MATH 510 - Introduction to Analysis I - Fall 2009
Homework # 6 (continuous functions)

Choose 3 problems from the list below to be turned in on Tuesday October 27, 2009.

1. Rudin Chapter 4 # 1.

2. Rudin Chapter 4 # 3 (zero set is closed).

3. Rudin Chapter 4 # 4 (continuity and density).

4. Rudin Chapter 4 # 18 (bizarre example).

5. Rudin Chapter 4 # 23-24 (convex functions).

6. (Qual Jan 2002 # 2, Jan 2003 # 3) Let E, F be metric spaces, and let f be a function defined on E into F.

(a) Give the "$\epsilon - \delta$" of continuity.

(b) Show that $f : E \to F$ is continuous at c if and only if given any sequence $\{x_n\}$ in E convergent to c, then $f(x_n)$ converges to $f(x)$ when $n \to \infty$.

(c) Suppose $f^{-1}(V)$ is open in E for every open V in F. Show that f is continuous on E.

7. Suppose f is a continuous mapping from a metric space X to a metric space Y.

(a) **(Qual Aug 1997 # 2)** Prove that if K is a compact subset of X, then $f(K)$ is a compact subset of Y. Discuss a consequence that could be stated (but not proved) in a first year calculus text.

(b) **(Qual Aug 2004 # 1(b))** If $G \in X$ is an open set in X is it necessarilly true that $f(G)$ is open? Give an example.

8. (Qual Jan 2007 # 3) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at $x = 0$, and suppose that

$$f(x + y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Show that $f(x) = cx$ for some $c \in \mathbb{R}$.

9. (Qual Jan 2009 #4) Let (X, d) be a compact metric space and F a mapping $F : X \to X$ such that

$$d\big(F(x), F(y)\big) < d(x, y) \quad \text{for all} \quad x, y \in X.$$

(a) Show that the function $g : X \to [0, \infty)$, defined by $g(x) = d\big(x, F(x)\big)$, is a continuous function whose minimum must be zero.

(b) Show that the equation $F(x) = x$ has exactly one solution, i.e., F has exactly one fixed point.

10. (Qual Aug 2003 # 2) Let V be a vector space over the reals.

(a) Define a norm $\|\cdot\| : V \to \mathbb{R}$ on V, and show it is continuous on V.

(b) Let C^0 be the vector space of real valued continuous functions on $\mathbb{R}$, define $\|\cdot\|_a : C^0 \to \mathbb{R}$ for each $f \in C^0$ by

$$\|f\|_a = \sup_{x \in \mathbb{R}} (1 + |x|^a)|f(x)|, \quad \text{where} \quad a > 0.$$

Show that for each $a > 0$, $\|\cdot\|_a$ is a norm, and show that $(C^0, \|\cdot\|_a)$ is a complete metric space.

11. (Qual Jan 2006 # 2) Let $C([0, 1])$ be the metric space of continous real-valued functions on $[0, 1]$, with the uniform metric (i.e. $d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$). Denote by B the closed unit ball in $C([0, 1])$, that is,

$$B = \{f \in C([0, 1]) : \|f\|_\infty = d(f, 0) = \sup_{x \in [0,1]} |f(x)| \leq 1\}.$$

Show that B is not compact. **Hint:** Consider the sequence of functions $x, x^2, x^3, \ldots$.