The Borel-Cantelli Group

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1 Borel-Cantelli Lemma

Exercise 16 is the introduction of the Borel-Cantelli Lemma using Lebesque measure. An approach using probability measure will be introduced later in the course.

Definition 1.1. *Let $\{E_1, E_2, E_3, \ldots\}$ be a sequence of sets. From this, we can define a decreasing sequence of sets $B_1 \supseteq B_2 \supseteq B_3 \ldots$ as follows:*

$$B_n = \bigcup_{k=n}^{\infty} E_k$$

Let $E = \{x \colon x \in E_k$ for infinitely many $k\}$ be defined to mean

$$E = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k := \limsup_{n\to\infty}(E_n)$$

Lemma 1.1. *Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that*

$$\sum_{k=1}^{\infty} m(E_k) < \infty$$

and

$$E = \limsup_{k\to\infty}(E_k).$$

Then E is measurable and $m(E) = 0$.

Proof. To show that E is measurable, note that

$$B_n = \bigcup_{k=n}^{\infty} E_k.$$

Each set B_i is the countable union of measurable sets and is therefore measurable. Furthermore,

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \bigcap_{n=1}^{\infty} B_n.$$

The set E is the countable intersection of measurable sets and is therefore measureable.

It remains to be shown that $m(E) = 0$. Note that

$$\sum_{k=1}^{\infty} m(E_k) = M < \infty.$$

Define the two sequences

$$s_n = \sum_{k=1}^{n-1} m(E_k)$$

and

$$t_n = \sum_{k=n}^{\infty} m(E_k).$$

Note that $m(E_k) \geq 0$ for all k. Therefore $\{s_n\}$ is a monotone increasing sequence and $\{t_n\}$ is a monotone decreasing sequence with the following properties:

$$\lim_{n\to\infty} s_n = M$$

and the "tail"

$$\lim_{n\to\infty} t_n = 0.$$

In other words,

$$\forall \epsilon > 0, \exists N \quad \text{such that} \quad n > N \Rightarrow t_n < \epsilon$$

By the properties of measurability,

$$E = \bigcap_{n=1}^{\infty} B_n$$

$$m(E) = m\left(\bigcap_{n=1}^{\infty} B_n\right) \leq m(B_n) \;\; \forall n \in \mathbb{N}$$

$$B_n = \bigcup_{k=n}^{\infty} E_k$$

$$m(B_n) = m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \sum_{k=n}^{\infty} m(E_k) = t_n.$$

$$0 \leq m(E) \leq m(B_n) \leq t_n \leq \epsilon$$

$$\Rightarrow m(E) = 0$$

2 Exercise 17

From Stein and Sakarchi we have: Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ with $|f_n(x)| < \infty$ a.e. x . Then $\exists$ a sequence c_n of positive real numbers such that:

$$\frac{f_n(x)}{c_n} \to 0 \; a.e. \; x$$

Proof. First we show that $\exists\, k > 0$ where for the set

$$A_k^n = \{x\colon\ |f_n(x)| > k\} \;\; then$$

$$m\left(A_k^n\right) < \epsilon$$

Assume otherwise, then for some fixed n, $\exists\, \epsilon > 0$ such that $\forall\, k > 0$,

$$m\{x\colon\ |f(x)| > k\} \geq \epsilon$$

then when we let $k \to \infty$, we have

$$lim_{k\to\infty} m\{x\colon\ |f_n(x)| > k\} \geq \epsilon,$$

which forms a decreasing sequence. We define the sequence as follows

$$A_* = \{x\colon\ |f_n(x)| = \infty\}$$

$$A_{k+1} = \{x\colon\ |f_n(x)| > k+1\}$$

$$A_k = \{x\colon\ |f_n(x)| > k\}$$

The sequence is then decreasing so that $A_* \subset \ldots A_{k+1} \subseteq A_k \ldots$.

Then $m(A_1) \geq m(A_2) \geq m(A_3) \ldots$ and further define

$$\lim_{n\to\infty} m(A_n) = M > \epsilon > 0.$$

Then by Corollary 3.3 from our text which states: for a decreasing sequence where E_k decreases to E and the $m(E_k) < \infty$ for some k, then

$$m(E) = \lim_{N \to \infty} m(E_N)$$

We know this applies to our sequence because we are given that $f_n(x)$ is a countable family of measurable functions and $x \in [0,1]$. Furthermore, since $A_k \downarrow A_*$ and by the measurability of each A_k and by the boundednes of the measure of the sequence, that is, $m(A_k) < \infty$, we can apply Corollary 3.3 which gives us:

$$\lim_{k \to \infty} m(A_k) = m(A_{k_*})$$

When we combine these results with what we were given, by the definition of almost everywhere in our text; $|f_n(x)| < \infty$ $a.e.$ x, implies

$$m(A_*) = m\{x\colon\ |f_n(x)| > \infty\} = 0$$

this gives us a contradiction. Our premise was that the measure of the set had to be $\geq \epsilon$ for $\epsilon > 0$, however, we have shown that:

$$m\{x\colon\ |f_n(x)| > k\} < \epsilon$$

Note: $\Rightarrow m(E_n) \leq \epsilon \Rightarrow m\left(\bigcap_{n \to \infty} E_n\right) \leq \epsilon$ We can now define $\epsilon = \frac{1}{2^n}$ and let $k = \frac{c_n}{n}$. this gives us a sequence of sets, $\{E_n\}$ which we have shown to be a countable family of sets where:

$$E_n = \left\{x\colon\ \left|\frac{f_n(x)}{c_n}\right| > \frac{1}{n}\right\}$$

In addition, we know the $m(E_n) \leq \frac{1}{2^n}$ as constructed for each of the sequences, which implies $\sum_{n=1}^{\infty}(E_n) \leq 1$. We can then define the set of E as follows

$$E = \left\{\limsup_{n \to \infty} E_n\right\}.$$

$$\Rightarrow E = \left\{x\colon\ \left|\frac{f_n(x)}{c_n}\right| > \frac{1}{n}\ for\ i.o.\ n\right\}$$

