MATH 565 - Introduction to Harmonic Analysis
Homework # 4

1. The Theta function is defined for $s > 0$ to be

$$\Theta(s) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 s}.$$

The Riemann Zeta funcion is defined to be for $s > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The Gamma function is defined for $s > 0$ to be

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx.$$

Show that they are connected via the following identity for $s > 1$,

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^{\infty} t^{\frac{s}{2}-1}(\Theta(t) - 1)dt.$$

2. The Heisenberg uncertainty principle states that if $\psi \in \mathcal{S}$ and is normalized in $L^2(\mathbb{R})$, that is $\|\psi\|_2 = 1$, then

$$\left(\int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx\right)\left(\int_{-\infty}^{\infty} \xi^2 |\hat{\psi}(\xi)|^2 d\xi\right) \geq \frac{1}{16\pi^2}.$$

Verify that identity holds if and only if $\psi(x) = Ae^{-Bx^2}$, for $B > 0$ and $A^2 = \sqrt{2B/\pi}$.
Verify that the following inequality also holds for all $x_0, \xi_0 \in \mathbb{R}$,

$$\left(\int_{-\infty}^{\infty} (x - x_0)^2 |\psi(x)|^2 dx\right)\left(\int_{-\infty}^{\infty} (\xi - \xi_0)^2 |\hat{\psi}(\xi)|^2 d\xi\right) \geq \frac{1}{16\pi^2}.$$

Due on Thursday March 3rd, 2005.