frac{1}{2^k} \) and let \( k = \frac{\epsilon n}{n} \). this gives us a sequence of sets, \( \{E_n\} \) which we have shown to be a countable family of sets where:

\[
E_n = \left\{ x : \left| \frac{f_n(x)}{c_n} \right| > \frac{1}{n} \right\}
\]

In addition, we know the \( m(E_n) \leq \frac{1}{2^k} \) as constructed for each of the sequences, which implies \( \sum_{n=1}^{\infty} (E_n) \leq 1 \). We can then define the set of \( E \) as follows

\[
E = \left\{ \limsup_{n \to \infty} E_n \right\}.
\]

\[\Rightarrow E = \left\{ x : \left| \frac{f_n(x)}{c_n} \right| > \frac{1}{n} \ for \ i.o. \ n \right\}
\]

It is left to show that \( x \) appears in infinitely many \( E_n \), then by the Borel-Cantelli Lemma, \( m(E) = 0 \). If this is true, then \( \frac{f_n(x)}{c_n} \to 0 \ a.e. \ x \)
Proof. Given some $x$, we define convergence of the sequence to be
\[
\lim_{n \to \infty} \left| \frac{f_n(x)}{c_n} \right| \to 0 \implies \forall \epsilon > 0 \exists N > 0 \text{ s.t. } \forall n > N, \left| \frac{f_n(x)}{c_n} \right| < \epsilon,
\]
If we assume this is not true, and this sequence does not converge to zero then we need to show that
\[
\exists \epsilon > 0, \forall N > 0 \exists n_N > N \text{ s.t. } \left| \frac{f_{n_N}(x)}{c_{n_N}} \right| > \epsilon
\]
If this is true then we know there is an $N_1$ s.t. $\frac{1}{N_1} < \epsilon$ and $\exists n_{N_1} > N$ where
\[
\left| \frac{f_{n_{N_1}}(x)}{c_n} \right| > \frac{1}{n_{N_1}} \implies x \in E_{n_{N_1}}
\]
We can continue in this way and show that $\exists N_2$ s.t. $N_2 > n_{N_1}$ further $\exists n_{N_2} > N_2 > N_1$. Now if we let $n_{N_1} = N_2$ we can continue to find $n_{N_k} = N_{k+1}$ for infinitely many $k$. Then by the Borel–Cantelli Lemma since this occurs infinitely often and since from part one $\sum m(E_n) < \infty$ then we can say:
\[
\frac{f_n(x)}{c_n} \to 0 \text{ a.e. } x
\]

3 Problem 1

Prove that the set of those $x \in \mathbb{R}$ such that there exist infinitely many fractions $p/q$ with relatively prime integers $p$ and $q$ such that
\[
\left| x - \frac{p}{q} \right| \leq \frac{1}{q^{2+\epsilon}}
\]
is a set of measure zero.

Proof. For any integers $p$ and $q$, define the interval $E_{\frac{p}{q}} = \left[ \frac{p}{q} + \frac{1}{q^{2+\epsilon}}, \frac{p}{q} - \frac{1}{q^{2+\epsilon}} \right]$.
It follows that $m(E_{\frac{p}{q}}) = \frac{2}{q^{2+\epsilon}}$ and
\[
\left| x - \frac{p}{q} \right| \leq \frac{1}{q^{2+\epsilon}} \iff x \in E_{\frac{p}{q}}.
\]
For every integer $n \in \mathbb{N}$, define $A_{\frac{p}{q}}$ to be the set of rational numbers of the form $\frac{p}{q}$ where $\frac{p}{q} \in [n, n+1]$. Each $A_{\frac{p}{q}}$ is a subset of $\mathbb{Q}$ and is therefore
countable. For each integer \( q' \), there are exactly \( q' + 1 \) elements of \( A_n \) of the form \( \frac{p}{q'} \). For example, the set \( A_0 \) is the set

\[
A_0 = \left\{ \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{3}{3}, \ldots \right\}.
\]

Consequently, \( \{E_{k_n}\}_{k_n \in A_n} \) is a countable family of measurable subsets in \( \mathbb{R} \) and

\[
\sum_{k_n \in A_n} m(E_{k_n}) < \sum_{q=1}^{\infty} (q + 1) \frac{2}{q^{2+\epsilon}} \leq \infty.
\]

Define the sets \( E_{(n)} = \{ x \in [n, n + 1] : x \in E_{\frac{p}{q}} \text{ for infinitely many } \frac{p}{q} \in A_n \} \) and \( E^* = \{ x \in \mathbb{R} : x \in E_{\frac{p}{q}} \text{ for infinitely many } \frac{p}{q} \in \mathbb{Q} \} \). By the Borel-Cantelli Lemma, for any integer \( n \), the set \( E_{(n)} \) is measurable and \( m(E_{(n)}) = 0 \). In other words, the set of those \( x \in [n, n + 1] \) such that there exist infinitely many fractions \( p/q \in [n, n + 1] \) with relatively prime integers \( p \) and \( q \) such that

\[
\left| x - \frac{p}{q} \right| \leq \frac{1}{q^{2+\epsilon}}
\]

is a set of measure zero. Clearly, \( \cup_{n=0}^{\infty} E_{(n)} \subset E^* \). To prove \( E^* = \cup_{n=0}^{\infty} E_{(n)} \), it suffices to show that \( E^* \subset \cup_{n=0}^{\infty} E_{(n)} \).

Assume \( \exists x \in \mathbb{R} \) such that \( x \in E^* \) but \( x \notin \cup_{n=0}^{\infty} E_{(n)} \). There must exist some integer \( n \) such that \( x \in [n, n + 1] \). In addition, \( x \) must also lie in infinitely many intervals \( E_{\frac{p}{q}} \), each of which must satisfy the requirement that \( n - 1 < \frac{p}{q} < n + 2 \). It follows that \( x \in (E_{(n-1)} \cup E_{(n)} \cup E_{(n+1)}) \subset \cup_{n=0}^{\infty} E_{(n)} \), which is a contradiction. This proves that

\[
E^* = \bigcup_{n=0}^{\infty} E_{(n)}
\]

and

\[
m(E^*) = \sum_{n=1}^{\infty} m(E_{(n)}) = 0
\]

which is the desired result.