MATH 565: Introduction to Harmonic Analysis - Spring 2008
Homework/Project # 3

We have three teams that will work on different components analyzing the discrete and finite Hilbert transforms. Each team will also do a homework problem of the three listed in class some weeks ago, as assigned below (part (b), you are welcome to try all of them not just yours!)

Consider the “discrete” Hilbert transform

\[ H = \{ h_m \}_{m,n \in \mathbb{Z}}, \quad h_{mn} = \begin{cases} \frac{1}{m-n} & m \neq n \\ 0 & m = n \end{cases} \]

whose action on a doubly infinite sequence \( x = \{ x_n \}_{n \in \mathbb{Z}} \) is given by

\[ (Hx)(i) = \sum_{j \in \mathbb{Z}, j \neq i} \frac{x_j}{i-j}, \quad i \in \mathbb{Z}. \]

Consider also a finite dimensional analogue, defined for \( z = (z_{-N}, \ldots, z_{-1}, z_0, z_1, \ldots, z_N) \in \mathbb{C}^{2N+1} \) by,

\[ (Hz)(i) = \sum_{|k| \leq N, k \neq i} \frac{x_j}{i-j}, \quad |i| \leq N. \]

**Finite Team 1** (Jorge, Oleksandra, Daewon):

(a) Check that \( H_N \) is bounded in \( \ell^2(\mathbb{Z}, N) \) with bounds independent of the dimension. Show that the \( \ell^1(\mathbb{Z}, N) \) norms grow with the dimension, find the rate of growth. Find applications of the discrete/finite Hilbert transforms to showcase. Consider the finite Fourier transform and its inverse, and compare the inverse finite Fourier transform of the Fourier multiplier given by multiplication by \( \text{isgn}(n) \)

(b) Given \( f \in L^p(\mathbb{R}^d) \) show that

\[ \lim_{|h| \to \infty} \| f + \tau_h f \|_p = 2^{1/p} \| f \|_p \]

**Fourier Team 2** (Jaime, Taylor, Arnab)

(a) Provide Fourier arguments to show that the discrete Hilbert transform \( H \) is bounded in \( \ell^2(\mathbb{Z}) \), is it an isometry perhaps up to a constant?)

(b) Prove a weighted version of Schur’s Lemma: Let \( k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \) be a measurable function. Suppose \( w_1, w_2 \) are measurable functions in \( \mathbb{R}^d \), and \( M_1, M_2 \) positive constants such that

\[ \int w_1(x) |k(x, y)| dx \leq M_1 w_2(y), \quad \text{a.e. y} \]

\[ \int w_2^{p/p'}(x) |k(x, y)| dy \leq M_2 w_1^{p/p'}(x), \quad \text{a.e. x}. \]

Then the integral operator \( T f(x) = \int f(y) K(x, y) dy \) is bounded in \( L^p \), moreover,

\[ \| T f \|_p \leq M_1^{1/p} M_2^{1/p'} \| f \|_p. \]

**Cotlar’s Team 3** (Jean, Jim, Hector)

(a) Use Cotlar’s Lemma to show that the discrete Hilbert transform \( H \) is bounded in \( \ell^2(\mathbb{Z}) \).

(b) Let \( k_\varepsilon \) be the truncated continuous Hilbert kernel, \( k_\varepsilon(x) = \frac{1}{2} \chi_{|k|<\varepsilon} \). Verify that \( k_\varepsilon \ast k_\varepsilon \in L^1(\mathbb{R}) \) with \( L^1 \)-bound independent of \( \varepsilon > 0 \).

Presentations and reports due on the Review and final’s Weeks May 5-9, May 12-17, 2008.