
1. (HE Exercise 3, p. 12) Prove the parallelogram law:

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2),$$

for two arbitrary complex numbers $z, w$. Interpret the equality geometrically.

2. (HE Exercise 5, p. 13) If $|\alpha| < 1$ and $|z| \leq 1$, show that

$$|z - \alpha| \leq 1.$$ 

When does equality hold?

3. (GK Exercise 7, p. 21) Let $p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ have real coefficients $a_j \in \mathbb{R}$, $0 \leq j \leq n$. Prove that if $z_0$ is a root pf the polynomial $p$ (i.e. $p(z_0) = 0$), then also $\overline{z_0}$ is a root of $p$. Give a counterexample when the coefficients are not all real.

4. (GK Exercise 8, p. 21) A field $F$ is said to be ordered if there is a distinguished subset $P \subset F$ such that:

   (i) if $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$;
   
   (ii) if $a \in F$ then precisely one of the following holds: $a \in P$ or $-a \in P$ or $a = 0$.

Verify that $\mathbb{R}$ is an ordered field when $P$ is chosen to be the strictly positive real numbers. Prove that $\mathbb{C}$ is not an ordered field.

5. (GK Exercise 9, p. 21) Show that the function

$$\phi(z) = \frac{1 - z}{1 + z}$$

maps the open unit disc $D = \{z : |z| < 1\}$ one-to-one onto the upper half plane $U = \{z = x + iy : y > 0\}$. (The map $\phi$ is called the Cayley transform.)

6. (HE Exercise 14(a), p. 14) Derive the identity

$$1 + \cos \theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin [(n + \frac{1}{2})\theta]}{2 \sin \frac{\theta}{2}} = \frac{\cos \frac{n\theta}{2} \sin \frac{(n+1)\theta}{2}}{\sin \frac{\theta}{2}},$$

for $0 < \theta < 2\pi$.

This homework is due on Monday August 26.