$\square$

It is left to show that x appears in infinitely many E_n, then by the Borel-Cantelli Lemma, $m(E) = 0$. If this is true, then $\frac{f_n(x)}{c_n} \to 0$ $a.e.$ x

Proof. Given some x, we define convergence of the sequence to be

$$\lim_{n\to\infty}\left|\frac{f_n(x)}{c_n}\right| \to 0 \;\Rightarrow \forall\epsilon > 0\;\exists\, N > 0\; s.t.\; \forall\, n > N,\; \left|\frac{f_n(x)}{c_n}\right| < \epsilon,$$

If we assume this is not true, and this sequence does not converge to zero then we need to show that

$$\exists\, \epsilon > 0, s.t.\; \forall\, N > 0\; \exists\, n_N > N\; s.t.\; \left|\frac{f_{n_N}(x)}{c_{n_N}}\right| > \epsilon$$

If this is true then we know there is an N_1 $s.t.$ $\frac{1}{N_1} < \epsilon$ and $\exists n_{N_1} > N$ where

$$\left|\frac{f_{n_{N_1}}(x)}{c_n}\right| > \frac{1}{n_{N_1}} \Rightarrow x \in E_{n_{N_1}}$$

We can continue in this way and show that $\exists\, N_2$ $s.t.$ $N_2 > n_{N_1}$ $further$ $\exists\, n_{N_2} > N_2 > N_1$. Now if we let $n_{N_1} = N_2$ we can continue to find $n_{N_k} = N_{k+1}$ for infinitely many k. Then by the $Borel - Cantelli\ Lemma$ since this occurs infinitly often and since from part one $\sum m(E_n) < \infty$ then we can say:

$$\frac{f_n(x)}{c_n} \to 0\; a.e.\; x$$

□

3 Problem 1

Prove that the set of those $x \in \mathbb{R}$ such that there exist infinitely many fractions p/q with relatively prime integers p and q such that

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{q^{2+\epsilon}}$$

is a set of measure zero.

Proof. For any integers p and q, define the interval $E_{\frac{p}{q}} = \left[\frac{p}{q} + \frac{1}{q^{2+\epsilon}}, \frac{p}{q} - \frac{1}{q^{2+\epsilon}}\right]$. It follows that $m\left(E_{\frac{p}{q}}\right) = \frac{2}{q^{2+\epsilon}}$ and

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{q^{2+\epsilon}} \Leftrightarrow x \in E_{\frac{p}{q}}.$$

For every integer $n \in \mathbb{N}$, define $A_{\frac{p}{q}}$ to be the set of rational numbers of the form $\frac{p}{q}$ where $\frac{p}{q} \in [n, n+1]$. Each $A_{\frac{p}{q}}$ is a subset of $\mathbb{Q}$ and is therefore

countable. For each integer q', there are exactly $q'+1$ elements of A_n of the form $\frac{p}{q'}$. For example, the set A_0 is the set

$$A_0 = \left\{ \frac{0}{1}, \frac{1}{1}, \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \ldots \right\}.$$

Consequently, $\{E_{k_n}\}_{k_n \in A_n}$ is a countable family of measurable subsets in $\mathbb{R}$ and

$$\sum_{k_n \in A_n} m(E_{k_n}) < \sum_{q=1}^{\infty} (q+1) \frac{2}{q^{2+\epsilon}} \leq \infty.$$

Define the sets $E_{(n)} = \left\{ x \in [n, n+1] : x \in E_{\frac{p}{q}} \text{ for infinitely many } \frac{p}{q} \in A_n \right\}$ and $E^* = \left\{ x \in \mathbb{R} : x \in E_{\frac{p}{q}} \text{ for infinitely many } \frac{p}{q} \in \mathbb{Q} \right\}$. By the Borel-Cantelli Lemma, for any integer n, the set $E_{(n)}$ is measurable and $m(E_{(n)}) = 0$. In other words, the set of those $x \in [n, n+1]$ such that there exist infinitely many fractions $p/q \in [n, n+1]$ with relatively prime integers p and q such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^{2+\epsilon}}$$

is a set of measure zero. Clearly, $\cup_{n=0}^{\infty} E_{(n)} \subset E^*$. To prove $E^* = \cup_{n=0}^{\infty} E_{(n)}$, it suffices to show that $E^* \subset \cup_{n=0}^{\infty} E_{(n)}$.

Assume $\exists x \in \mathbb{R}$ such that $x \in E^*$ but $x \notin \cup_{n=0}^{\infty} E_{(n)}$. There must exist some integer n such that $x \in [n, n+1]$. In addition, x must also lie in infinitely many intervals $E_{\frac{p}{q}}$, each of which must satisfy the requirement that $n - 1 < \frac{p}{q} < n + 2$. It follows that $x \in \left(E_{(n-1)} \cup E_{(n)} \cup E_{(n+1)} \right) \subset \cup_{n=0}^{\infty} E_{(n)}$, which is a contradiction. This proves that

$$E^* = \bigcup_{n=0}^{\infty} E_{(n)}$$

and

$$m(E^*) = \sum_{n=1}^{\infty} m(E_{(n)}) = 0$$

which is the desired result. $\square$