MATH 561 - COMPLEX ANALYSIS I
Homework # 2:

HE = Hahn-Epstein's book, GK = Greene-Krantz's book.

1. (HE Chapter 2, ex. 3)

(a) Let $\{a_n\}$ be a sequence of positive real numbers. Show that

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \le \liminf_{n\to\infty} (a_n)^{1/n} \le \limsup_{n\to\infty} (a_n)^{1/n} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Therefore if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists, so does $\lim_{n\to\infty} (a_n)^{1/n}$, and they coincide.

(b) Give an example where equalities don't hold. Give an example where $\lim_{n\to\infty} (a_n)^{1/n}$ exists but $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist.

2. (HE Chapter 2, ex. 7) A conditionally convergent series of real numbers can be reordered so that the new series adds up to any prescribed real number or even diverges. Discuss the validity of the complex version of this result.

3. (HE Chapter 2, ex. 12) Let a_n and b_n be non-zero complex numbers for $n = 1, 2, 3, \ldots$ Suppose $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \ell$ exists, and $\ell \neq 0, \infty$. Show that if one of the series $\sum_{n=1}^{\infty} a_n$ or $\sum_{n=1}^{\infty} b_n$ converges absolutely, then so does the other. In particular, the power series $\sum_{n=1}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} b_n z^n$ have the same radii of convergence. What if $\ell = 0$ or $\ell = \infty$?

4. (HE Chapter 2, exs. 15, 16, 17, 18, 24)

(a) Discuss the convergence and uniform convergence of the sequence $\{nz^n\}_{n=1}^{\infty}$.

(b) Discuss the convergence and uniform convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2(1-z^n)}$.

(c) Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+|z|}$ is not absolutely convergent but it is uniformly convergent in the whole complex plane.

(d) Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{z}{(1+|z|)^n}$ is absolutely convergent for all z, but it is not convergent on all compact subsets of the complex plane (*locally uniformly*).

(e) Show that the series $\displaystyle\sum_{n=1}^{\infty} e^{-nz}$ converges uniformly on compact subsets of the open right half-plane $\{z \in C : \mathrm{Re} z > 0\}$. Find the sum of the series.

(f) Discuss the convergence (pointwise, absolute, uniform) of the series:

$$\sum_{n=1}^{\infty} \left(\frac{z+i}{z-1}\right)^n.$$

5. (HE Chapter 2, ex. 27)

(a) (Qual Aug 1996) Suppose $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$ has the property that the series $\sum_{n=0}^{\infty} f^{(n)}(c)$ converges. Show that f is an *entire* function (i.e. a function that can be expressed as power series with radius of convergence $R = \infty$).

(b) What if the convergence of $\sum_{n=0}^{\infty} f^{(n)}(c)$ is replaced by the assumption that the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{(n+1)^{2002}} z^n$$

has a positive radius of convergence?

6. (HE Chapter 2, ex. 30) (Qual Aug 1996) Show that an entire function that takes real values on the real axis and purely imaginary values on the imaginary axis must be an odd function:

$$f(-z) = -f(z), \quad \text{for all } z \in C.$$

7. Compute the radius of convergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} z^n$ **(Qual Aug 1994)** .

(b) $\sum_{n=0}^{\infty} (n + a^n) z^n$, where $a \in C$ **(Qual Aug 1994)** .

(c) The Taylor series around zero for the function $z \cot z$ **(Qual Aug 1994)** .

(d) $\sum_{n=1}^{\infty} \frac{(3n)!}{(3n)^{3n}} z^n$ **(Qual Jan 1994)**.

(e) $\sum_{n=1}^{\infty} (3 + (-1)^n)^n z^n$ **(Qual Jan 1994)**.

(c) The Taylor series around zero for the function $\frac{z}{e^z - 1}$ **(Qual Jan 1994)**.

This homework is due on Wednesday September 11th.