Choose 3 problems from the list to turn them on Tuesday November 3rd, 2009.

1. Rudin Chapter 4 # 8 (unif. cont. on bounded set implies bounded).
2. Rudin Chapter 4 # 11 (unif. cont. iff Cauchy into Cauchy).
3. Rudin Chapter 4 # 14 (fixed point).

4. (Qual Fall 2009 # 2)
   (a) Show that the function \( f(x) = \sin \left( \frac{\pi}{x} \right) \) is continuous on the interval \((0, 1)\).
   (b) Is \( f \) uniformly continuous on \((0, 1)\)?
   (c) For a real-valued function \( g \) defined on a metric space \((X, d)\) let
       \[
       w(r) = \sup \{|g(x) - g(x')| : d(x, x') \leq r\}.
       \]
       Show that \( g \) is a uniformly continuous function if and only if \( \lim_{r \to 0} w(r) = 0 \).

5. (Qual Aug 1999 # 6) Let \( S \) be a bounded open interval \( S = (a, b) \). Let \( \overline{S} = [a, b] \) be its closure. Let \( f \) be a function defined on \( S \).
   (b) Show that if \( f \) is uniformly continuous on \( S \) then it can be extended continuously to \( \overline{S} \).
   (c) Give an example of a continuous function on \( S \) that cannot be extended continuously to \( \overline{S} \).

6. (Qual Aug 2002 # 4) The space \( C([a, b]) \) of real-valued continuous functions over the interval \([a, b]\) is a complete metric space with the distance induced by the sup norm,
   \[
   d(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)|.
   \]
   (a) Given a real-valued function \( F \) defined on \( C([a, b]) \), define continuity and uniform continuity of \( F \).
   (b) Let \( F : C([a, b]) \to \mathbb{R} \) be given by \( F(f) = \int_{a}^{b} f(t) \, dt \). Show that \( F \) is uniformly continuous.
       (Operate here with integrals freely.)

7. (a) (Qual Aug 2008 # 2) Let \( K \) be a compact subset of \( \mathbb{R} \), and let \( f \) be a real-valued function defined on \( K \). Denote by \( \Gamma_f \) the graph of \( f \), a subset of \( \mathbb{R}^2 \), more precisely,
   \[
   \Gamma_f = \{(x, y) \in \mathbb{R}^2 : x \in K, y = f(x)\}.
   \]
   Show that \( f \) is continuous on \( K \) if and only if its graph \( \Gamma_f \) is a compact subset of \( \mathbb{R}^2 \).
Let $K$ be a compact metric space with metric $d$ and let $f$ be a continuous real-valued function defined on $K$. Prove that the graph of the function 

$$\Gamma_f = \{(x, y); x \in K, y = f(x)\}$$

is a compact set in the metric space $(K \times \mathbb{R}, \rho)$, where

$$\rho((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + |y_1 - y_2|.$$ 

8. (Qual Jan 1999 # 3) Let $f : [a, b] \to \mathbb{R}$, where $[a, b]$ is a compact interval.

(a) Define what it means that $f$ is uniformly continuous.

(b) Prove that if $f$ is continuous, then it is uniformly continuous.

(c) Show that the graph of $f$, $G_f := \{(x, y) : x \in [a, b], y = f(x)\}$, can be covered with a finite number of rectangles such that the area of their union is smaller than $\epsilon$ for any given $\epsilon > 0$.  

(Remark: This implies that $G_f$ has measure zero in $\mathbb{R}^2$.)

9. (Qual Jan 2006 # 3) Let $(X, d)$, and $(Y, \rho)$ be metric spaces, $f : X \to Y$ a continuous function. Prove that if $X$ is compact and $f$ is one-to-one and onto (bijective) then $f^{-1} : Y \to X$ is continuous. Will the statement remain true if $X$ is not compact? Explain.

10. (Qual Aug 2008 #3, Rudin Chapter 4 #20) Let $(X, d)$ be a metric space. Let $E$ be a non-empty subset of $X$. Define the distance from $x \in X$ to $E$ by

$$\rho_E(x) = \inf_{y \in E} d(x, y).$$

(a) Prove that $\rho_E(x) = 0$ if and only if $x$ belongs to the closure of $E$, denoted $\overline{E}$.

(b) Prove that $\rho_E$ is a uniformly continuous function on $X$.

(Hint: show that $|\rho_E(x_1) - \rho_E(x_2)| \leq d(x_1, x_2)$.)

11. (Qual Fall 2009 # 4)

(a) The projection map $p : \mathbb{R}^2 \to \mathbb{R}$ is given by $p(x, y) = x$. Determine if the projection map is (i) a continuous map, (ii) an open map, or (iii) a closed map. We are using the standard metrics in the corresponding Euclidean spaces.

(b) Let $f : X \to Y$ be a continuous map between the metric spaces $(X, d)$ ad $(Y, \rho)$. If $K \subset X$ is compact, is it true that $f(K)$ is compact? Explain.

12. (Qual Aug 2000 # 5(b) ) Let $(X, \rho)$ be a compact metric space, and let $f : X \to X$ be an isometry, i.e., $\rho(f(x), f(y)) = \rho(x, y)$ for every $x, y \in X$. Show that $f$ is bijective (that is a one-to-one and onto function).