MATH 510 - Introduction to Analysis I - Fall 2009
Homework # 6 (continuous functions)

Choose 3 problems from the list below to be turned in on Tuesday October 27, 2009.

1. Rudin Chapter 4 # 1.

2. Rudin Chapter 4 # 3 (zero set is closed).

3. Rudin Chapter 4 # 4 (continuity and density).

4. Rudin Chapter 4 # 18 (bizarre example).

5. Rudin Chapter 4 # 23-24 (convex functions).

6. (Qual Jan 2002 # 2, Jan 2003 # 3) Let $E, F$ be metric spaces, and let $f$ be a function defined on $E$ into $F$.
   
   (a) Give the “$\epsilon - \delta$” of continuity.

   (b) Show that $f : E \rightarrow F$ is continuous at $c$ if and only if given any sequence $\{x_n\}$ in $E$ converging to $c$, then $f(x_n)$ converges to $f(x)$ when $n \rightarrow \infty$.

   (c) Suppose $f^{-1}(V)$ is open in $E$ for every open $V$ in $F$. Show that $f$ is continuous on $E$.

7. Suppose $f$ is a continuous mapping from a metric space $X$ to a metric space $Y$.
   
   (a) (Qual Aug 1997 # 2) Prove that if $K$ is a compact subset of $X$, then $f(K)$ is a compact subset of $Y$. Discuss a consequence that could be stated (but not proved) in a first year calculus text.

   (b) (Qual Aug 2004 # 1(b)) If $G \in X$ is an open set in $X$ is it necessarily true that $f(G)$ is open? Give an example.

8. (Qual Jan 2007 # 3) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous at $x = 0$, and suppose that

   \[ f(x + y) = f(x) + f(y) \quad \text{for all} \quad x, y \in \mathbb{R}. \]

Show that $f(x) = cx$ for some $c \in \mathbb{R}$. 
9. (Qual Jan 2009 #4) Let \((X,d)\) be a compact metric space and \(F : X \to X\) such that

\[
d(F(x), F(y)) < d(x, y) \quad \text{for all} \quad x, y \in X.
\]

(a) Show that the function \(g : X \to [0, \infty)\), defined by \(g(x) = d(x, F(x))\), is a continuous function whose minimum must be zero.

(b) Show that the equation \(F(x) = x\) has exactly one solution, i.e., \(F\) has exactly one fixed point.

10. (Qual Aug 2003 #2) Let \(V\) be a vector space over the reals.

(a) Define a norm \(\| \cdot \| : V \to \mathbb{R}\) on \(V\), and show it is continuous on \(V\).

(b) Let \(C^0\) be the vector space of real valued continuous functions on \(\mathbb{R}\), define \(\| \cdot \|_a : C^0 \to \mathbb{R}\) for each \(f \in C^0\) by

\[
\| f \|_a = \sup_{x \in \mathbb{R}} (1 + |x|^a)|f(x)|, \quad \text{where} \quad a > 0.
\]

Show that for each \(a > 0\), \(\| \cdot \|_a\) is a norm, and show that \((C^0, \| \cdot \|_a)\) is a complete metric space.

11. (Qual Jan 2006 #2) Let \(C([0,1])\) be the metric space of continuous real-valued functions on \([0,1]\), with the uniform metric (i.e. \(d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|\)). Denote by \(B\) the closed unit ball in \(C([0,1])\), that is,

\[
B = \{ f \in C([0,1]) : \| f \|_\infty = d(f, 0) = \sup_{x \in [0,1]} |f(x)| \leq 1 \}.
\]

Show that \(B\) is not compact. **Hint:** Consider the sequence of functions \(x, x^2, x^3, \ldots\).